R-ratio from Lattice QCD

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Outline

- Inverse Problem
- R-ratio on the Lattice
- a_μ^{HVP} on the Lattice
- Summary and Outlook

Inverse Problem

The two-point correlation functions in finite volume can be expressed as:

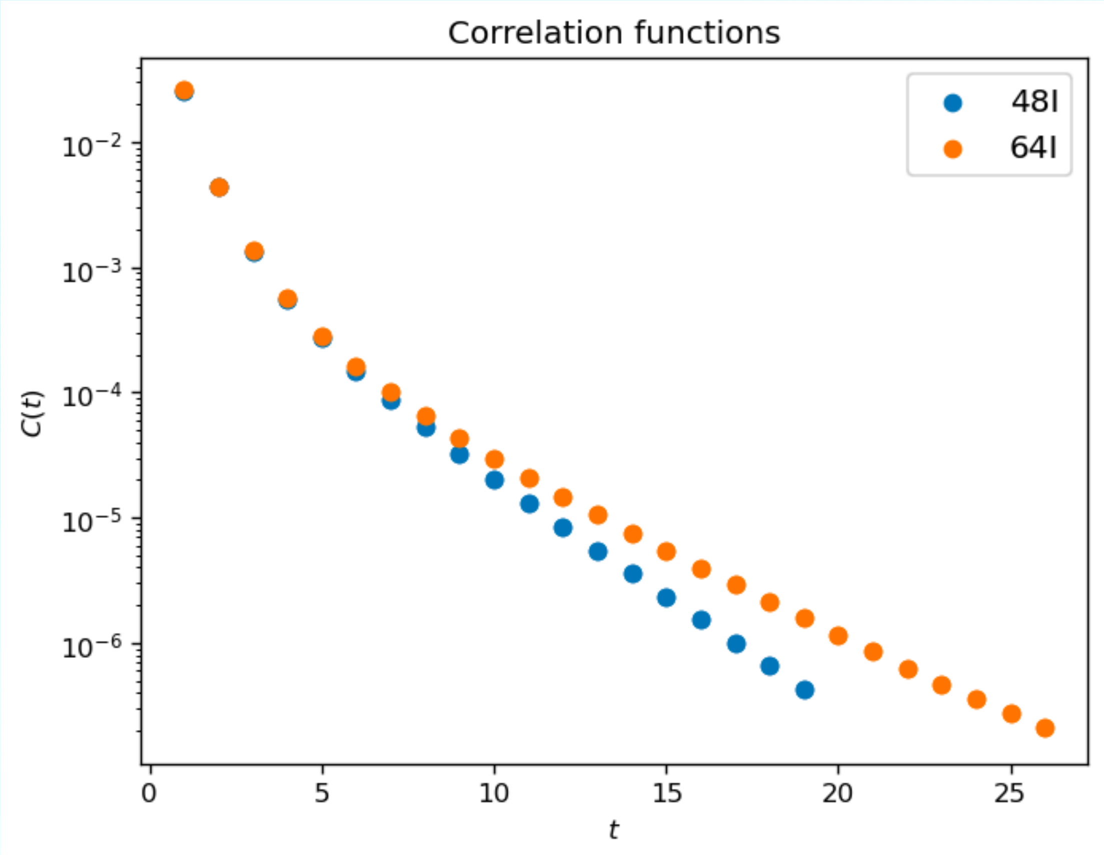
$$C_2(t) = \int d\omega \rho_L(\omega) e^{-\omega t}$$

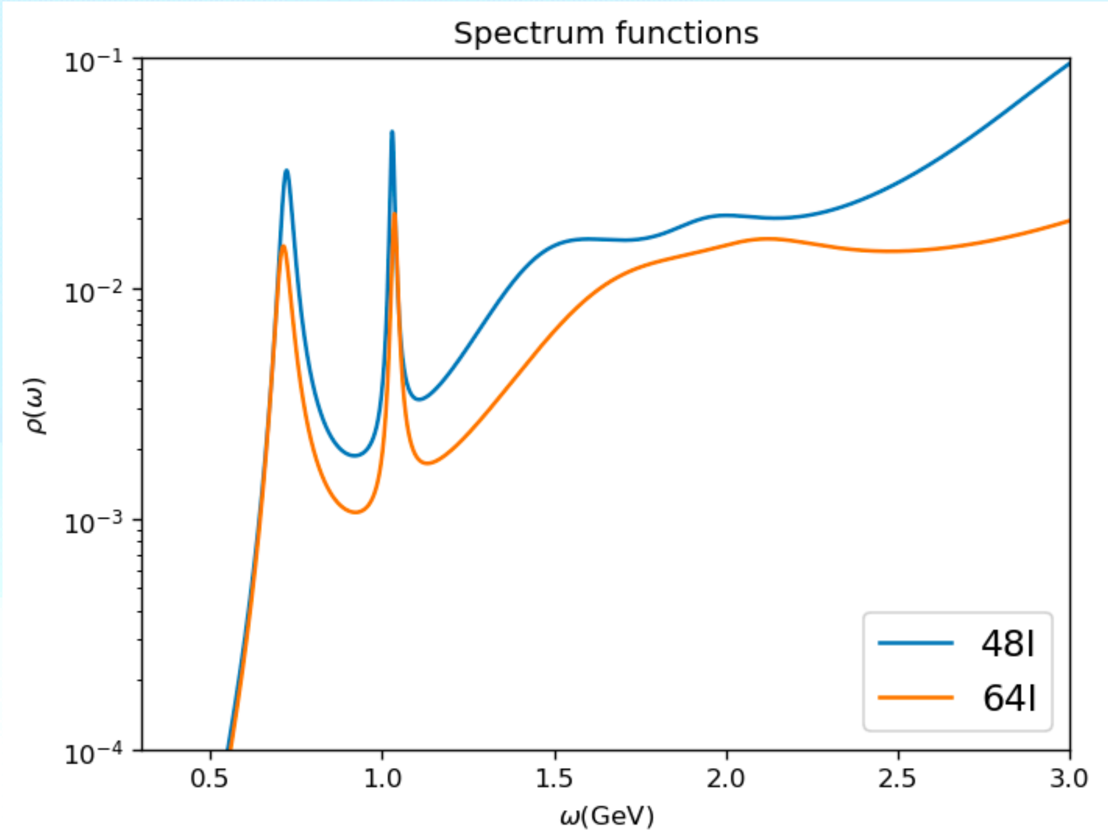
$\rho_L(\omega)$ is the spectral function at finite volume, $e^{-\omega t}$ is the kernel function.

Generally, the number of lattice data is much less that the number of ω we want, so it is called an inverse problem("ill-posed" problem).

Bayesian Reconstruction (BR) method is one of the most effective methods for solving the inverse problems.

$$P[\rho | D, m] = \frac{P[D | \rho, I]}{P[D | m]} \int d\alpha P[\alpha | D, m]$$



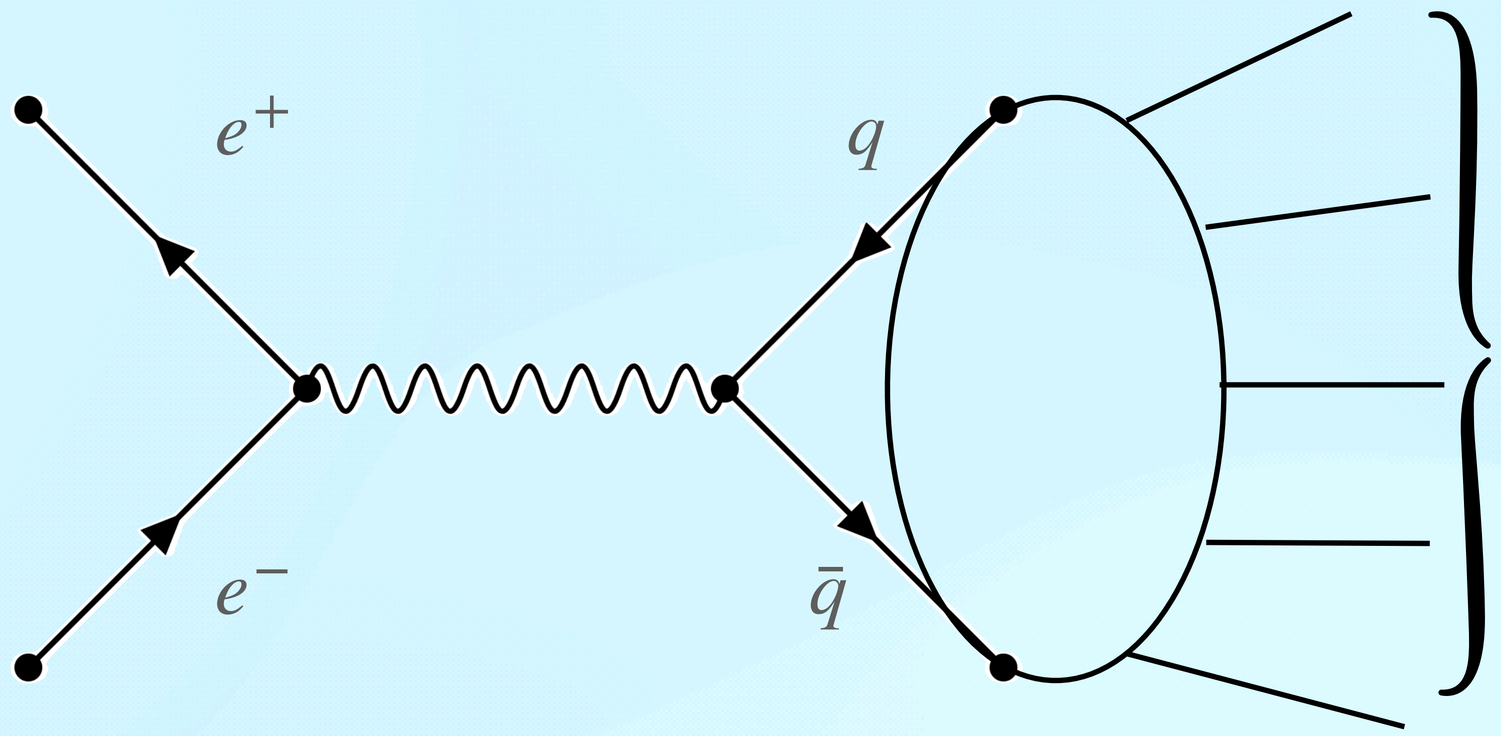


- Studying hadronic spectroscopy (Calculate the spectral functions)
- Studying deep inelastic scatterings (Calculate the hadronic tensor)
- R-ratio、hadronic decay widths etc.

Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)
J. Liang et al., PRD. 101, 114503 (2020)
C. Alexandrou et al.(ETMC), PRL. 130, 241901 (2023).
M. T. Hansen et al., PRD. 96.094513(2017).
J. Karpie, et al., jhep. 04, 057 (2019).

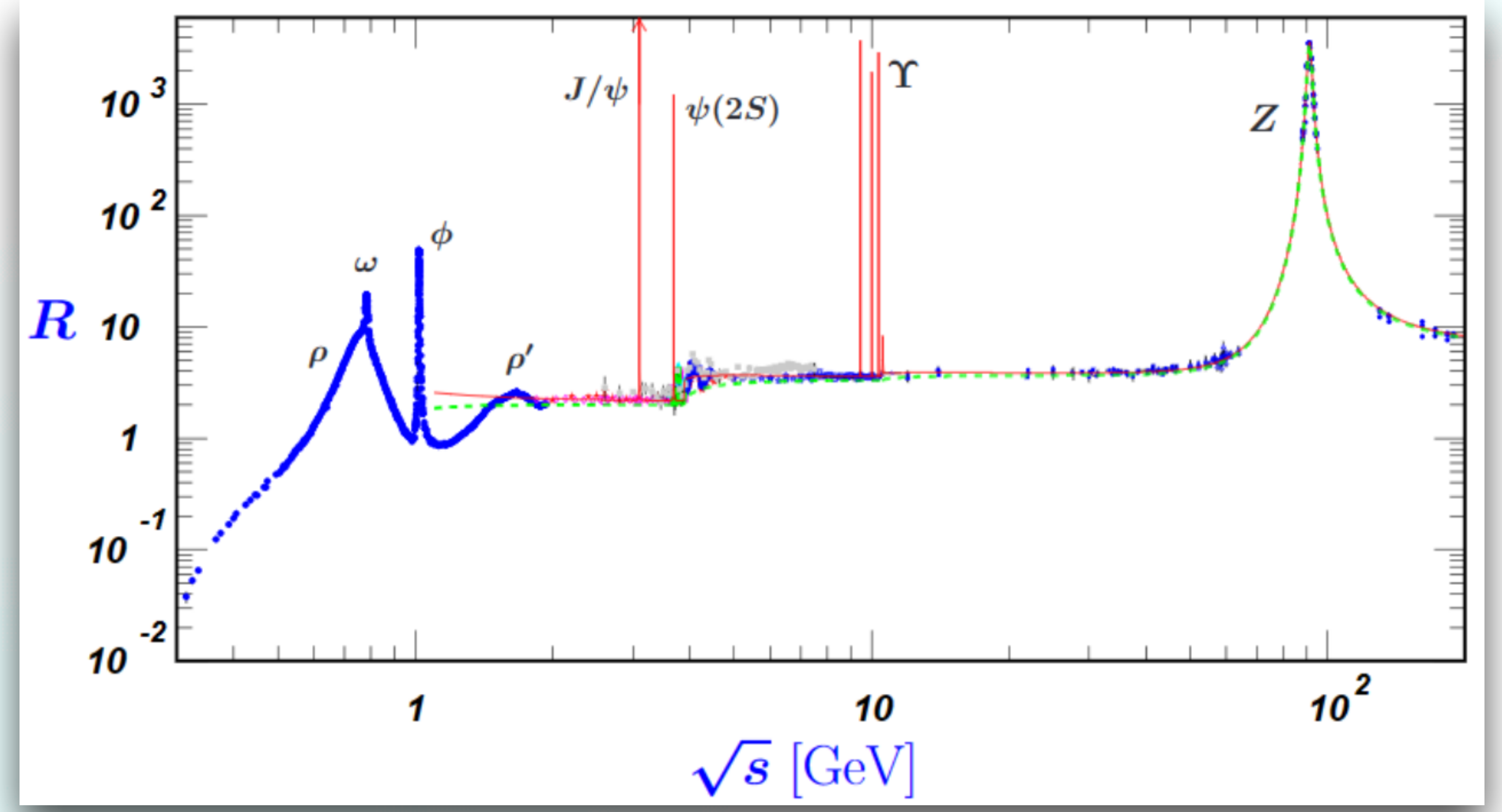
Motivation

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



Hadrons

$$R = N_c \sum_{i=1}^{n} Q_i^2 = \begin{cases} \frac{2}{3}N_c & \text{for } q = u, d, s; \\ \frac{10}{9}N_c & \text{for } q = u, d, s, c; \\ \frac{11}{9}N_c & \text{for } q = u, d, s, c, b; \end{cases}$$



