第六届粒子物理天问论坛

The PWA amplitudes in the covariant L-S scheme

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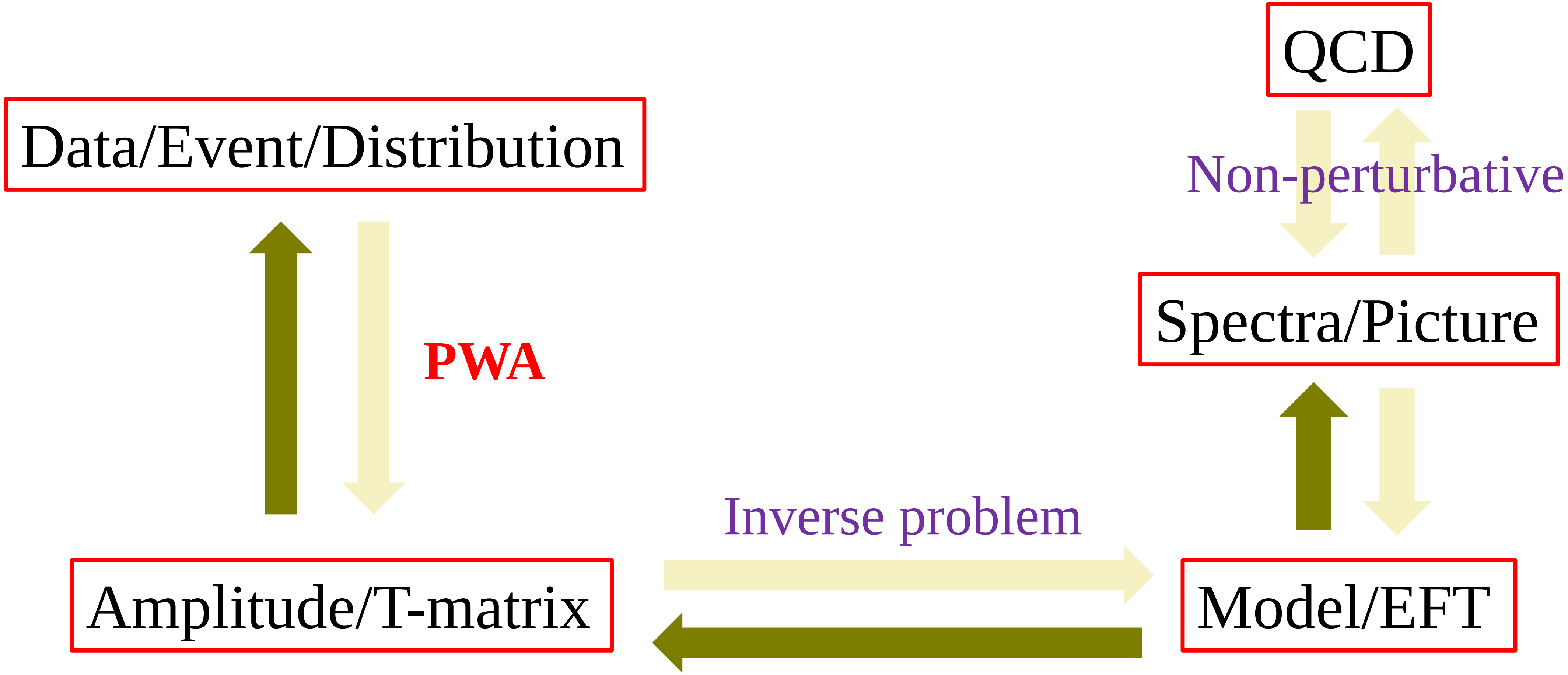
Introduction



Introduction

Data/Event/Distribution

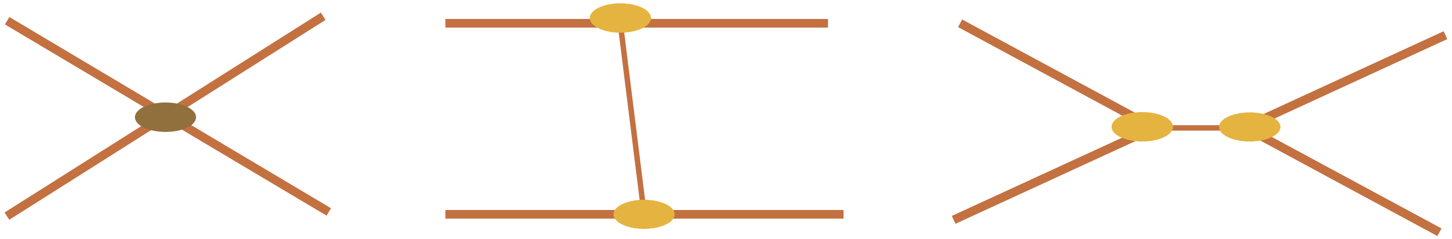
PWA

Amplitude/T-matrix

Amplitude will include: Vertex function and Topology structure
Topology structure: the unitary, crossing symmetry, analytical.
(tree, one-loop, … , refer to coupled channel.)
Vertex function: the Lorentz covariant.

All Lorentz structures, Data-driven, Model independent.

Multi-particle vertex can be derived recursively from the three-particle vertex.



