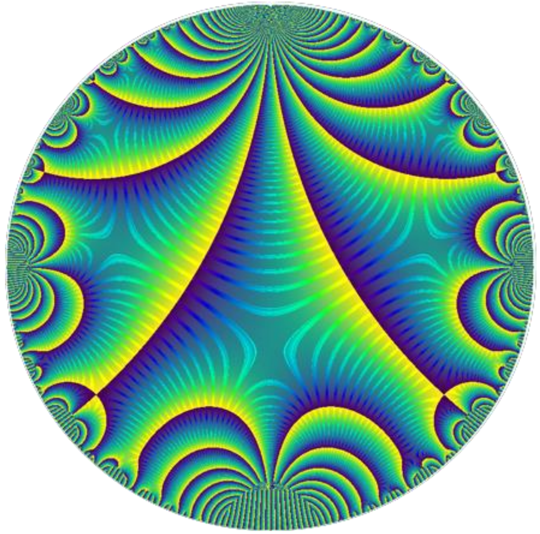
Modular cosmology



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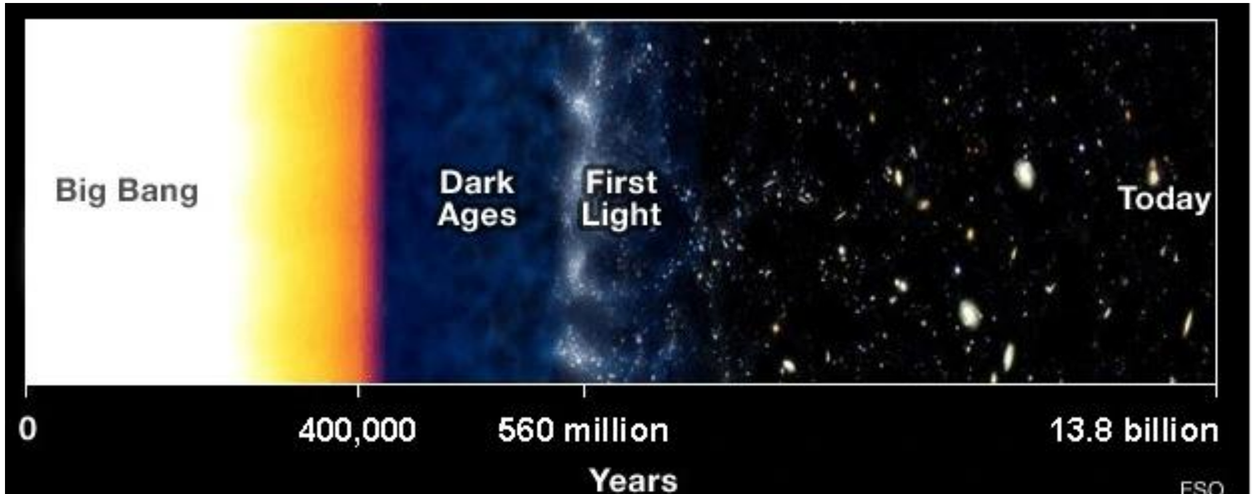
based on: Gui-Jun Ding, Si-Yi Jiang, Wenbin Zhao,arXiv:2405.06497
Gui-Jun Ding, Si-Yi Jiang, Yong Xu, Wenbin Zhao, arXiv:2411.xxxxx (in preparation)