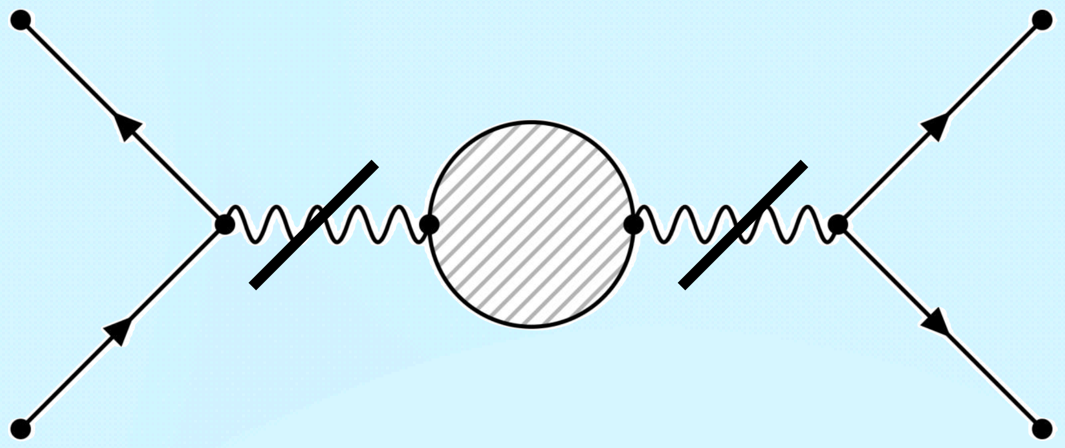
V. V. Ezhela et al. arXiv:hep-ph/0312114 2004.

- R-ratio is a basic experimental measurement and has very important physical significance.
- Good playground to check the BR algorithms solving the inverse problem. *J. Liang et al., PRD101, 114503 (2020)* *Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)*
- Closely related to the HVP contribution to muon g - 2.

$$a_\mu^{\mathrm{Had}}[\mathrm{LO}] = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds \frac{K(s)}{s} R^{(0)}(s)$$

P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2021)

R-ratio on the Lattice



$$C_2(t) = \left\langle J_\mu^{\mathrm{em}}(t)\ J_\mu^{\mathrm{em}}(0) \right\rangle = \int d\omega \rho(\omega) e^{-\omega t}$$

BR ↓ inverse problem

$$\rho_L(\omega) = \sum_n A_n \delta(\omega, \omega_n)$$

↓ smearing

$$\rho_{\sigma,L}(E) = \int_0^\infty d\omega \Delta_\sigma(E,\omega)\rho_L(\omega)$$

$$\Delta_\sigma(\omega, E) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\mathrm{E}-\omega)^2}{2\sigma^2}\right)$$

↓

$$R_\sigma(\omega) = \frac{12\pi^2}{\omega^2}\rho_{\sigma,L}(\omega)$$

- For finite lattice size the spectrum is discrete, in principle, lattice spectrum functions are delta functions.
- Physically, the spectral function is the result of a continuous.
- We can compare the experimental results with the smeared lattice spectrum function.

Ensemble details

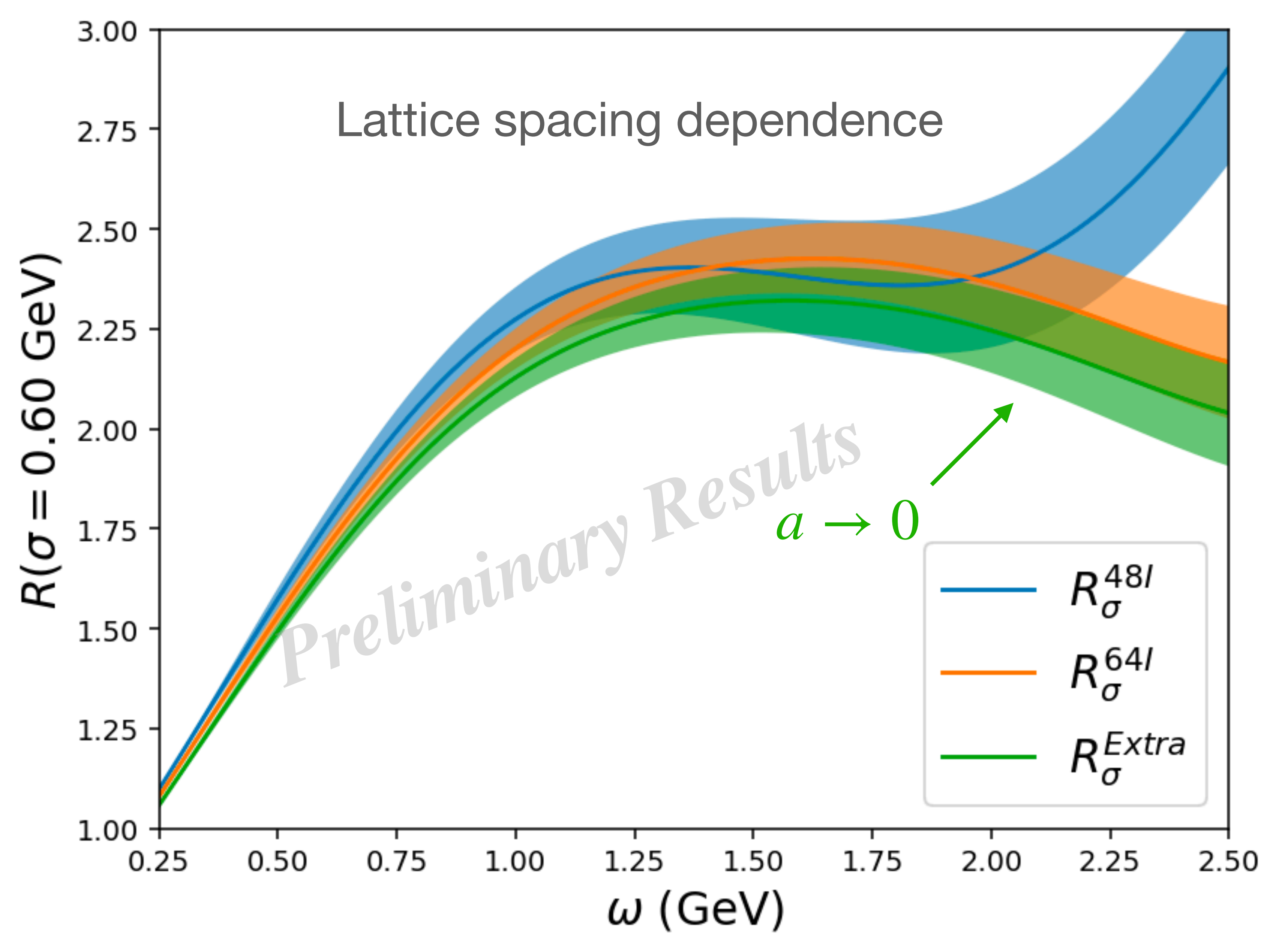
Label	L/T	Mpi (MeV)	a (fm)	L (fm)
48I	48/96	139	0.11406	5.47
64I	64/128	139	0.08365	5.35
24D	24/64	141	0.1940	4.656
32D	32/64	141	0.1940	6.208
48D	48/96	141	0.1940	9.312

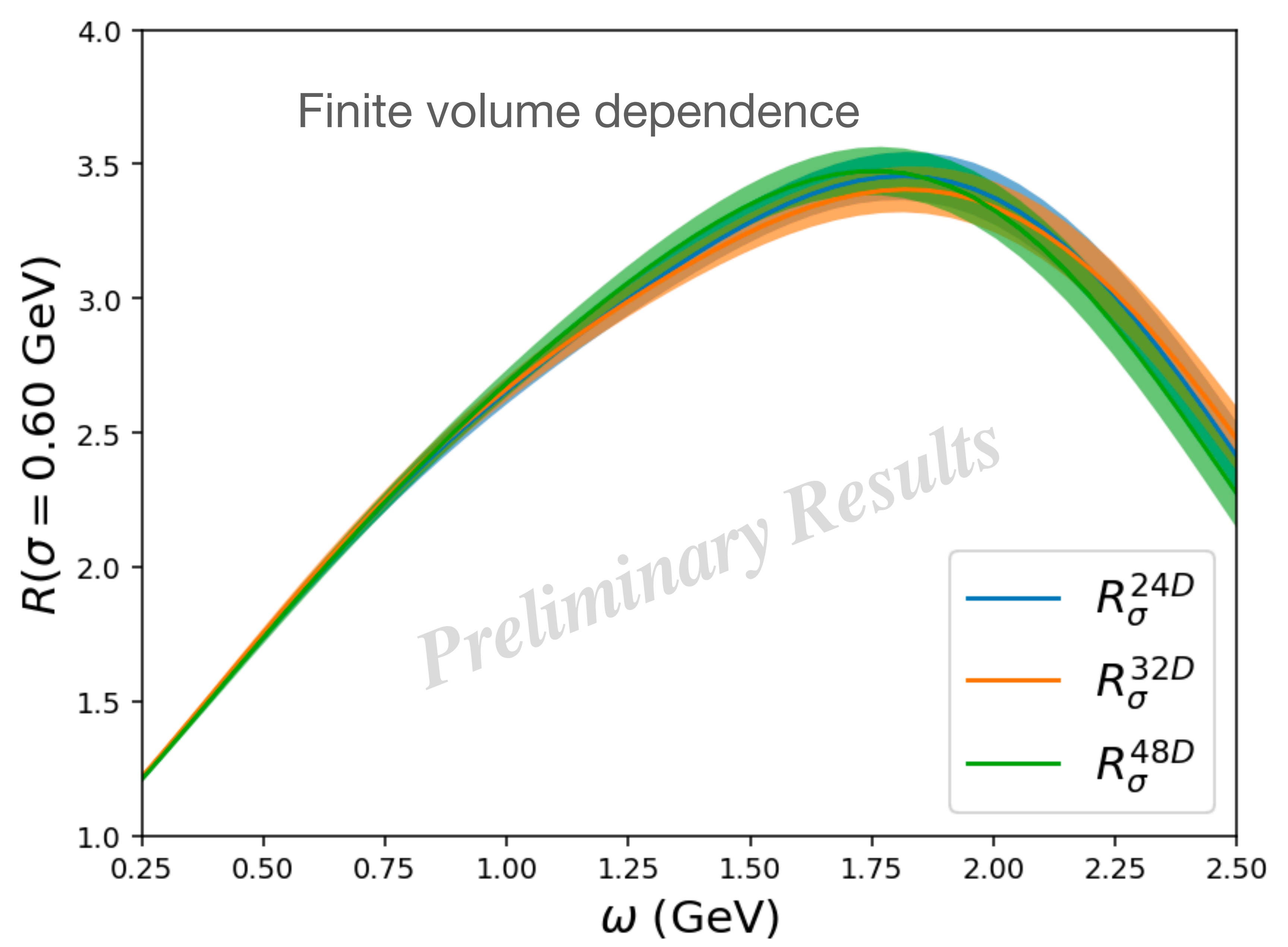
- Overlap fermions on RBC/UKQCD domain wall gauge ensembles at the physical point with different lattice spacings and volumes.
- High-precision current-current correlation functions for both u/d and s, but no charm and no disconnected insertions for now.