Introduction

Data/Event/Distribution

PWA

Amplitude/T-matrix

Three-particle vertex!

Multi-particle vertex can be derived recursively from the three-particle vertex.

Covariant tenser $T^{a_1a_2\cdots}_{b_1b_2\cdots} \xrightarrow{g\in G} \tilde{T}^{a_1a_2\cdots}_{b_1b_2\cdots} = D_{b_1}{}^{b_1'}(g)\, D_{b_2}{}^{b_2'}(g)\, D^{a_1}{}_{a_1'}(g)\, D^{a_2}{}_{a_2'}(g)\,\cdots\, T^{a_1'a_2'\cdots}_{b_1'b_2'\cdots},$

order-2 COVT X_{ij} $\quad [i]\otimes[j] = [k_1]\oplus[k_2]\oplus\cdots$

$$X_{ij} = C_{k_1}\, T^{k_1}_{ij} + C_{k_2}\, T^{k_2}_{ij} + \cdots, \quad \text{with} \quad C_{k_n} \xrightarrow{g\in G} \tilde{C}_{k_n} = D_{k_n}{}^{k_n'}(g)\, C_{k_n'}$$

$$T^{k_m}_{ij}\, T^{ij}_{k_n} \propto \delta_{mn}\, \delta_{k_n}{}^{k_m}$$

INVTEN cannot be further decomposed

INVTENs are also called irreducible tensors (IRTENs)

Invariant tenser(INVT) $\tilde{T}^{a_1a_2\cdots}_{b_1b_2\cdots} = T^{a_1a_2\cdots}_{b_1b_2\cdots}$

Order-0 INVT: scalar

Order-1 INVT: 0 vector $\quad D_1(g)\,T = T\,D_1(g)$

Order-2 INVT: identity $\quad D_1(g)\,T = T\,D_2(g)$

Order-3 INVT: CGC

Order-n>3 INVT: just constructed by Order-3 INVT.

Order-3 IRTs $(C^s_{s_1s_2})^m_{m_1m_2} \xrightarrow{g\in \mathrm{SU}(2)} D^{(s)m}_{m'}(g^{-1})\, D^{(s_1)m_1'}_{m_1}(g)\, D^{(s_2)m_2'}_{m_2}(g)\, (C^s_{s_1s_2})^{m'}_{m_1'm_2'} = (C^s_{s_1s_2})^m_{m_1m_2}$

Generation

Three-particle vertex!

1➔2+3

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;s_1,s_2,s_3)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\,\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p}_1;s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2;s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3;s_3).$$

From ($\mathbf{k}_1$, $\mathbf{p}_2^*$, $\mathbf{p}_3^*$) to ($\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$) by Lorentz invariant, where for massive particle, $\mathbf{k}_1=\mathbf{0}$, for massless particle $\mathbf{k}_1=(0,0,|k|)$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;s_1,s_2,s_3)=\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)$$

Standard momentum

Weinberg, The Quantum theory of fields. Vol. 1: Foundations, Cambridge University Press (2005)

In the ($\mathbf{k}_1$, $\mathbf{p}_2^*$, $\mathbf{p}_3^*$) frame, we can define L and S.

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)=\underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\,D_{\alpha_2}{}^{\beta_2}\left(h_{\mathbf{p}_2^*}\right)\,D_{\alpha_3}{}^{\beta_3}\left(h_{\mathbf{p}_3^*}\right)}_{\text{pure-orbitial part}}\underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1;s_1)\,u_{\beta_2}^{\sigma_2}(\mathbf{k}_2;s_2)\,u_{\beta_3}^{\sigma_3}(\mathbf{k}_3;s_3)}_{\text{pure-spin part}},$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\,\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1;s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{k}_2;s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{k}_3;s_3)$$

Generation

Three-particle vertex!

1➔2+3

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,S) = \boxed{D_{\alpha_2}{}^{\beta_2}(R_{12})\,D_{\alpha_3}{}^{\beta_3}(R_{13})}\ \boxed{\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1;s_1)\,u_{\beta_2}^{\sigma_2}(\mathbf{k}_2;s_2)\,u_{\beta_3}^{\sigma_3}(\mathbf{k}_3;s_3)}\ \boxed{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)}$$

Lorentz Transformation

$$D_\alpha{}^{\alpha'}(h_\mathbf{p}) = D_\alpha{}^{\alpha''}(R_{\hat{\mathbf{p}}})\ D_{\alpha''}{}^{\alpha'''}(B_{|\mathbf{p}|})\ D_{\alpha'''}{}^{\alpha'}\left(R_{\hat{\mathbf{p}}}^{-1}\right)$$

$$D_\alpha{}^{\alpha'}(R_{\hat{\mathbf{p}}}) = U_\alpha^{lr}\ U_{l'r'}^{\alpha'}\ D_l^{(s_L)l''}(\theta_3)\ D_{l''}^{(s_L)l'}(\theta_2)\ D_r^{(s_R)r''}(\theta_3)\ D_{r''}^{(s_R)r'}(\theta_2)$$

$$D_\alpha{}^{\alpha'}(B_{|\mathbf{p}|}) = U_\alpha^{lr}\ U_{l'r'}^{\alpha'}\ D_l^{(s_L)l'}(-i\vartheta_3)\ D_r^{(s_R)r'}(i\vartheta_3)$$

$$R_{1i} = h_{\mathbf{p}_i^*}^{-1}\cdot h_{\mathbf{p}_1}^{-1}\cdot h_{\mathbf{p}_i}$$