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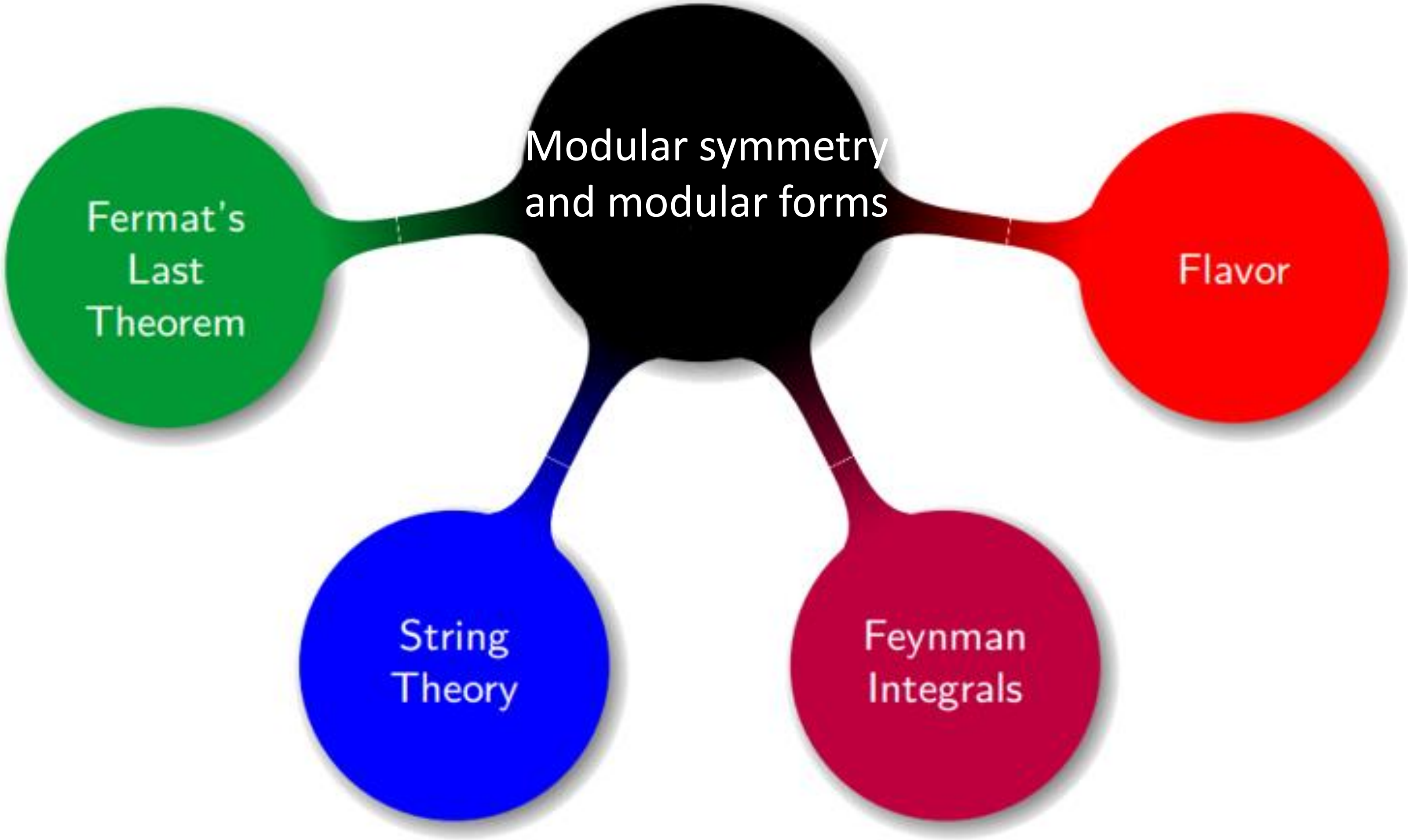
Outline:

- Introduction to modular symmetry
- Modular inflation
- Reheating from inflaton decay
- Baryon asymmetry from non-thermal leptogenesis
- Summary

Modular symmetry and modular forms are well-known in Mathematics and some areas of physics.