Wang G et al. PRD, 2023, 107(3): 034513.

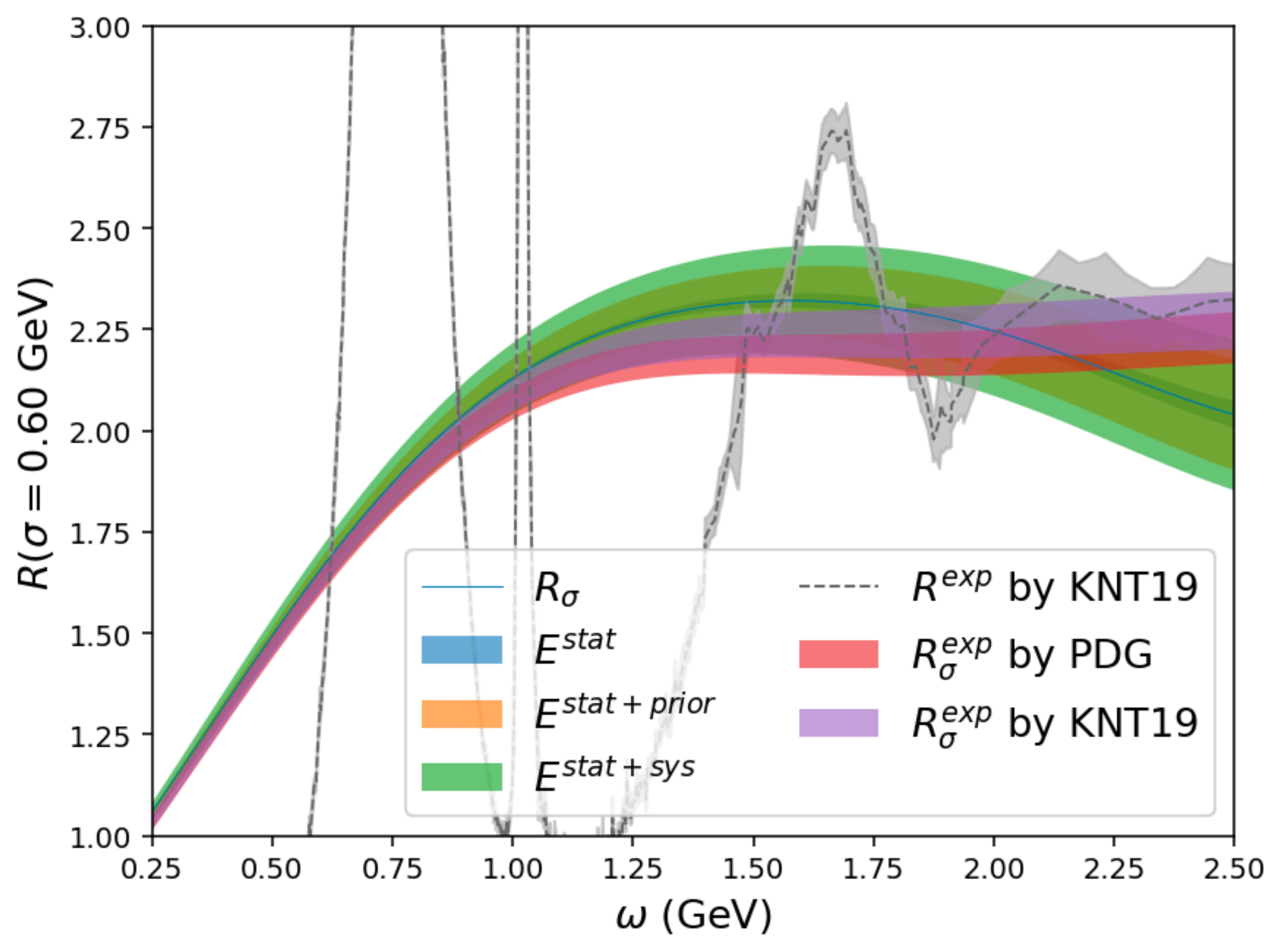
Systematic uncertainty control

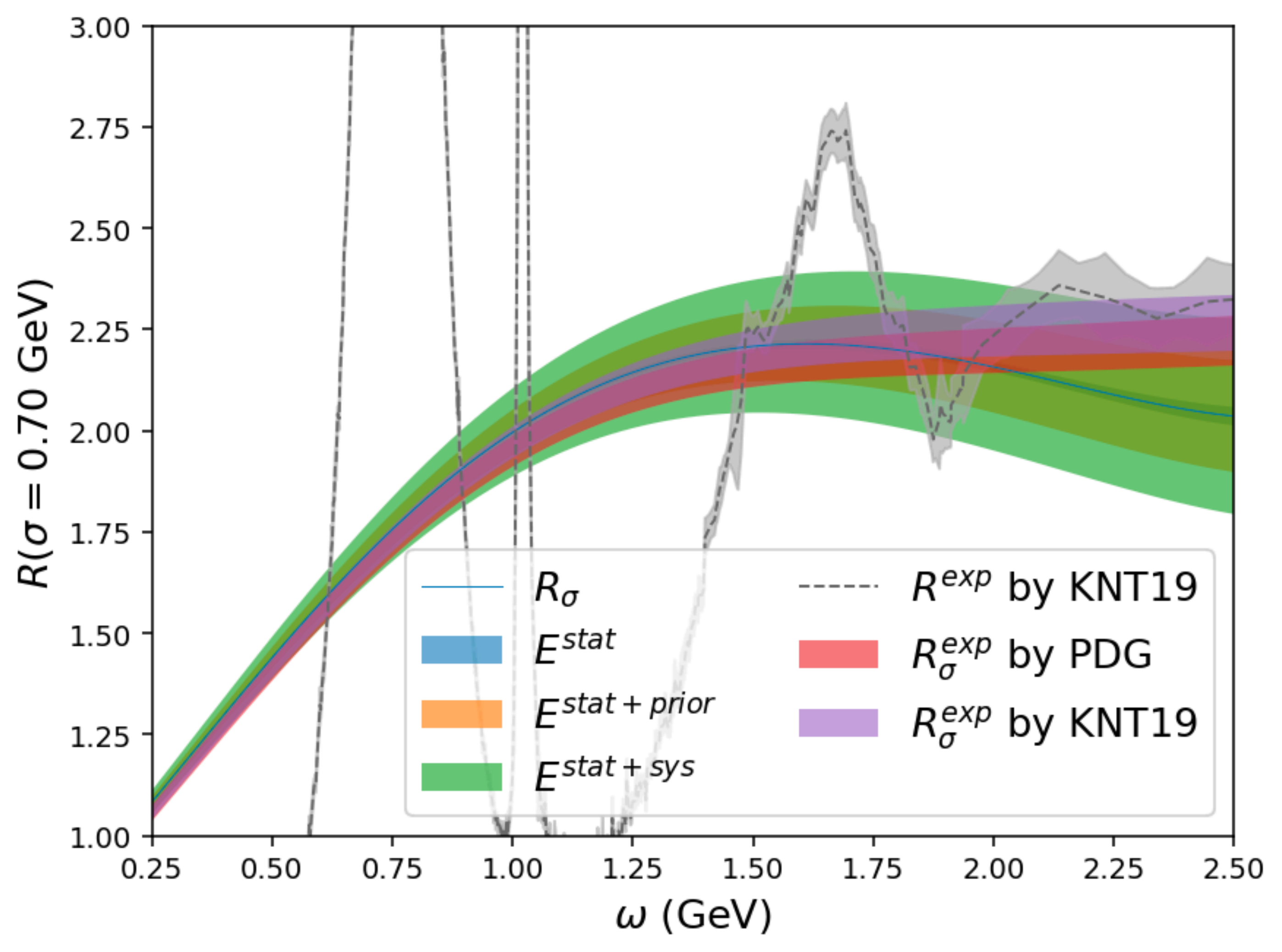
- We extrapolate our results to the continuous limit.
- The effect of finite volume on the results is negligible.