(for massive particles)

$$\tilde{R}_{1i} = R_{\hat{\mathbf{p}}_i^*}^{-1}\cdot R_{1i}\cdot R_{\hat{\mathbf{p}}_i}$$

(for massless particles)

Standard k^μ	Little group $L_{p,k}$	Generators of $L_{p,k}$ =
$(\pm\|m\|,0,0,0)$	SO(3)	$J_1,\ J_2,\ J_3$
$(\pm\|\mathbf{k}\|,0,0,\|\mathbf{k}\|)$	ISO(2)	$J_3,\ (J_2+K_1),\ (J_1-K_2)$
$(0,0,0,\|\mathbf{k}\|)$	SO(1,2)	$J_3,\ K_1,\ K_2$
$(0,0,0,0)$	SO(1,3)	$J_1,\ J_2,\ J_3,\ K_1,\ K_2,\ K_3$

$$L_p/\mathrm{SO}(3) \ni h_\mathbf{p} = R_{\hat{\mathbf{p}}}\cdot B_{|\mathbf{p}|}\cdot R_{\hat{\mathbf{p}}}^{-1},$$

$$L_p/\mathrm{ISO}(2) \ni \tilde{h}_\mathbf{p} = R_{\hat{\mathbf{p}}}\cdot B_{|\mathbf{p}|}$$

$D_b^a(g)$ ($g \in L_p$) is a Lorentz transformation matrix;
h_p ($\in L_p$) is a pure-boost transformation;

$$u_\alpha^\sigma(\mathbf{p};s) = D_\alpha{}^\beta(h_\mathbf{p})\,u_\beta^\sigma(\mathbf{k};s), \qquad \bar{u}_\sigma^\alpha(\mathbf{p};s) = D_\beta{}^\alpha(h_\mathbf{p}^{-1})\,\bar{u}_\sigma^\beta(\mathbf{k};s).$$

Generation

Three-particle vertex!

1→2+3

Lorentz index, the index of specific irreps of L_p

Spin index

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,S) = D_{\alpha_2}{}^{\beta_2}(R_{12})\,D_{\alpha_3}{}^{\beta_3}(R_{13})\;\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1;s_1)\,u_{\beta_2}^{\sigma_2}(\mathbf{k}_2;s_2)\,u_{\beta_3}^{\sigma_3}(\mathbf{k}_3;s_3)\;\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)$$

$$L_p \simeq SU(2)_L \otimes SU(2)_R$$

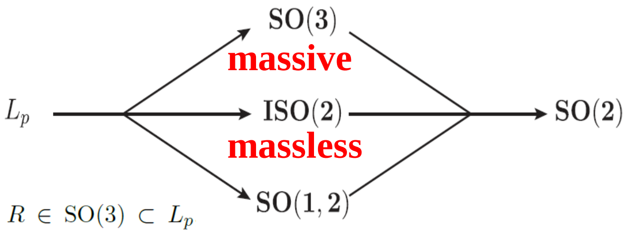
$$[s_L,s_R] \equiv \begin{cases} (s_L,s_R) & \text{for } s_L = s_R \\ (s_L,s_R)\oplus(s_R,s_L) & \text{for } s_L \neq s_R \end{cases}$$

Representation	Physical correspondence
$(0,0)$	Scalar
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	Dirac spinor (spin 1/2)
$(\frac{1}{2},\frac{1}{2})$	Lorentz four-vector
$(1,0)\oplus(0,1)$	Maxwell electromagnetic fields
$(\frac{3}{2},0)\oplus(0,\frac{3}{2})$	Weinberg spinor (spin 3/2)
$(1,1)$	Lorentz order-2 traceless symmetric tensor
$(1,\frac{1}{2})\oplus(\frac{1}{2},1)$	Rarita-Schwinger spinor (spin 3/2)
$(2,0)\oplus(0,2)$	Einstein gravitational fields
$\vdots$	$\vdots$

$$[\alpha] = (s_L,s_R) = (s_L,0)\otimes(0,s_R) \equiv [l]\otimes[r]$$

$$1 \leq \alpha \leq (2s_L+1)(2s_R+1) \qquad \alpha \to (l+s_L)(2s_R+1)+r+s_R+1$$

In the standard momentum frame, there is a subgroup of L_p to make k unchanged.



$$D_\alpha{}^\beta(R)\,u_\beta^\sigma(\mathbf{k};s) = u_\alpha^{\sigma'}(\mathbf{k};s)\,D_{\sigma'}^{(s)\sigma}(R),$$

$$D_\alpha{}^\beta(R)\,\bar{u}_\sigma^\alpha(\mathbf{k};s) = \bar{u}_{\sigma'}^\beta(\mathbf{k};s)\,D_\sigma^{(s)\sigma'}(R)$$