Modular symmetry and modular forms

Fermat's Last Theorem

Flavor

String Theory

Feynman Integrals

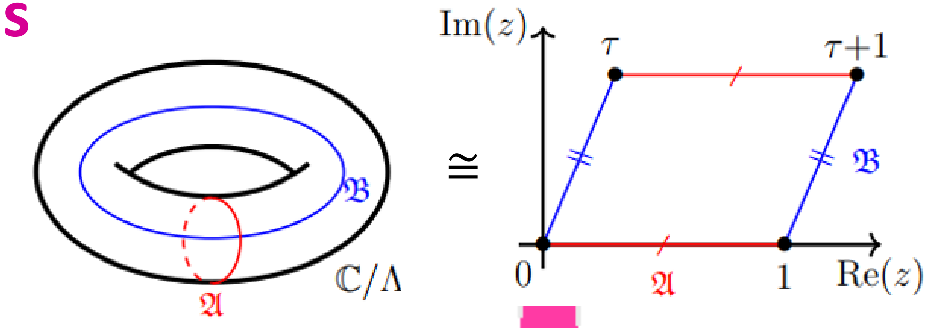
Credit: Michael Ratz, Moriond 2024

Modular symmetry motivated by string compactification

a single complex flavon τ parametrizing shape of torus

- Modular action **[Feruglio, 1706.08749]**

$$\tau \to \gamma\tau = \frac{a\tau+b}{c\tau+d} \qquad SL(2,Z)=\left\{\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}$$



$SL(2,Z)$ on torus T^2

- Field **Non-linear** transformation

$$\psi \to \boxed{(c\tau+d)^{-k}}\ \boxed{\rho(\gamma)}\,\psi$$

weight $k\in\mathbb{Z}$ ρ is a unitary representation of Γ_N or Γ'_N

- Superpotential

$$\mathcal{W}=\sum Y_{I_1 I_2\ldots I_n}(\tau)\,\psi_{I_1}\psi_{I_2}\ldots\psi_{I_n}$$

Yukawa coupling $Y_{I_1 I_2 \ldots I_n}(\tau)$ only depends on τ, and it is modular form of level N:

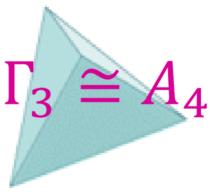
$$Y_{I_1 I_2\ldots I_n}(\tau)\to Y_{I_1 I_2\ldots I_n}(\gamma\tau)=(c\tau+d)^{k_Y}\rho_Y(\gamma)Y_{I_1 I_2\ldots I_n}(\tau)$$

finite modular groups as G_f

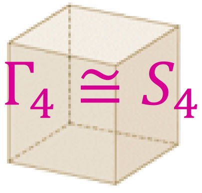
$$G_f=\begin{cases}\Gamma_N\equiv SL(2,Z)/\pm\Gamma(N)\\ \Gamma'_N\equiv SL(2,Z)/\Gamma(N)\end{cases}$$

Principal congruence subgroup of level N

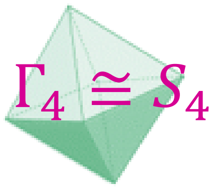
$$\Gamma(N)=\{\gamma\in SL(2,Z)\,|\,\gamma=1_2 \mod N\}$$