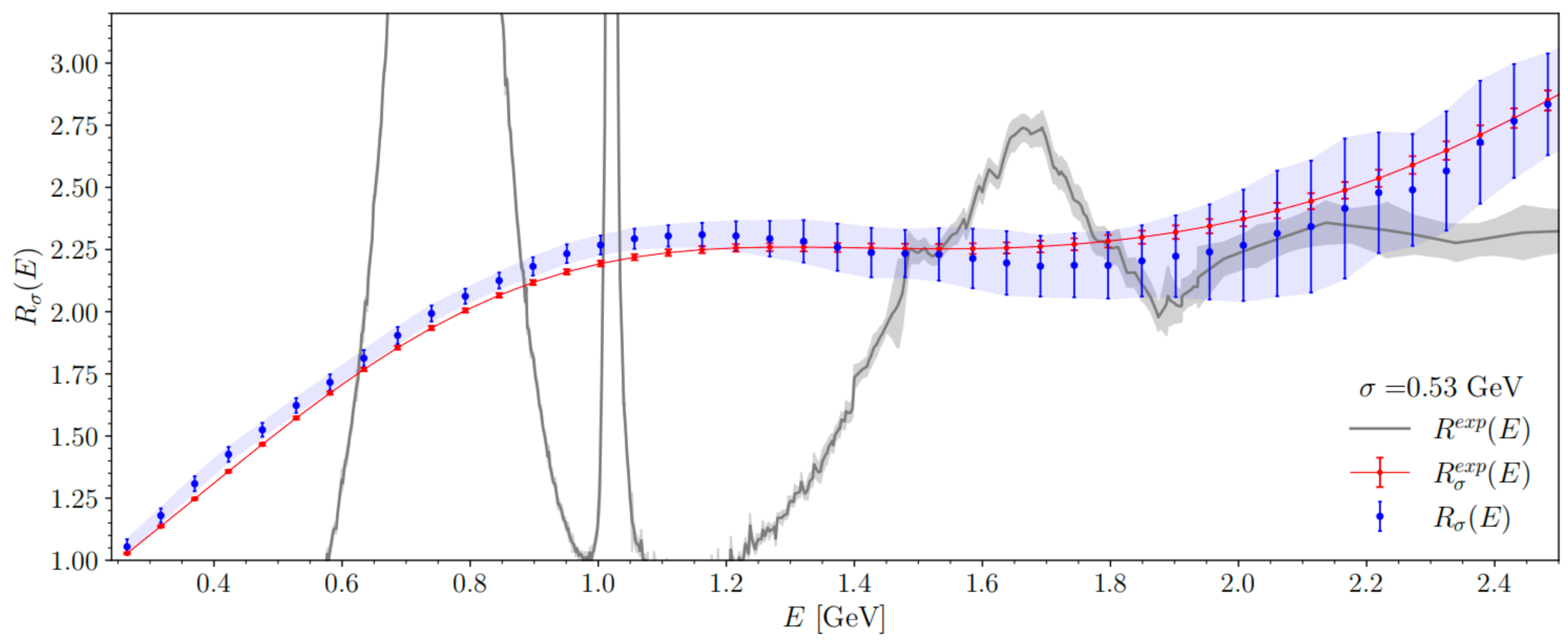


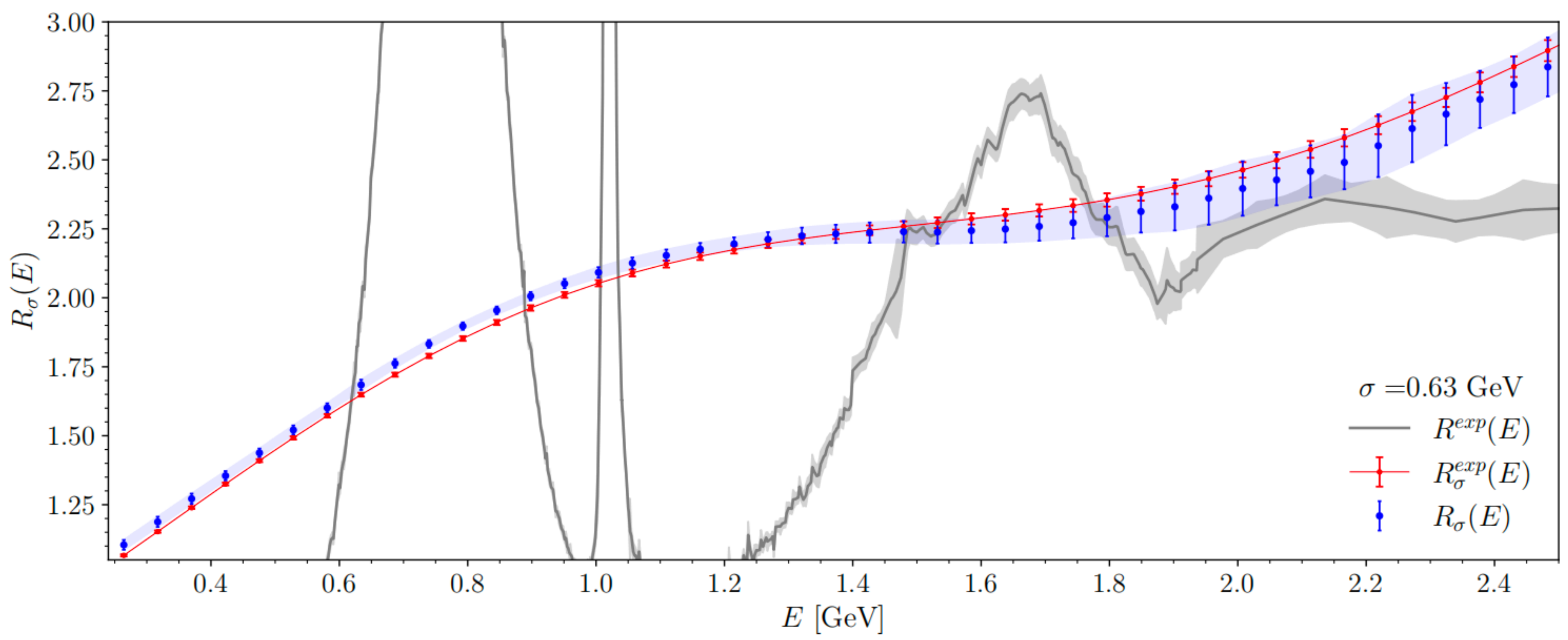
Preliminary results





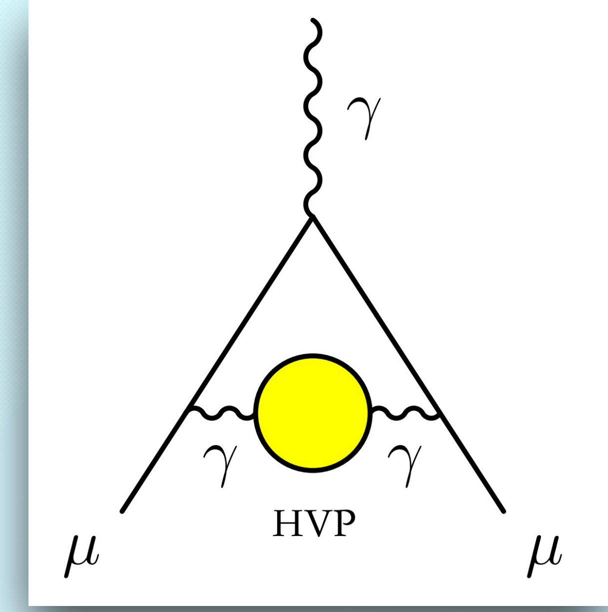
- We considered the statistical error and system error caused by the prior、lattice spacing and finite volume, and obtained results for two smearing parameters.
- We compute the R-ratio with energy up to 2.5 GeV.





C. Alexandrou et al.(ETMC), PRL. 130, 241901 (2023).

HVP contribution to muon g - 2



$$a_\mu^{\text{HVP}}[\text{LO}] = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{thr}}^{\infty} ds \frac{K(s)}{s^2} R(s)$$

$$\sqrt{s} = \omega \qquad R(\omega) = \frac{12\pi^2 \rho(\omega)}{\omega^2}$$

$$a_\mu^{\text{HVP}}[\text{LO}] = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{2m_\mu}^{\infty} d\omega \frac{24\pi^2 K(\omega)\rho(\omega)}{\omega^5}$$

Target function $T(\omega) = \dfrac{24\pi^2 K(\omega)\rho(\omega)}{\omega^5}$

$$C_2(t) = \int d\omega \rho(\omega) e^{-\omega t}$$

$$N(\omega, t) = \frac{\omega^5 e^{-\omega t}}{24\pi^2 K(\omega)}$$

$$C_2(t) = \int d\omega T(\omega) N(\omega, t)$$

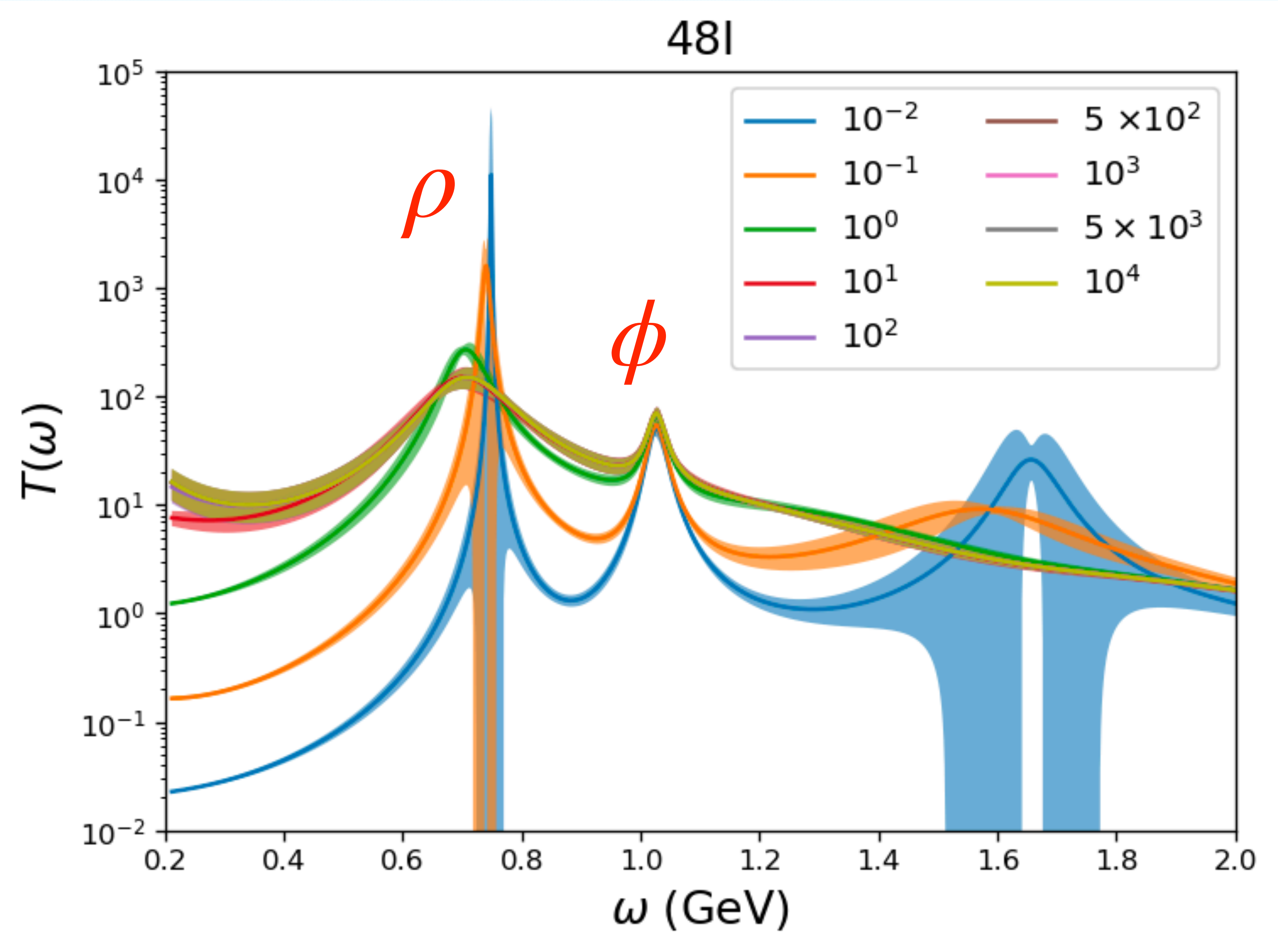
New Kernel

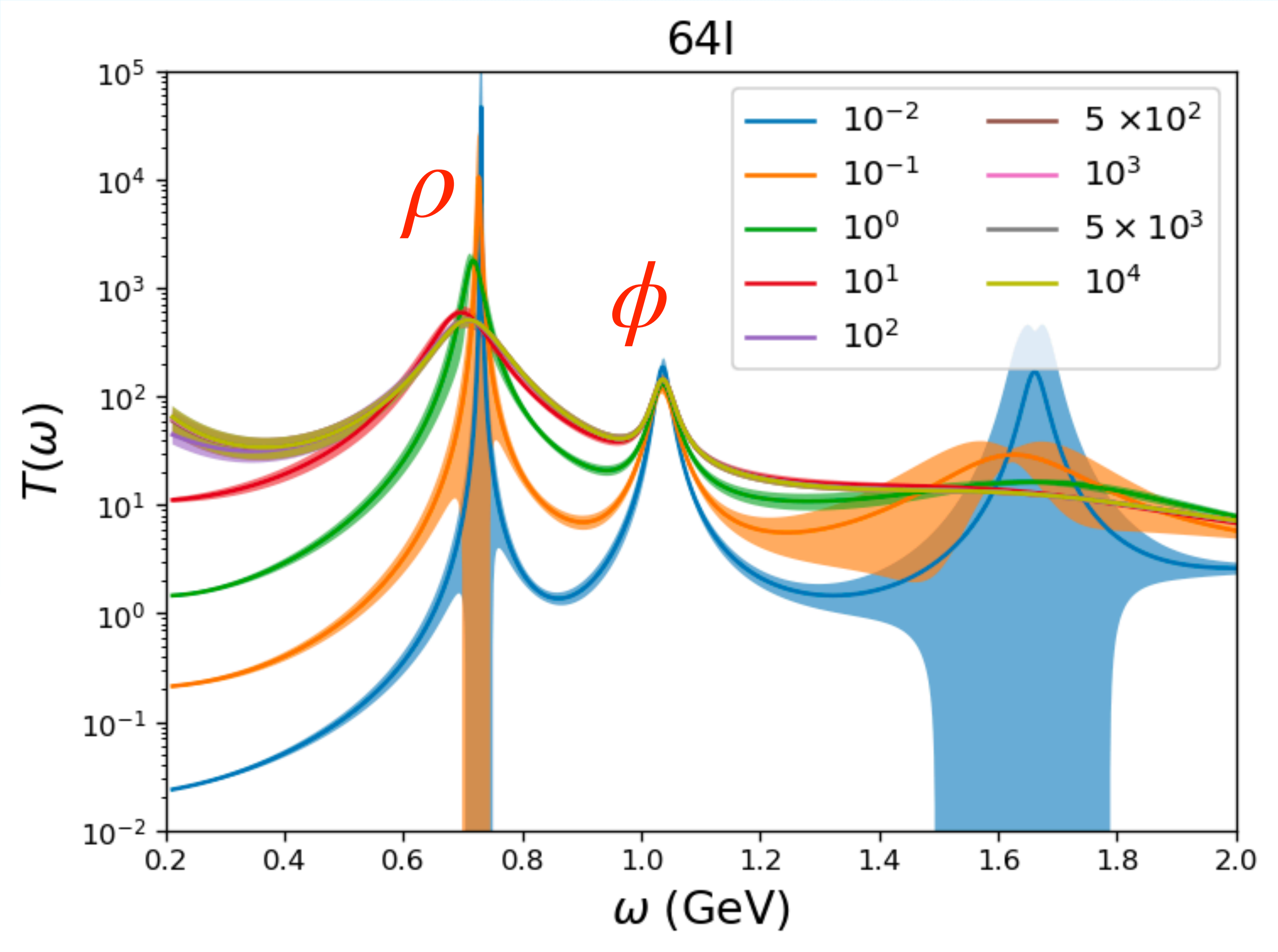
We also use BR method to solve the inverse problem and calculate the target function.

Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

Preliminary results

$$C_2(t) = \int d\omega T(\omega) N(\omega, t)$$

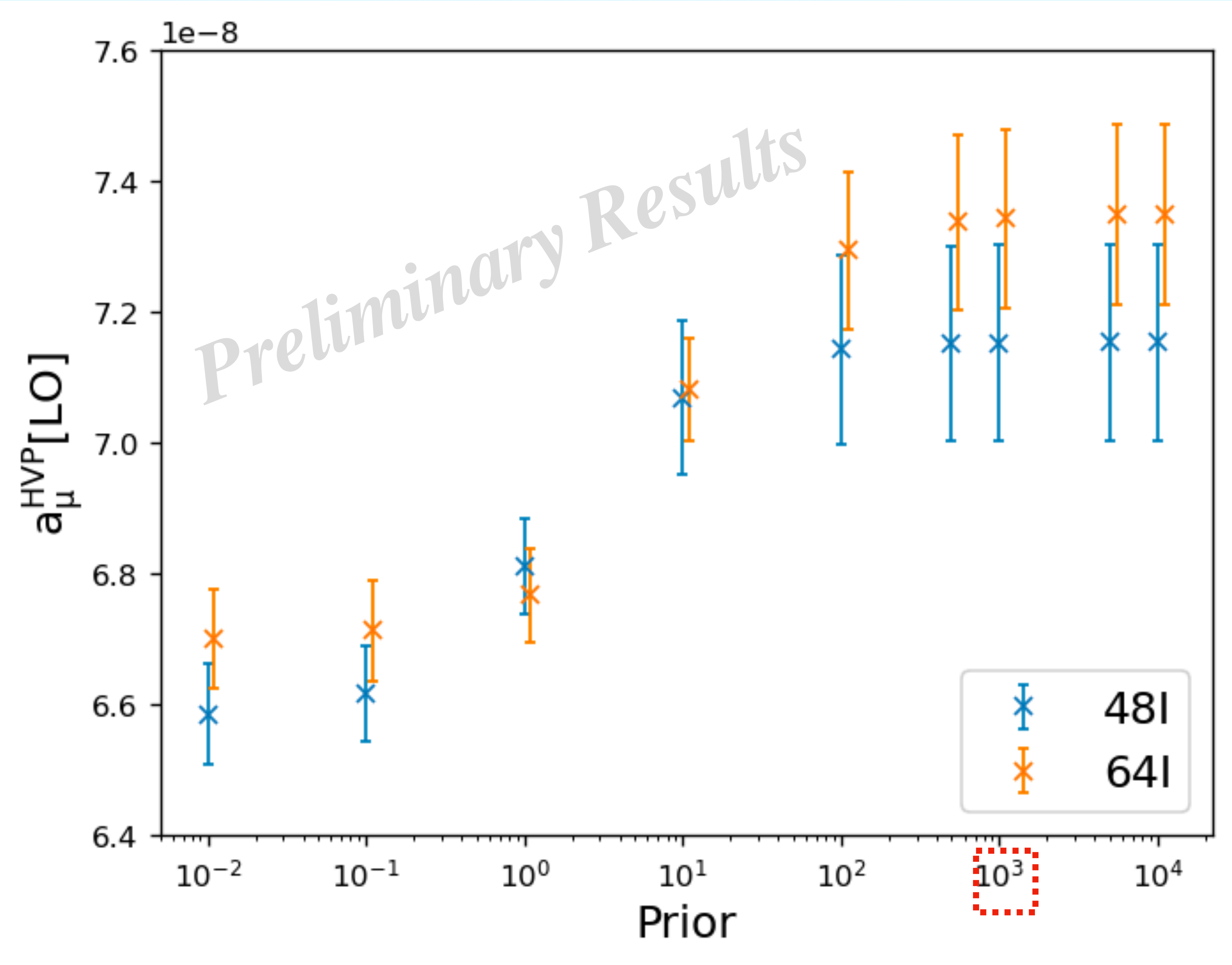




BR method can properly extract the ρ and ϕ resonance

Results are stable when the prior larger than 10^2

Prior dependence



$$a_\mu^{\mathrm{HVP}}[\mathrm{LO}] = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{2m_\mu}^{\infty} d\omega T(\omega)$$

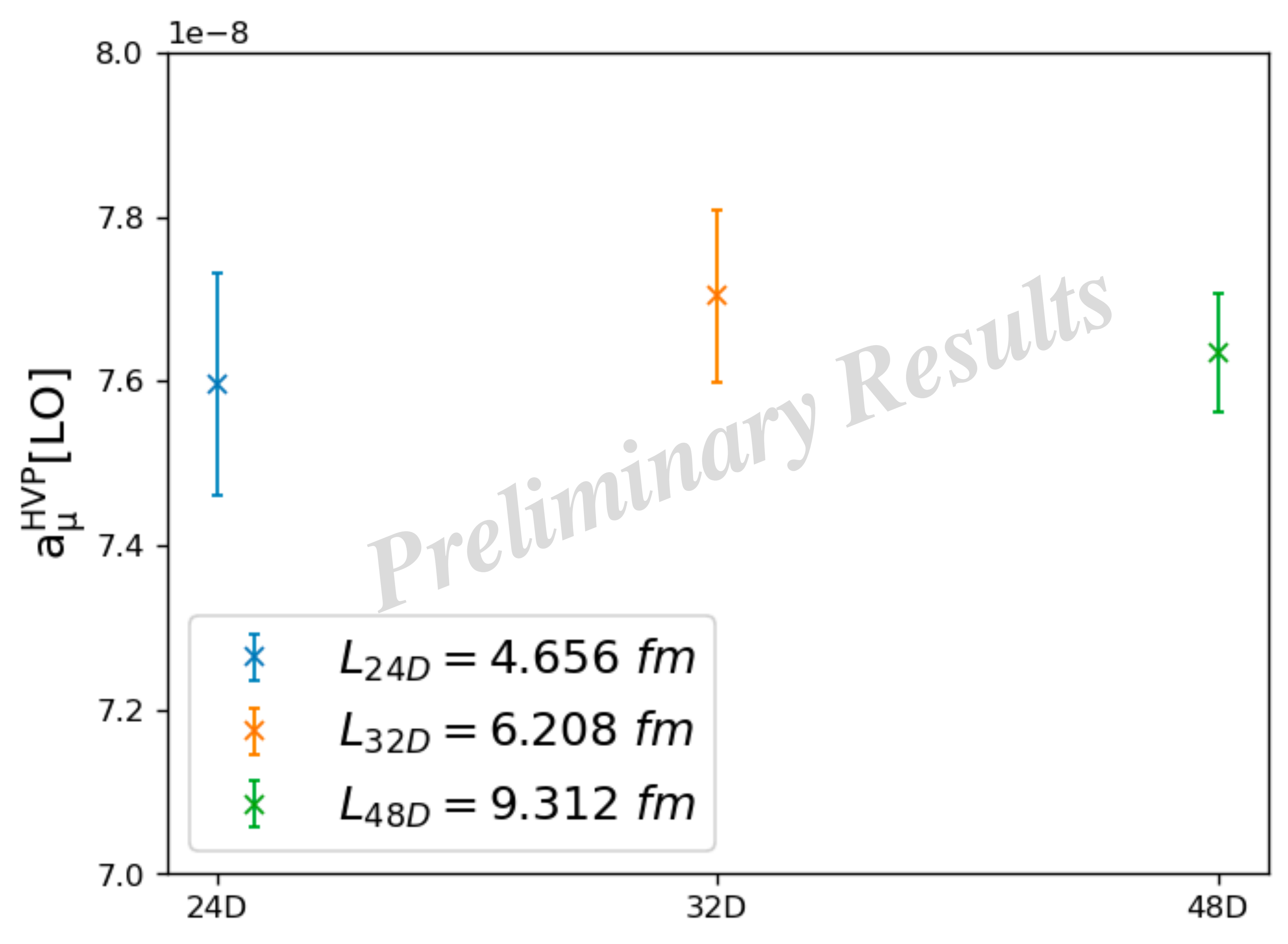
For two different lattice spacing ensembles, as the prior increases, a_μ tends to be stable.

We choose a prior of 10^3 for extrapolation.

Volume dependence

$$a_\mu^{\mathrm{HVP}}[\mathrm{LO}] = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{2m_\mu}^{\infty} d\omega T(\omega)$$

Symbol	a(fm)	L(fm)
24D	0.1940(19)	4.656
32D	0.1940(19)	6.208
48D	0.1940(19)	9.312

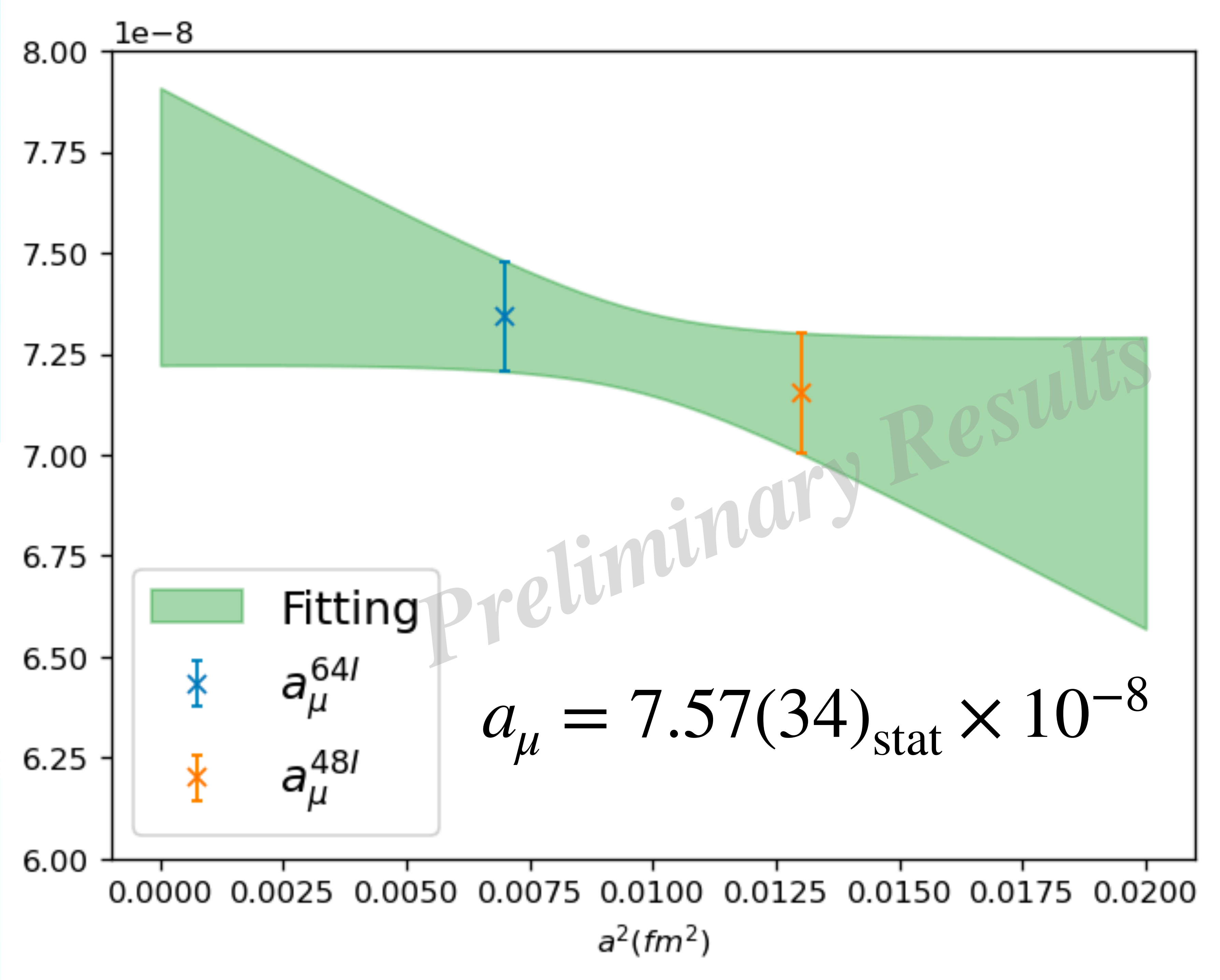


The central values will be within the error range of the three results.

The influence of finite volume effects on the results can be negligible.

Continuum extrapolation

For the linear a^2 fitting: $y = ka^2 + a_\mu$.



The results when the prior is 10^3:

$$a_\mu^{64\rm I} = 7.34(14)_{\rm stat} \times 10^{-8}$$

$$a_\mu^{48\rm I} = 7.15(15)_{\rm stat} \times 10^{-8}$$

Window method:

$$a_\mu^{W=(0.4-1.0)\ \rm fm} = 2.07(2) \times 10^{-8} + 0.268(1) \times 10^{-8}$$

Wang G et al. PRD, 2023, 107(3): 034513.

BMW: $a_\mu^{\rm con,l\ and\ s}$[LO] $= 6.871(21)_{\rm tot} \times 10^{-8}$

Sz. Borsanyi et al. Nature 593.7857 (Apr. 2021).

Summary and Outlook

- We demonstrate a systematic approach based on BR to tackle the inverse problem with sophisticated error control.
- We present a new method based on BR for addressing inverse problems and calculate the R-ratio from lattice correlators.
- We provide a very promising alternative way for calculating the a_μ^{HVP} and test the effect of prior and finite volume effects on a_μ^{HVP}.
- Next, we will perform a more detailed and comprehensive analysis of a_μ^{HVP}, along with a rigorous error calculation.

Summary and Outlook

- We demonstrate a systematic approach based on BR to tackle the inverse problem with sophisticated error control.
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Thank you!

Numerical Calculation

○ Bayesian Reconstruction Method(BR)

$$P[\rho\,|\,D, \alpha, m] \propto e^{Q(\rho)}$$

$$Q = \alpha S - L - \gamma (L - N_\tau)^2$$

$$S = \sum_{\omega} \left[1 - \frac{\rho(\omega)}{m(\omega)} + \log\left(\frac{\rho(\omega)}{m(\omega)} \right) \right] \Delta\omega$$

$$P[\rho\,|\,D, m] = \frac{P[D\,|\,\rho, I]}{P[D\,|\,m]} \int d\alpha P[\alpha\,|\,D, m]$$

- Hyper parameter α is integrated over;
- Maximum search is in the entire parameter space($O(10^3)$)
- High precision architecture is needed(e.g.,512-bit floating point number).