A specific form of the spin wave function by calculating the IRT of the little group SO(3)

$$u_\alpha^\sigma(\mathbf{k};s) \equiv u_\alpha^\sigma(\mathbf{k};(s_L,s_R),s) = U_\alpha^{lr}\;u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s)$$

$$\bar{u}_\sigma^\alpha(\mathbf{k};s) \equiv \bar{u}_\sigma^\alpha(\mathbf{k};(s_L,s_R),s) = U_{lr}^\alpha\;\bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s)$$

$$U_\alpha^{lr} = \delta_\alpha{}^{[(l+s_L)(2s_R+1)+r+s_R+1]},$$

$$U_{lr}^\alpha = \delta^\alpha{}_{[(l+s_L)(2s_R+1)+r+s_R+1]}.$$

$$D_l^{(s_L)l'}(R)\;D_r^{(s_R)r'}(R)\;u_{l'r'}^\sigma(\mathbf{k};(s_L,s_R),s) = u_{lr}^{\sigma'}(\mathbf{k};(s_L,s_R),s)\,D_{\sigma'}^{(s)\sigma}(R),$$

$$D_{l'}^{(s_L)l}(R)\;D_{r'}^{(s_R)r}(R)\;\bar{u}_\sigma^{l'r'}(\mathbf{k};(s_L,s_R),s) = \bar{u}_{\sigma'}^{lr}(\mathbf{k};(s_L,s_R),s)\,D_\sigma^{(s)\sigma'}(R).$$

$$u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s) = (C_{s_L s_R}^s)_{lr}^\sigma,$$

$$\bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s) = (C_s^{s_L s_R})_\sigma^{lr}$$

Generation

Three-particle vertex!

1→2+3

Lorentz index, the index of specific irreps of L_p

Spin index

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,S) = D_{\alpha_2}{}^{\beta_2}(R_{12})\,D_{\alpha_3}{}^{\beta_3}(R_{13})\;\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1;s_1)\,u_{\beta_2}^{\sigma_2}(\mathbf{k}_2;s_2)\,u_{\beta_3}^{\sigma_3}(\mathbf{k}_3;s_3)\;\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)$$

Massive bosons : $[\alpha]\equiv(s_L,s_R)=\left(\frac{s}{2},\frac{s}{2}\right)\quad(1\le\alpha\le(2s_L+1)(2s_R+1))$

$$u_\alpha^\sigma(\mathbf{k};s)\equiv u_\alpha^\sigma(\mathbf{k};(s_L,s_R),s)=U_\alpha^{lr}\;u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s)\qquad U_\alpha^{lr}=\delta_\alpha{}^{[(l+s_L)(2s_R+1)+r+s_R+1]}\qquad U_{lr}^\alpha=\delta^\alpha{}_{[(l+s_L)(2s_R+1)+r+s_R+1]}$$

$$\bar{u}_\sigma^\alpha(\mathbf{k};s)\equiv\bar{u}_\sigma^\alpha(\mathbf{k};(s_L,s_R),s)=U_{lr}^\alpha\;\bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s)\qquad u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s)=(C^s_{s_Ls_R})^\sigma_{lr}\qquad \bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s)=(C_s^{s_Ls_R})^{lr}_\sigma$$

Massive fermions : $[\alpha]\equiv(s_L,s_R)\oplus(s_R,s_L)=\left(\frac{2s+1}{4},\frac{2s-1}{4}\right)\oplus\left(\frac{2s-1}{4},\frac{2s+1}{4}\right)\quad(1\le\alpha\le 2(2s_L+1)(2s_R+1))$

$$u_\alpha^\sigma(\mathbf{k};s)\equiv u_\alpha^\sigma(\mathbf{k};\chi,s)=\begin{cases}(U_L)_\alpha^{lr}\,u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s)\ \ \text{with}\ \chi=(s_L,s_R)\\ (U_R)_\alpha^{lr}\,u_{lr}^\sigma(\mathbf{k};(s_R,s_L),s)\ \ \text{with}\ \chi=(s_R,s_L)\end{cases}\qquad \begin{aligned}(U_L)_\alpha^{lr}&=\delta_\alpha{}^{[(l+s_L)(2s_R+1)+r+s_R+1]}\\(U_R)_\alpha^{lr}&=\delta_\alpha{}^{[(3s_L+l+1)(2s_R+1)+r+s_R+1]}\end{aligned}$$

$$\bar{u}_\sigma^\alpha(\mathbf{k};s)\equiv\bar{u}_\sigma^\alpha(\mathbf{k};\chi,s)=\begin{cases}(U_R)_{lr}^\alpha\,\bar{u}_\sigma^{lr}(\mathbf{k};(s_R,s_L),s)\ \ \text{with}\ \chi=(s_L,s_R)\\ (U_L)_{lr}^\alpha\,\bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s)\ \ \text{with}\ \chi=(s_R,s_L)\end{cases}\qquad \begin{aligned}(U_R)_{lr}^\alpha&=\delta^\alpha{}_{[(3s_L+l+1)(2s_R+1)+r+s_R+1]}\\(U_L)_{lr}^\alpha&=\delta^\alpha{}_{[(l+s_L)(2s_R+1)+r+s_R+1]}\end{aligned}$$