$\Gamma_3\cong A_4$

Tetrahedron


$\Gamma_4\cong S_4$

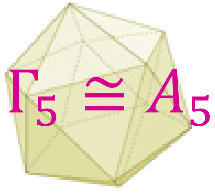
Hexahedron


$\Gamma_4\cong S_4$

Octahedron

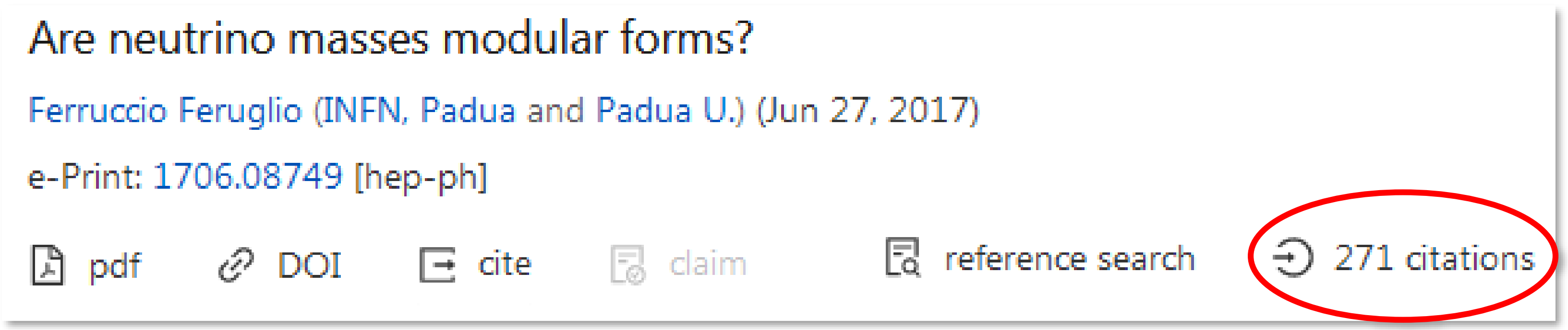

$\Gamma_5\cong A_5$

Dodecahedron


$\Gamma_5\cong A_5$

Icosahedron

Modular invariant flavor models

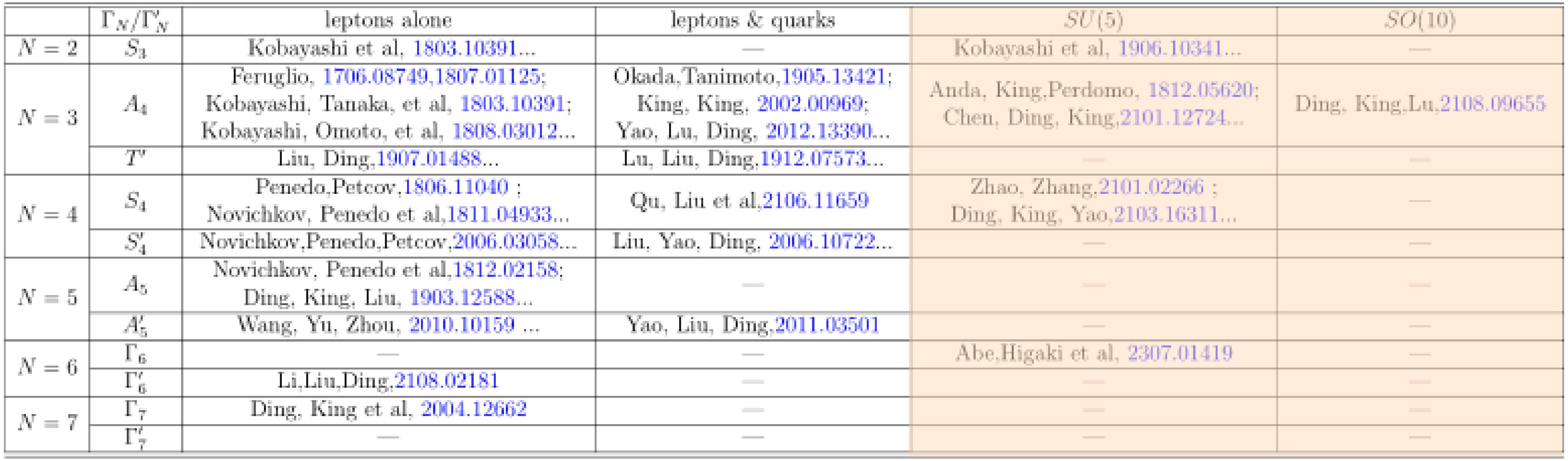


Are neutrino masses modular forms?