J. Liang et al., PRD101, 114503 (2020)
Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

Numerical Calculation

○ Smearing Method

- For finite lattice size the spectrum is discrete, in principle, lattice spectrum functions are delta functions.
- Experimentally, what we have a continuous result.
- We can compare the experimental results with the smeared lattice spectrum function.

$$C(t) = \sum_n A_n e^{-\omega_n t} = \int d\omega \rho_L(\omega) e^{-\omega t}$$

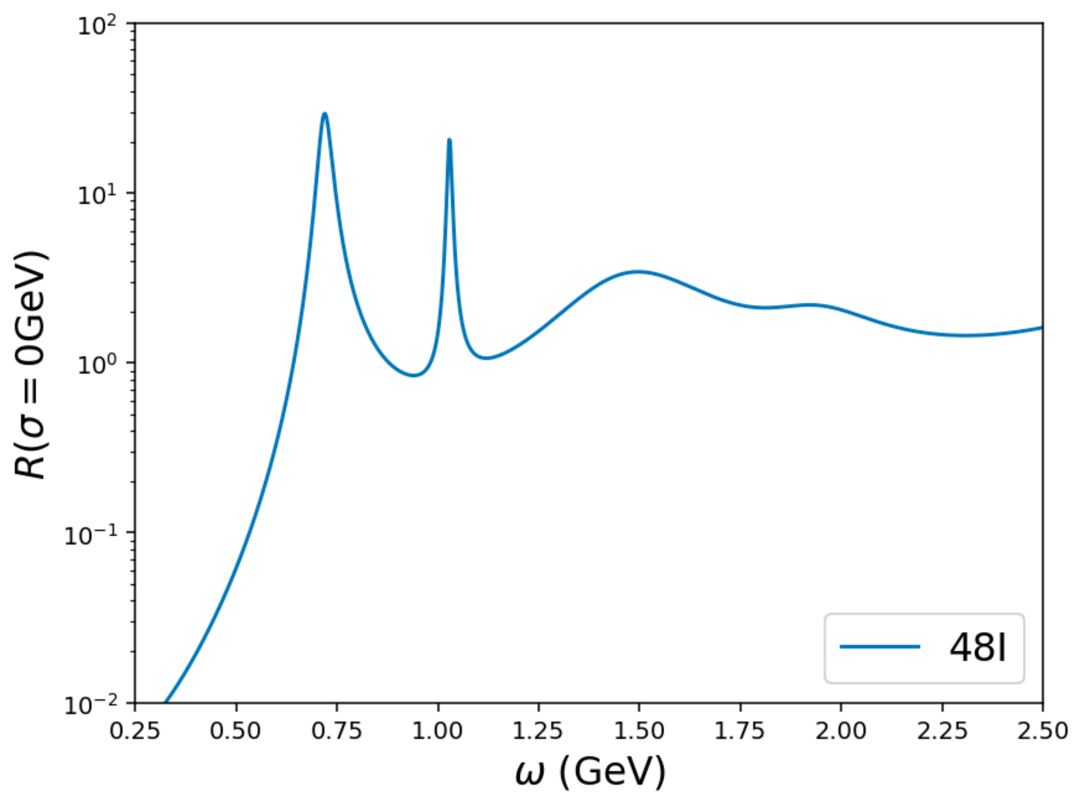
BR

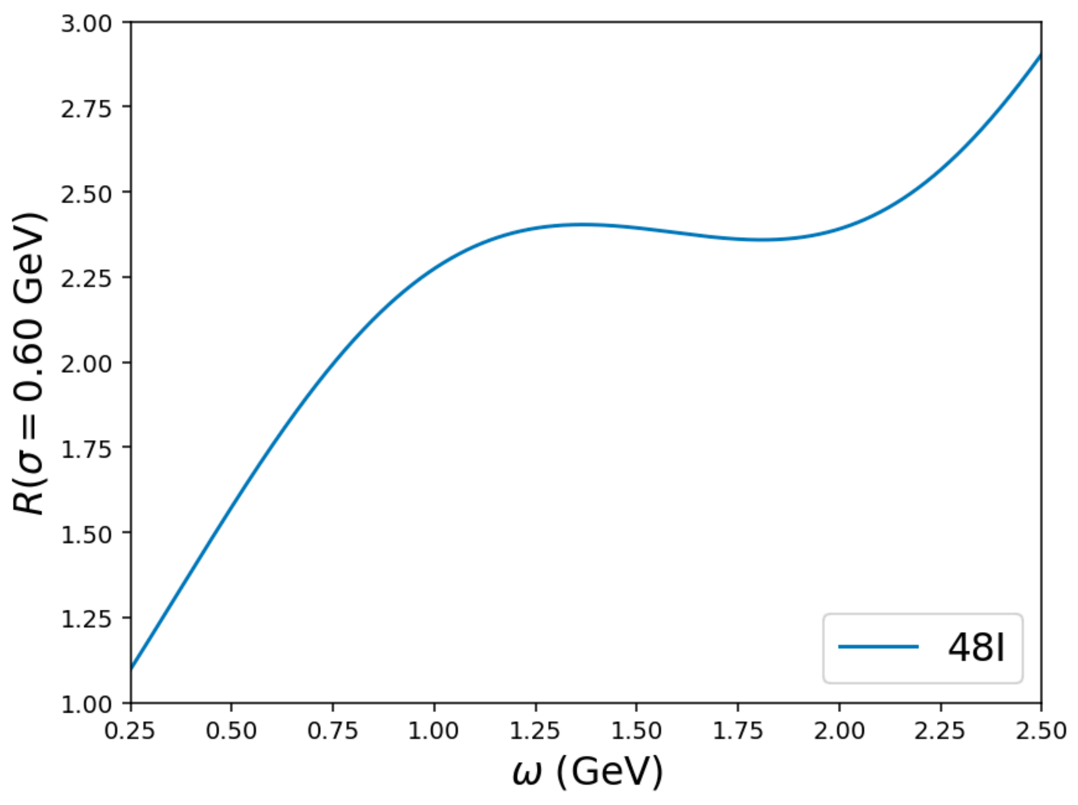
$$\rho_L(\omega) = \sum_n A_n \delta(\omega, \omega_n)$$

Smearing

$$\rho_{\sigma,L}(E) = \int_0^\infty d\omega \Delta_\sigma(E,\omega)\rho_L(\omega)$$

$$\Delta_\sigma(\omega, E) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\mathrm{E}-\omega)^2}{2\sigma^2}\right)$$





Numerical Calculation

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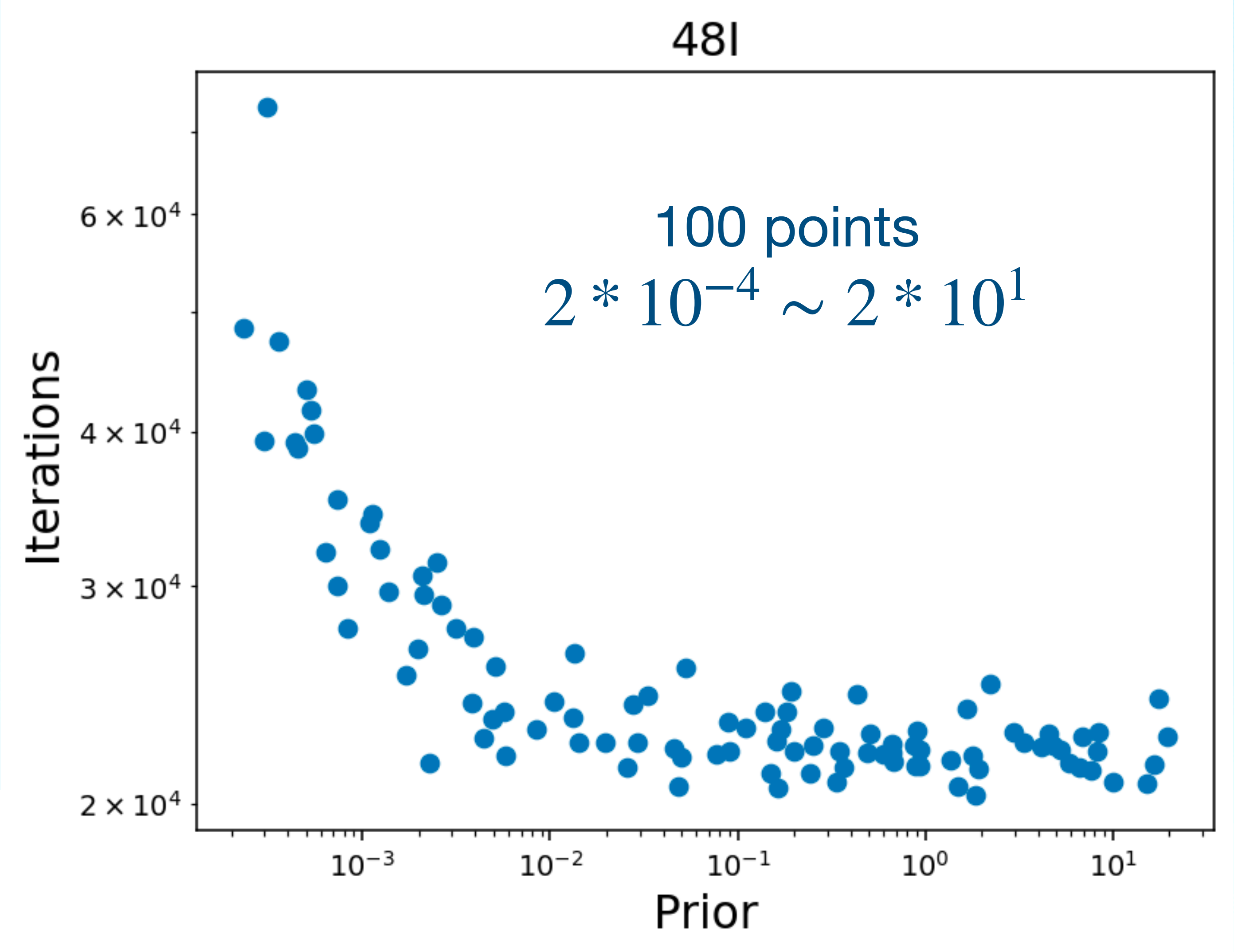
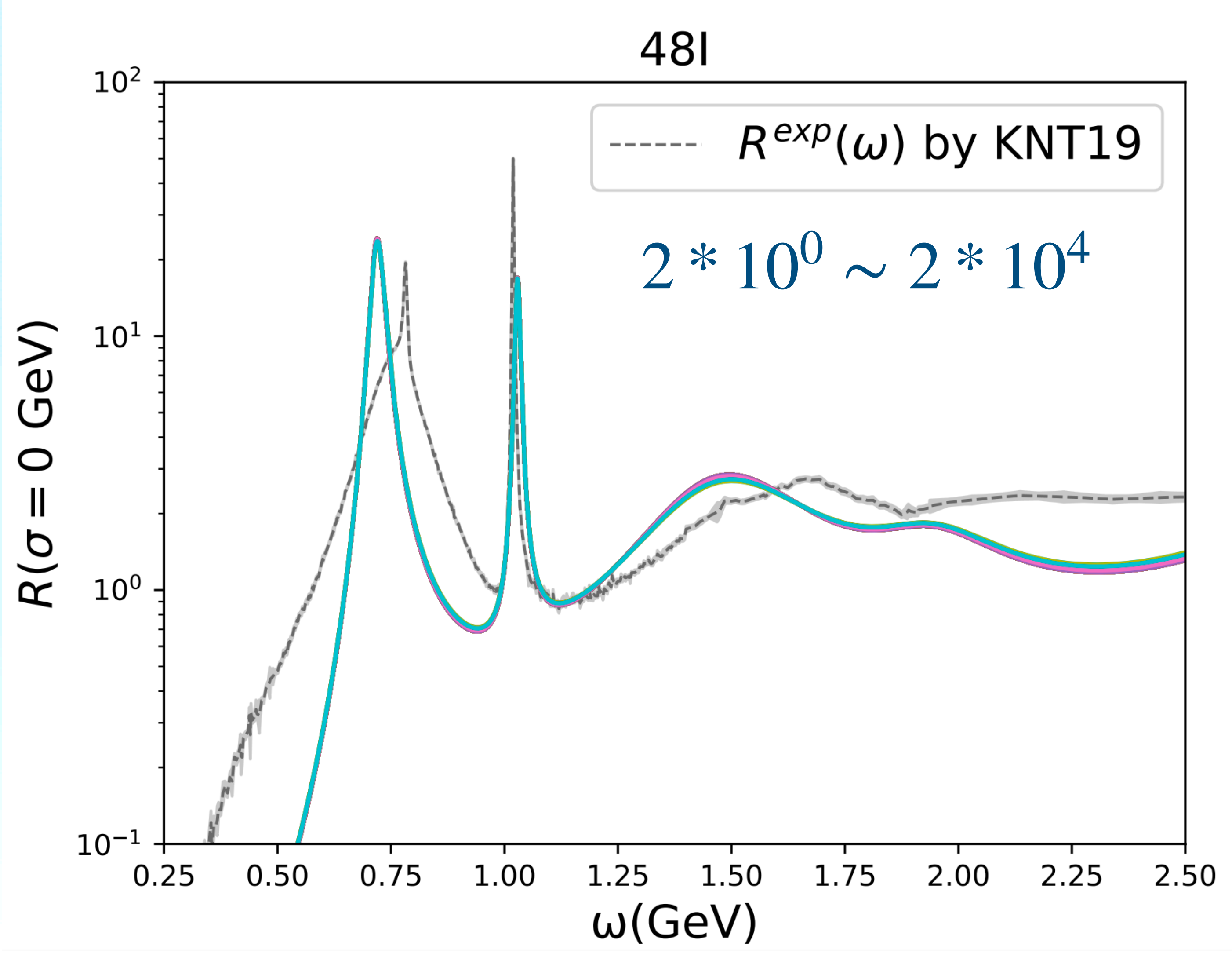
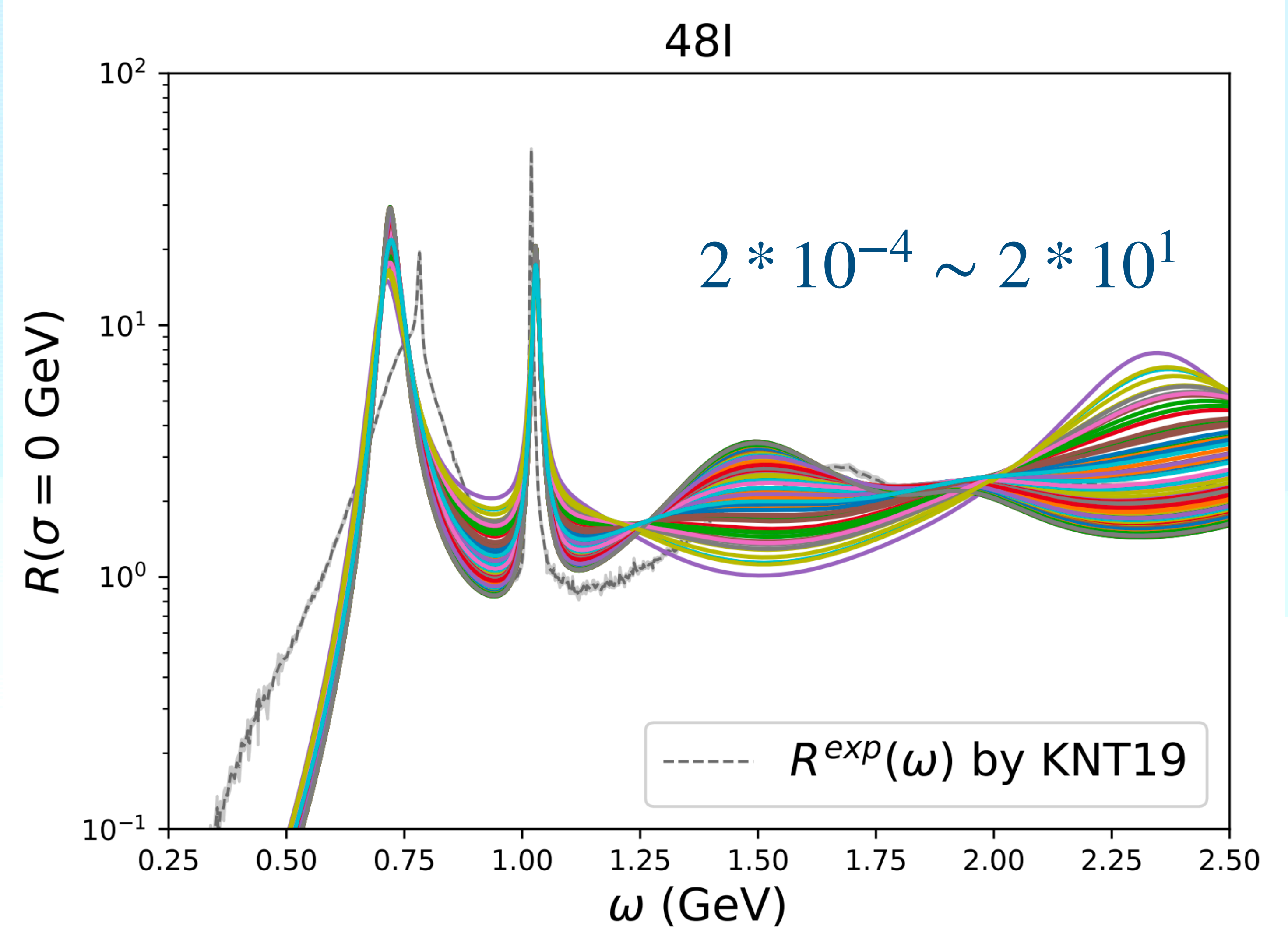