Massless particles (helicity $\sigma=\pm s$) **:** $[\alpha]\equiv[\zeta^s]=(s,0)\oplus(0,s)$

$$u_{\zeta^s}^\sigma(\mathbf{k};(s,0),s)=\delta^1_{\zeta^s}\,\delta^\sigma_{-s}\qquad u_{\zeta^s}^\sigma(\mathbf{k};(0,s),s)=\delta^{2(2s+1)}_{\zeta^s}\,\delta^\sigma_s$$

$$\bar{u}_\sigma^{\zeta^s}(\mathbf{k};(s,0)^*,s)=\delta_1^{\zeta^s}\,\delta_\sigma^{-s}\qquad \bar{u}_\sigma^{\zeta^s}(\mathbf{k};(0,s)^*,s)=\delta^{\zeta^s}_{2(2s+1)}\,\delta^s_\sigma$$

$$u_\alpha^\sigma(\mathbf{k};s)\equiv u_\alpha^\sigma(\mathbf{k};(s_L,s_R),s)=U_\alpha^{lr}\;u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s)$$

$$\bar{u}_\sigma^\alpha(\mathbf{k};s)\equiv\bar{u}_\sigma^\alpha(\mathbf{k};(s_L,s_R),s)=U_{lr}^\alpha\;\bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s)$$

$$U_\alpha^{lr}=\delta_\alpha{}^{[(l+s_L)(2s_R+1)+r+s_R+1]},$$

$$U_{lr}^\alpha=\delta^\alpha{}_{[(l+s_L)(2s_R+1)+r+s_R+1]}.$$

$$D_l^{(s_L)l'}(R)\;D_r^{(s_R)r'}(R)\,u_{l'r'}^\sigma(\mathbf{k};(s_L,s_R),s)=u_{lr}^{\sigma'}(\mathbf{k};(s_L,s_R),s)\,D_{\sigma'}^{(s)\sigma}(R),$$

$$D_{l'}^{(s_L)l}(R)\;D_{r'}^{(s_R)r}(R)\,\bar{u}_\sigma^{l'r'}(\mathbf{k};(s_L,s_R),s)=\bar{u}_{\sigma'}^{lr}(\mathbf{k};(s_L,s_R),s)\,D_\sigma^{(s)\sigma'}(R).$$

→

$$u_{lr}^\sigma(\mathbf{k};(s_L,s_R),s)=(C^s_{s_Ls_R})^\sigma_{lr},$$

$$\bar{u}_\sigma^{lr}(\mathbf{k};(s_L,s_R),s)=(C_s^{s_Ls_R})^{lr}_\sigma$$

Generation

Three-particle vertex!

1→2+3

Lorentz index, the index of specific irreps of L_p

Spin index

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,S) = D_{\alpha_2}{}^{\beta_2}(R_{12})\,D_{\alpha_3}{}^{\beta_3}(R_{13}) \quad \bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1;s_1)\,u_{\beta_2}^{\sigma_2}(\mathbf{k}_2;s_2)\,u_{\beta_3}^{\sigma_3}(\mathbf{k}_3;s_3) \quad \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)$$

Coupling Structure

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha^L\alpha^S}(\mathbf{k}_1;s_1,L,S)\;P_{\alpha^S}^{\alpha_2\alpha_3}(\mathbf{k}_1;S,s_2,s_3)\;\tilde{t}_{\alpha^L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

spin+orbit — spin — orbit

$$\tilde{t}_{\alpha^L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*) \equiv P_{\alpha^L}^{\beta_1\cdots\beta_L}(\mathbf{k}_1;L)\,(p_2^*-p_3^*)_{\beta_1}\cdots(p_2^*-p_3^*)_{\beta_L}$$

$$(p_2^*-p_3^*)_\beta = (U^{-1})_\beta{}^\mu\,(p_2^*-p_3^*)_\mu$$

$$P_{\alpha^L}^{\beta_1\cdots\beta_L}(\mathbf{k}_1;L) = P_{\alpha^L}^{\beta_1\alpha^{L-1}}(\mathbf{k}_1;L,1,L-1)\;P_{\alpha^{L-1}}^{\beta_2\cdots\beta_L}(\mathbf{k}_1;L-1)$$

$$P_{\alpha_1}^{\alpha_2\,\alpha_3}(\mathbf{k}_1;j_1,j_2,j_3) = \sum_{\chi_1,\chi_2,\chi_3} C_{\chi_1\chi_2\chi_3}\;P_{\alpha_1}^{\alpha_2\,\alpha_3}(\mathbf{k}_1;\chi_1,j_1;\chi_2,j_2;\chi_3,j_3)$$

$j_2+j_3=j_1$

$$(U^{-1})_\alpha{}^\mu = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}_\alpha^{\ \mu}$$

$$P_\beta^{\alpha_1\alpha_2}(\mathbf{p};\chi,s;\chi_1,s_1;\chi_2,s_2) = (C_s^{s_1s_2})_\sigma^{\sigma_1\sigma_2}\;u_\beta^\sigma(\mathbf{p};\chi,s)\,\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p};\chi_1^*,s_1)\;\bar{u}_{\sigma_2}^{\alpha_2}(\mathbf{p};\chi_2^*,s_2)$$

P is the project operator, which can be generate by the wave functions (IRT).
Project various angular momenta to one specific angular momentum.