Ferruccio Feruglio (INFN, Padua and Padua U.) (Jun 27, 2017)

e-Print: 1706.08749 [hep-ph]

- Bottom-up models for lepton and quark **[reviews: Kobayashi, Tanimoto, 2307.03384; Ding, King, arXiv:2311.09282]** **See talk by Fei Wang**



	Γ_N/Γ'_N	leptons alone	leptons & quarks	$SU(5)$	$SO(10)$
$N=2$	S_3	Kobayashi et al, 1803.10391...	—	Kobayashi et al, 1906.10341...	—
$N=3$	A_4	Feruglio, 1706.08749,1807.01125; Kobayashi, Tanaka, et al, 1803.10391; Kobayashi, Omoto, et al, 1808.03012...	Okada,Tanimoto,1905.13421; King, King, 2002.00969; Yao, Lu, Ding, 2012.13390...	Anda, King,Perdomo, 1812.05620; Chen, Ding, King,2101.12724...	Ding, King,Lu,2108.09655
	T'	Liu, Ding,1907.01488...	Lu, Liu, Ding,1912.07573...	—	—
$N=4$	S_4	Penedo,Petcov,1806.11040 ; Novichkov, Penedo et al,1811.04933...	Qu, Liu et al,2106.11659	Zhao, Zhang,2101.02266 ; Ding, King, Yao,2103.16311...	—
	S'_4	Novichkov,Penedo,Petcov,2006.03058...	Liu, Yao, Ding, 2006.10722...	—	—
$N=5$	A_5	Novichkov, Penedo et al,1812.02158; Ding, King, Liu, 1903.12588...	—	—	—
	A'_5	Wang, Yu, Zhou, 2010.10159 ...	Yao, Liu, Ding,2011.03501	—	—
$N=6$	Γ_6	—	—	Abe,Higaki et al, 2307.01419	—
	Γ'_6	Li,Liu,Ding,2108.02181	—	—	—
$N=7$	Γ_7	Ding, King et al, 2004.12662	—	—	—
	Γ'_7	—	—	—	—