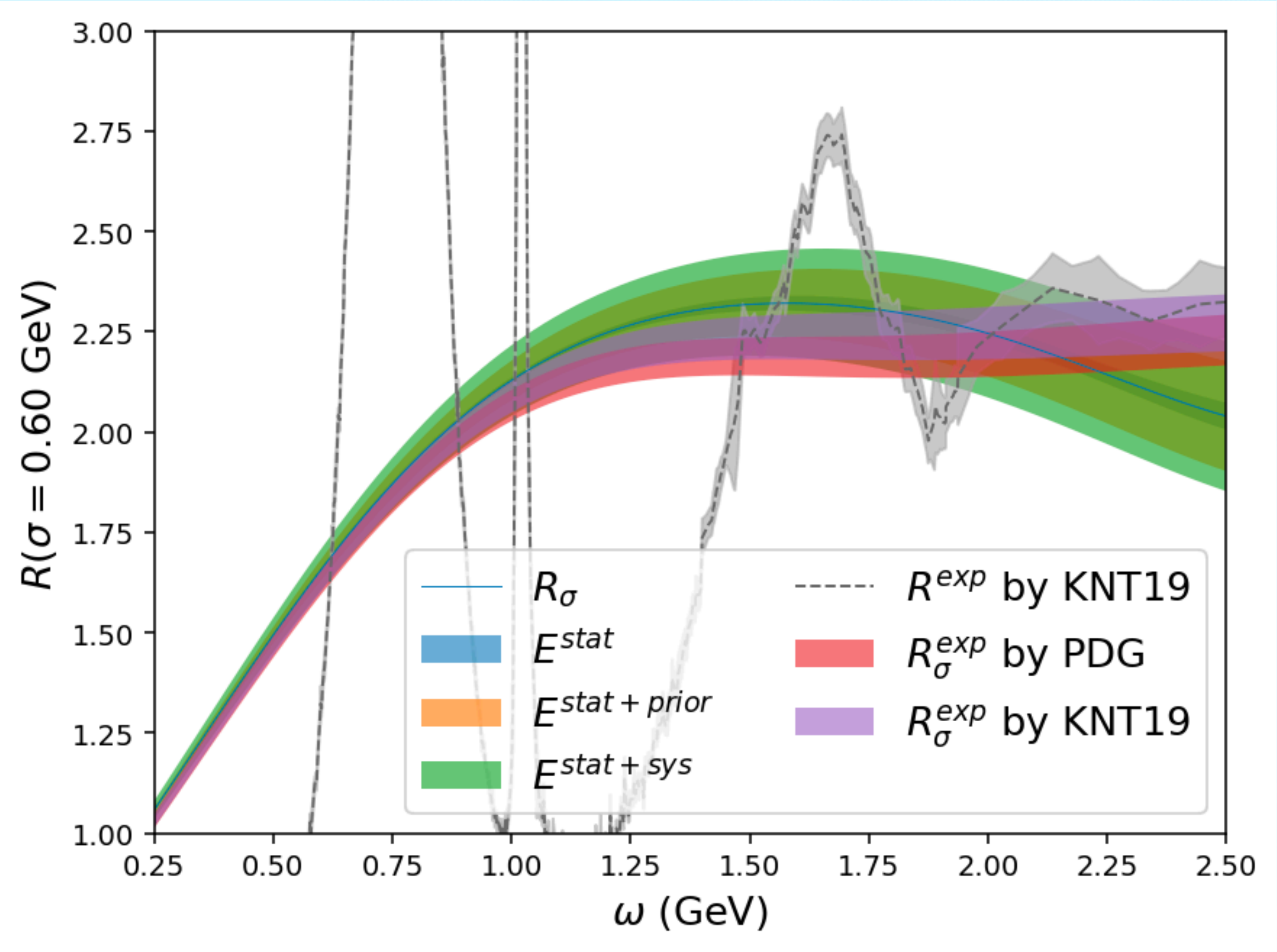
- We tested the number of iterations of different prior in the BR program.
- With the prior decreases, the number of iterations increases exponentially.

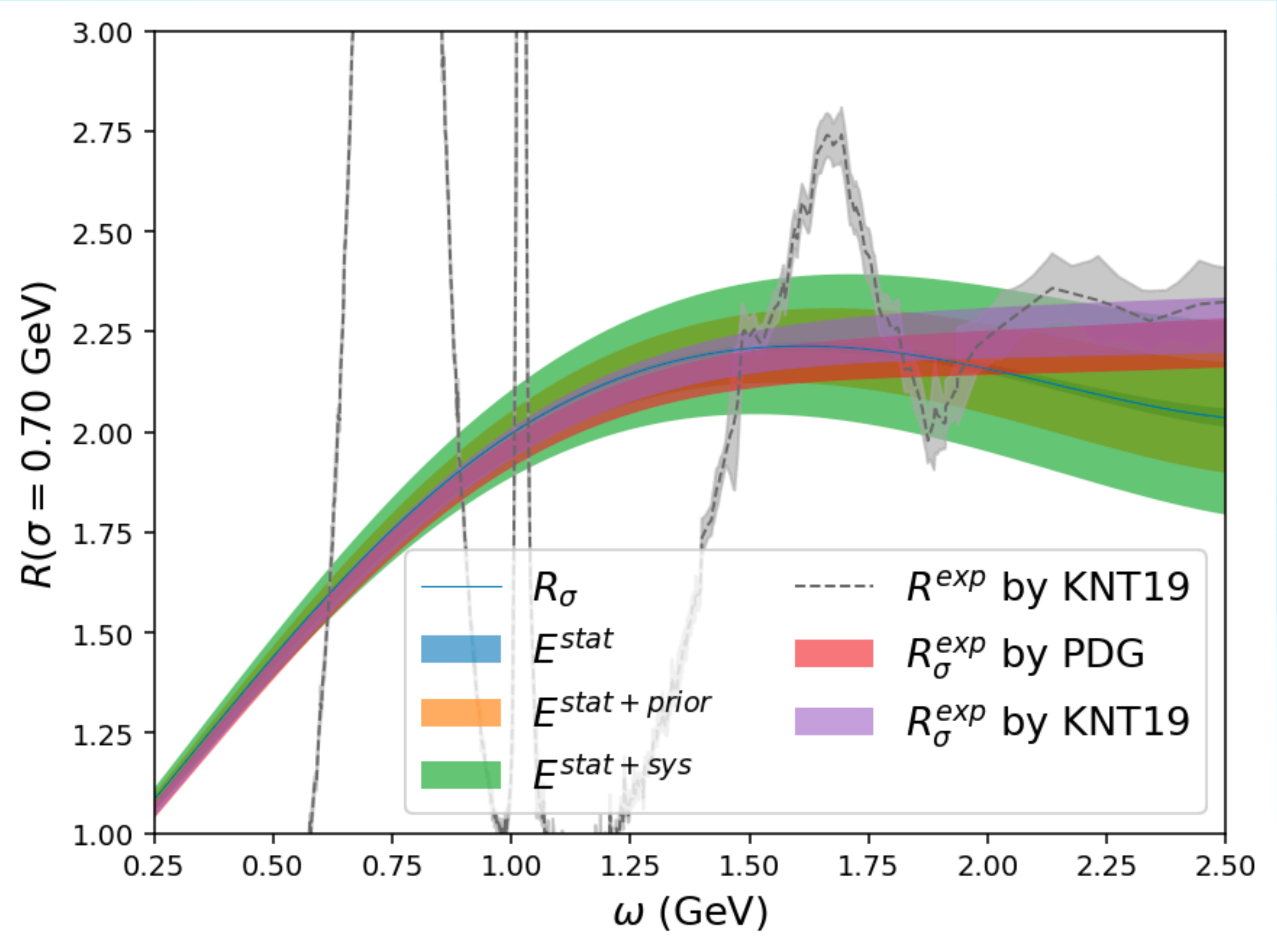
- BR is stable so long as the prior value is more than 2.

- When the prior of BR is less than 1, the lattice results change as the prior decreases.

The input variable function ‘prior’ of the BR program has a significant impact on the result of the R-ratio.