$$T_\beta^{\alpha_1\alpha_2} = \sum_{\chi,s} u_\beta^\sigma(\mathbf{k};\chi,s)\,\bar{u}_\sigma^{\alpha_1\alpha_2}(\mathbf{k};\chi^*,s)$$

$$[\alpha_1]\otimes[\alpha_2]\to[\beta]$$

Comparison

- **The pure-orbital (L) and pure-spin (S) component**

  - **C-scheme** $\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$

  - **H-scheme** $\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\, D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$

$$\begin{aligned}\mathcal{H}_{\lambda_1}^{\lambda_2\lambda_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) &= \mathcal{H}_{\sigma_1}^{\lambda_2\lambda_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\\ &= D_{\sigma_2}^{(s_2)\lambda_2}\left(R_{\hat{\mathbf{p}}_2^*}\right) D_{\sigma_3}^{(s_3)\lambda_3}\left(R_{\hat{\mathbf{p}}_3^*}\right) \mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\end{aligned}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = D_{\sigma_1}^{(s_1)\sigma_1'}\left(R_{\hat{\mathbf{p}}_2^*}\right) D_{\sigma_2'}^{(s_2)\sigma_2}\left(R_{\hat{\mathbf{p}}_2^*}^{-1}\right) D_{\sigma_3'}^{(s_3)\sigma_3}\left(R_{\hat{\mathbf{p}}_2^*}^{-1}\right) \mathcal{A}_{\sigma_1'}^{\sigma_2'\sigma_3'}(\mathbf{k}_1,\bar{\mathbf{p}}_2^*,\bar{\mathbf{p}}_3^*;L,S)$$

$$\begin{aligned}\mathcal{H}_{\sigma_1}^{\lambda_2\lambda_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) &= D_{\sigma_1}^{(s_1)\sigma_1'}\left(R_{\hat{\mathbf{p}}_2^*}\right) D_{\sigma_3}^{(s_3)\lambda_3}\left(R_{\hat{\mathbf{p}}_2^*}^{-1}\cdot R_{\hat{\mathbf{p}}_3^*}\right) \mathcal{A}_{\sigma_1'}^{\lambda_2\sigma_3}(\mathbf{k}_1,\bar{\mathbf{p}}_2^*,\bar{\mathbf{p}}_3^*;L,S)\\ &\equiv e^{i\Theta\left(R_{\hat{\mathbf{p}}_2^*},R_{\hat{\mathbf{p}}_3^*}\right)} D_{\sigma_1}^{(s_1)\sigma_1'}\left(R_{\hat{\mathbf{p}}_2^*}\right) \mathcal{F}_{\sigma_1'}^{\lambda_2\,\lambda_3}(\mathbf{k}_1,|\mathbf{p}_2^*|,|\mathbf{p}_3^*|;L,S)\end{aligned}$$

Helicity amplitude is equivalent with the covariant amplitude just with several rotation matrix.