$\langle\tau\rangle$ + few couplings

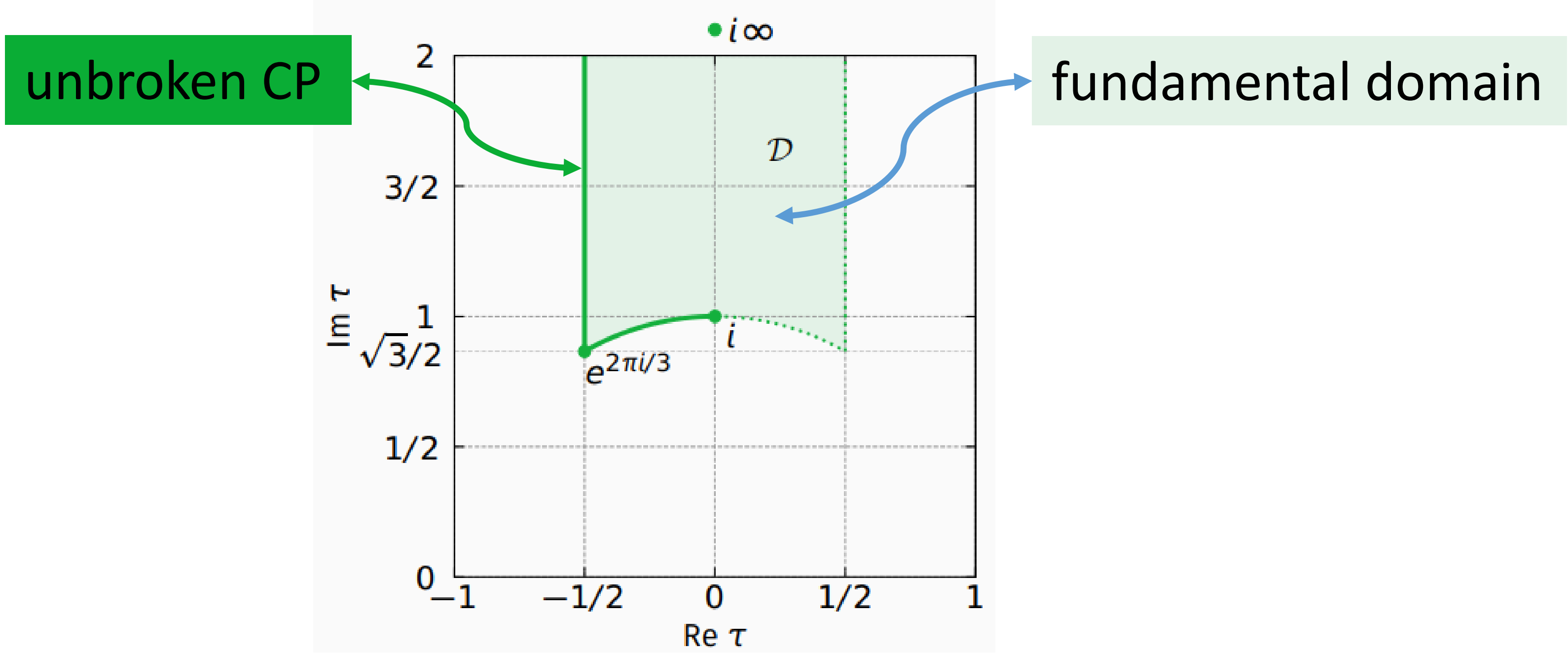
3 ν masses m_i

3 mixing angles θ_{ij}

3 CP phases $\delta_{CP}, \alpha_{21}, \alpha_{31}$

The complex modulus τ

- ➢ Modular symmetry can be thought as a gauge symmetry. With a gauge choice τ can be restricted to the **fundamental region**,



unbroken CP

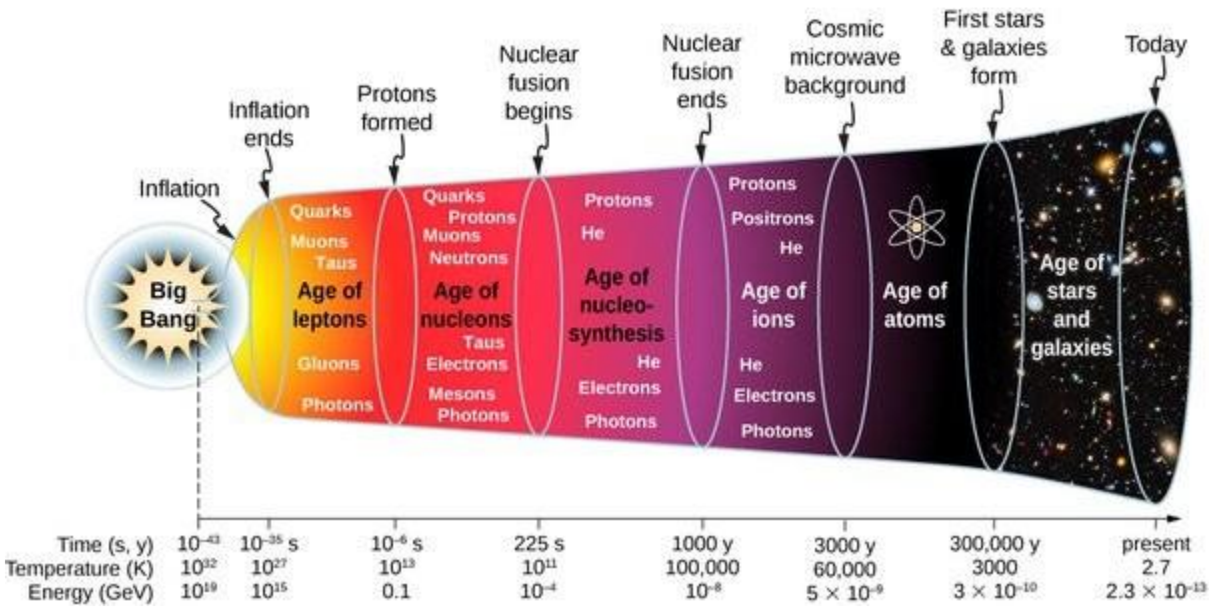
fundamental domain

- ➢ Modular symmetry is broken by the vacuum expectation value of τ

- Bottom-up approach: $\langle\tau\rangle$ is treated as a free parameter and its value is obtained by fitting the data
- Top-down approach: modulus stabilization, $\langle\tau\rangle$ is the global minimum of the modular invariant scalar potential for τ

The role of τ in particle physics and cosmology?