Preliminary Results





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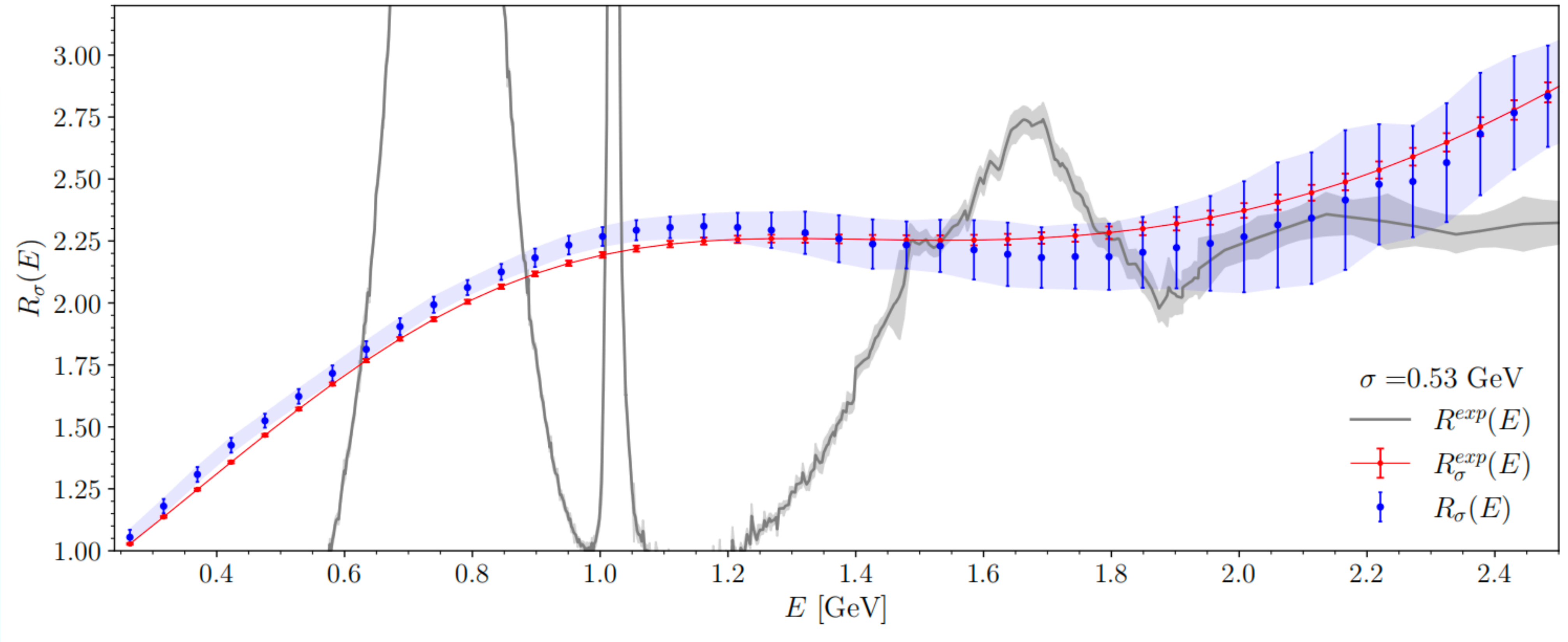
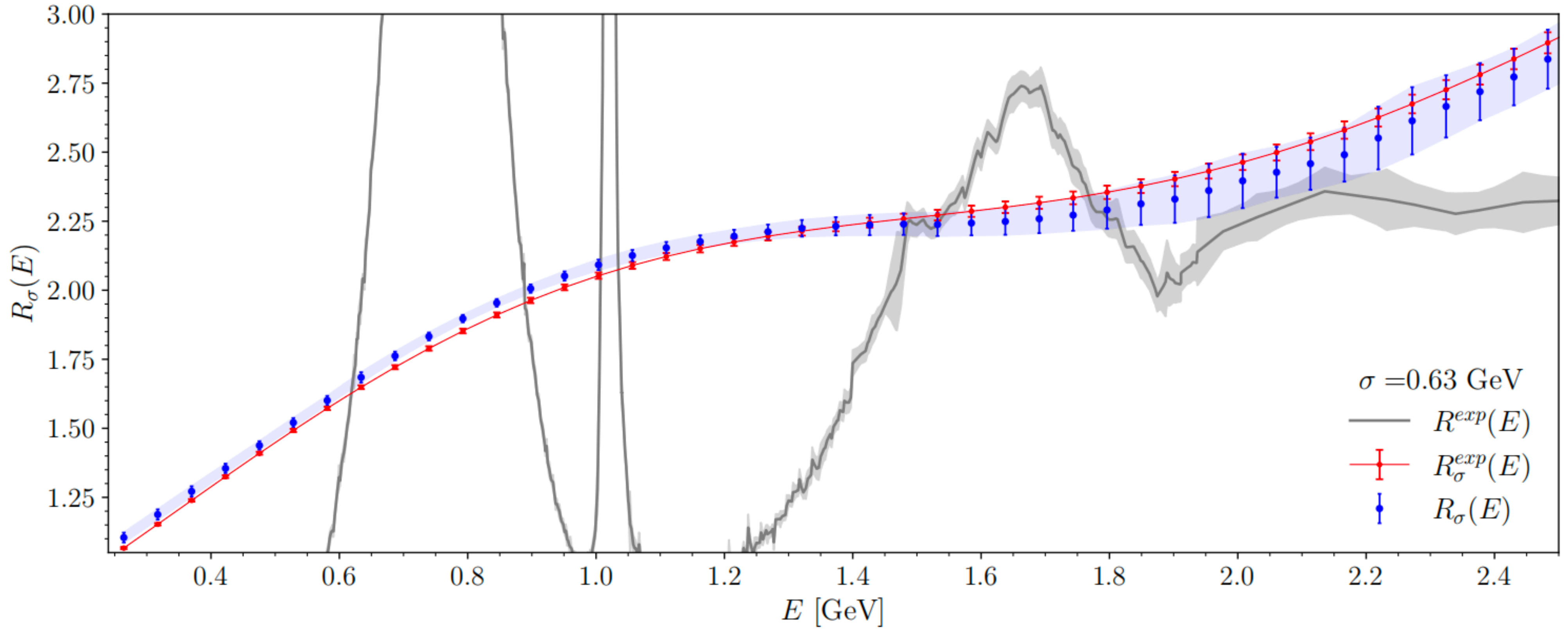




C. Alexandrou et al.(ETMC), PRL. 130, 241901 (2023).

Back up

$$\sigma(e^+e^- \to hadrons) \propto \sum_i Q_i^2 * 3 * \frac{4\pi\alpha^2}{3E_{cm}^2}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

○ Smearing Method

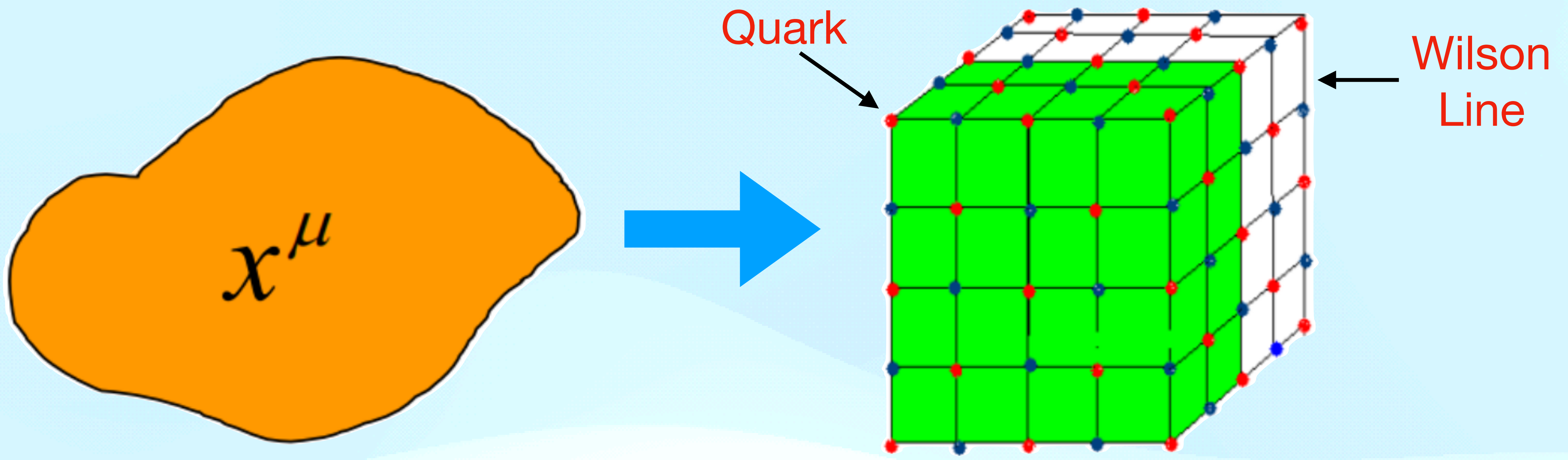
$$C(t) = \sum_n A_n e^{-\omega_n t} = \int d\omega \rho_L(\omega) e^{-\omega t}$$

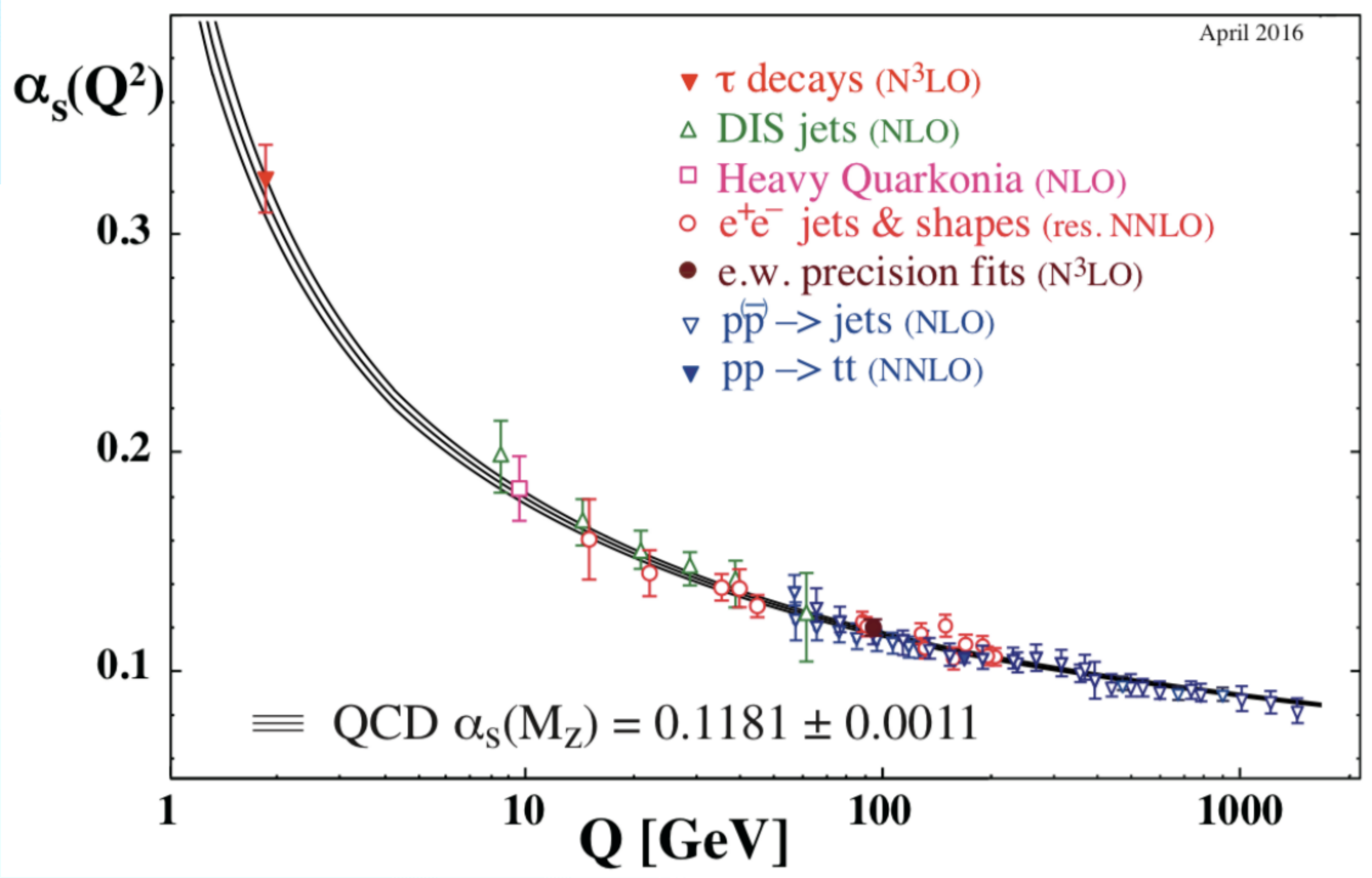
$$\rho_L(\omega) = \sum_n A_n \delta(\omega, \omega_n)$$

- For finite lattice size the spectrum is discrete, in principle, lattice spectrum functions are delta functions.
- Experimentally, what we have is a continuous spectrum function.
- We can compare the experimental results with the smeared lattice spectrum function.

Introduction to Lattice QCD

The most systematic theoretical method for studying non-perturbative QCD is Lattice QCD.





The path integral on the lattice:

- Space-time discretization→Finite dimensional integral
- Euclidean space time→Importance sampling

$$\begin{cases} Z = \int D\phi e^{-S_E[\phi]} \\ \langle O[\phi] \rangle = \frac{1}{Z} \int D\phi O[\phi] e^{-S_E[\phi]} \end{cases}$$

- Monte Carlo simulations →Calculate path integral.