Comparison

+helicity
—Cov_new

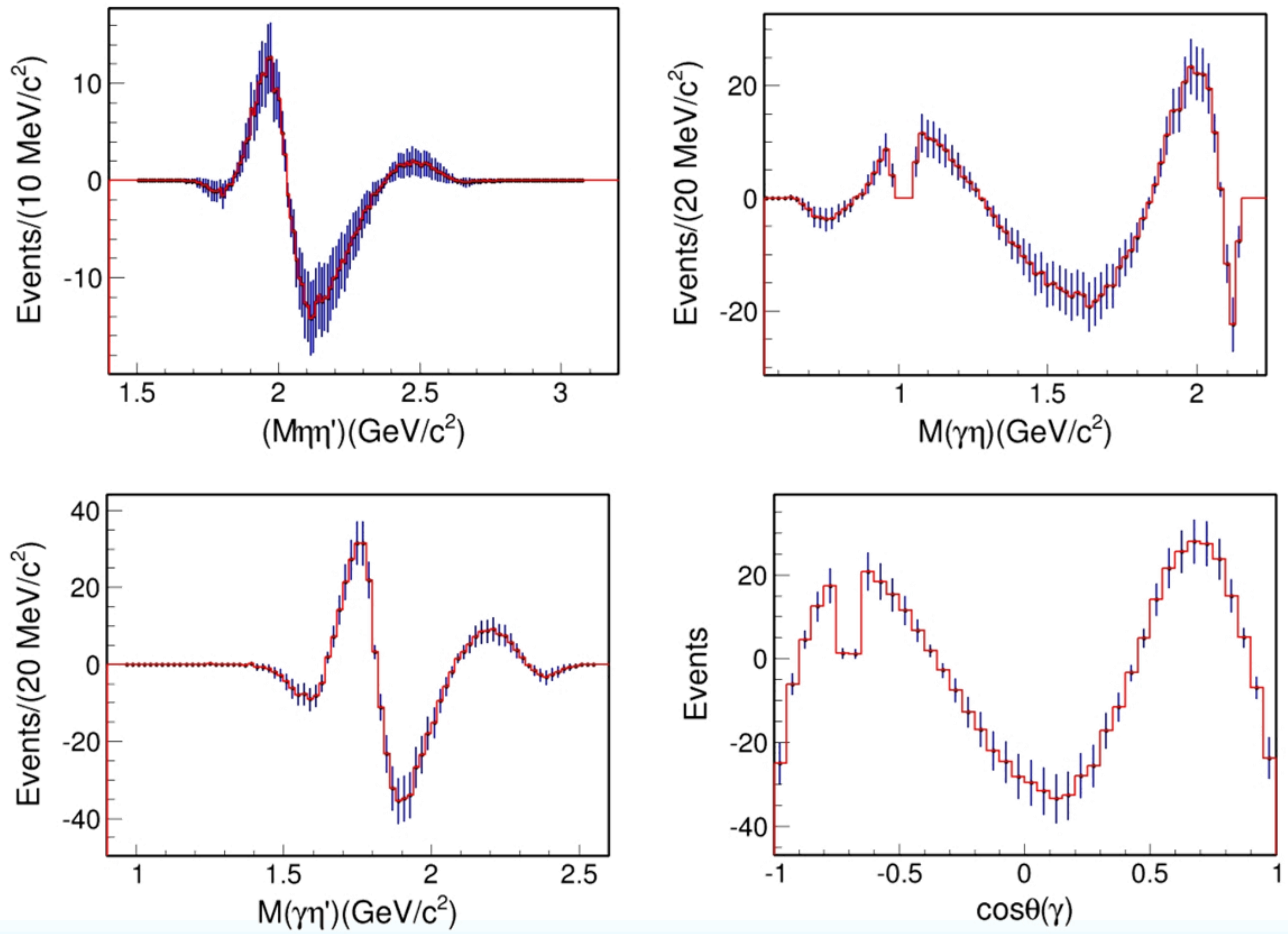
$J/\psi \to \gamma X,\ X \to \eta\eta'$

X: $J^{PC} = 0^{++}$, $M = 2.0\ GeV$, $\Gamma = 0.2\ GeV$

$J/\psi \to \eta Y,\ Y \to \gamma\eta'$

Y: $J^{PC} = 1^{+-}$, $M = 1.8\ GeV$, $\Gamma = 0.2\ GeV$

$J/\psi \to \gamma\eta\eta'$



Helicity amplitude is equivalent with the covariant amplitude just with several rotation matrix.

Application

$$1(s_1 = 1) \to 2(s_2 = 1) + 3(s_3 = 0)$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; L, S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S)\, D_{\alpha_2}{}^{\beta_2}(R_{12})\, D_{\alpha_3}{}^{\beta_3}(R_{13})\, \bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{k}_1; s_1)\; u_{\beta_2}^{\sigma_2}(\mathbf{k}_2; s_2)\; u_{\beta_3}^{\sigma_3}(\mathbf{k}_3; s_3)$$

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, 1) = P_{\alpha_1}^{\alpha^L\alpha^S}(\mathbf{k}_1; 1, L, 1)\, P_{\alpha^S}^{\alpha_2\alpha_3}(\mathbf{k}_1; 1, 1, 0)\, \tilde{t}_{\alpha^L}^{(L)}(\mathbf{k}_1, \mathbf{p}_2^* - \mathbf{p}_3^*)$$

$$P_{\alpha_1}^{\alpha^L\alpha^S}(\mathbf{k}_1; 1, L, 1) = \left(C_1^{L1}\right)_{\sigma_1}^{\sigma_L\sigma_S} u_{\alpha_1}^{\sigma_1}\left(\mathbf{k}_1; \left[\mu^1\right], 1\right)\, \bar{u}_{\sigma_L}^{\alpha^L}\left(\mathbf{k}_1; \left[\mu^L\right], L\right)\, \bar{u}_{\sigma_S}^{\alpha^S}\left(\mathbf{k}_1; \left[\mu^1\right], 1\right)$$

$$[\alpha_1] = \left[\mu^1\right] \quad \Rightarrow \quad \bar{u}_{\sigma_1}^{\alpha_1}\left(\mathbf{k}_1; \left[\mu^1\right], 1\right) = U_{lr}^{\alpha_1}\left(C_1^{\frac{1}{2}\frac{1}{2}}\right)_{\sigma_1}^{lr},$$

$$[\alpha_2] = \left[\mu^1\right] \quad \Rightarrow \quad u_{\alpha_2}^{\sigma_2}\left(\mathbf{k}_2; \left[\mu^1\right], 1\right) = U_{\alpha_2}^{lr}\left(C_{\frac{1}{2}\frac{1}{2}}^{1}\right)_{lr}^{\sigma_1},$$

$$[\alpha_3] = \left[\mu^0\right] \quad \Rightarrow \quad u_{\alpha_3}^{\sigma_3}\left(\mathbf{k}_3; \left[\mu^0\right], 1\right) = U_{\alpha_3}^{00}\left(C_{00}^{0}\right)_{00}^{\sigma_1}.$$

$$\tilde{t}_{\alpha^L}^{(L)}(\mathbf{k}_1, \mathbf{p}_2^* - \mathbf{p}_3^*) \equiv P_{\alpha^L}^{\beta_1\cdots\beta_L}(\mathbf{k}_1; L)\; (p_2^* - p_3^*)_{\beta_1} \cdots (p_2^* - p_3^*)_{\beta_L}$$

$$L = 0 \;: \qquad P_{\alpha^0}(\mathbf{k}_1; 0) = u_{\alpha^0}^{0}\left(\mathbf{k}_1; \left[\mu^0\right], 0\right),$$

$$L = 1 \;: \qquad P_{\alpha^1}^{\beta_1}(\mathbf{k}_1; 1) = P_{\alpha^1}^{\beta_1\alpha^0}(\mathbf{k}_1; 1, 1, 0)\; P_{\alpha^0}(\mathbf{k}_1; 0),$$

$$L = 2 \;: \qquad P_{\alpha^2}^{\beta_1\beta_2}(\mathbf{k}_1; 2) = P_{\alpha^2}^{\beta_1\alpha^1}(\mathbf{k}_1; 2, 1, 1)\; P_{\alpha^1}^{\beta_2}(\mathbf{k}_1; 1),$$

Application

$$1(s_1=1)\to 2(s_2=1)+3(s_3=0)$$

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,S)=\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\,D_{\alpha_2}{}^{\beta_2}(R_{12})\,D_{\alpha_3}{}^{\beta_3}(R_{13})\,\bar u^{\alpha_1}_{\sigma_1}(\mathbf{k}_1;s_1)\;u^{\sigma_2}_{\beta_2}(\mathbf{k}_2;s_2)\;u^{\sigma_3}_{\beta_3}(\mathbf{k}_3;s_3)$$

$$\Gamma^{\nu}_{\mu}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,1)=U_{\mu}{}^{\alpha_1}\,\left(U^{-1}\right)_{\alpha_2}{}^{\nu}\;\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,1),$$

$$\Gamma^{\nu}_{\mu}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;0,1)=-g_{\mu}{}^{\nu}+\frac{k_{1\mu}k_1^{\nu}}{m_1^2},$$

$$\Gamma^{\nu}_{\mu}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;1,1)=i\,g_{\mu\mu'}\left(\frac{k_1}{m_1}\right)_{\nu'}U_{\rho'}{}^{\alpha^1}\;\tilde t^{(1)}_{\alpha^1}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)\,\epsilon^{\mu'\nu'\rho'\nu}$$

$$\Gamma^{\nu}_{\mu}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;2,1)=U_{\mu}{}^{\alpha_1}\,\left(U^{-1}\right)_{\alpha_2}{}^{\nu}\;T^{\alpha_2\alpha^2}_{\alpha_1}\;\tilde t^{(2)}_{\alpha^2}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*),$$

$$\mathcal{A}^{\sigma_2}_{\sigma_1}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,1)=\bar\epsilon^{\mu}_{\sigma_1}(\mathbf{p}_1)\;\Gamma^{\rho}_{\mu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;L,1)\;D^{\nu}_{\rho}\left(h_{\mathbf{p}_1}\cdot h^{-1}_{\mathbf{p}_2^*}\cdot h^{-1}_{\mathbf{p}_1}\right)\;\epsilon^{\sigma_2}_{\nu}(\mathbf{p}_2)$$

It is exact the same as that in this reference.

B.S. Zou and D.V. Bugg, EPJA 16 (2003) 537

Application

- Currently commonly used PWA package (PKG) on BESIII
  - ☐ FDC-PWA： 协变有效拉氏量方法下协变振幅的自动化计算

    (王建雄研究员、平荣刚研究员@IHEP) [https://www1.ihep.ac.cn/wjx/pwa] (2000)
  - ☐ GPUPWA： 协变 L-S 方案下分波振幅的自动化计算 （刘北江研究员@IHEP）
    - ➢ 支持矢量介子的强衰变和辐射衰变过程 [https://sourceforge.net/projects/gpupwa/] (2011)
  - ☐ TF-PWA： 螺旋度方案下分波振幅的自动化计算 [https://tf-pwa.readthedocs.io] (2020)

    （蒋艺、刘寅睿、钱文斌教授、吕晓睿教授、郑阳恒教授@UCAS）
- Automatic calculation of PWF under the covariant L-S scheme
  - PKG for calculating PWF under C/H-scheme based on C++

    （景豪杰与吴蜀明博士@UCAS合作） [https://github.com/Wu-ShuMing/PWFs] (2024)
  - Crosscheck our PKG with the TF-PWA

    (与蒋艺@UCAS, 马润秋@IHEP 和王石@LZU合作)

TF-PWA

A general and user-friendly partial wave analysis framework

Hao Cai[1], Chen Chen[5], Shuangshi Fang[4], Haojie Jing[2], Yi Jiang[2], Pei-Rong Li[3], Beijiang Liu[4], Yin-Rui Liu[2], Xiao-Rui Lyu[2], Runqiu Ma[4], Rong-Gang Ping[4], Wenbin Qian[2], Rongsheng Shi[3], Mengzhen Wang[5], Shi Wang[4], Zi-Yi Wang[2], Jiajun Wu[2], Shuming Wu[2], Liming Zhang[5], Yang-Heng Zheng[2]

[1]WHU, [2]UCAS, [3]LZU, [4]IHEP, [5]THU

- Fast
  - GPU based
  - Vectorized calculation
  - Automatic differentiation
- General
  - Custom model available
- Easy to use
  - Simple configuration file
  - Automatics process
  - All necessary functions implemented

TensorFlow

https://gitlab.com/jiangyi15/tf-pwa

Open access and well supported

Summary

- Show a new PWA formalism for 3 particle vertex.
- Covariant ! L-S scheme!

- Benefit
- Use such vertex can include loop contribution easily!
- Also extend to coupled channel in future!

Thanks for attention !