Modular inflation



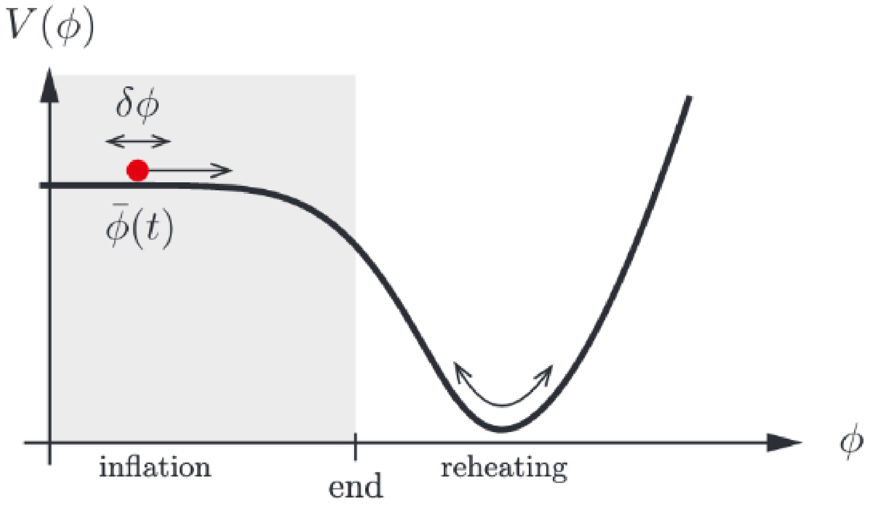
Accelerated expansion of the early universe favored by cosmic microwave background (CMB) observations

$$
\begin{aligned}
\ln\left(10^{10} A_s\right) &= 3.044 \pm 0.014\,(68\%\mathrm{CL})\,, \\
n_s &= 0.9649 \pm 0.0042\,(68\%\mathrm{CL})\,, \\
r &< 0.036\,(95\%\mathrm{CL})\,.
\end{aligned}
$$

A_s: scalar amplitude
n_s: spectral index
r: tensor-to-scalar ratio

[Planck Collaboration,1807.06211]

- Inflation can be realized by a slow-rolling scalar field (inflaton)



SM cannot explain inflation!

Modular inflation: modulus τ plays the role of inflaton, a plateau in the scalar potential is necessary

A cosmological probe of modular symmetry!

General modular invariant scalar potential for τ

- The **modular invariant** effective action in SUGRA for $\tau + S$ (dilaton) [Cvetic et al., Nucl. Phys. B 361 (1991)]

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) = \mathcal{K}(\tau,\bar{\tau},S,\bar{S}) + \ln|\mathcal{W}(\tau,S)|^2$$

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) \xrightarrow{\gamma} \mathcal{G}(\tau,\bar{\tau},S,\bar{S})$$

Kähler potential:

$$\mathcal{K}(\tau,\bar{\tau},S,\bar{S}) = K(S,\bar{S}) - 3\ln[-i(\tau-\bar{\tau})]$$

$$K(S,\bar{S}) = \underset{\substack{\uparrow \\ \text{Tree level}}}{-\ln(S+\bar{S})} + \underset{\substack{\uparrow \\ \text{Non-perturbative effects: important to stabilize dilaton } S}}{\delta k(S,\bar{S})}$$

Superpotential:

$$\mathcal{W}(\tau,S) = \Lambda_W^3 \frac{\Omega(S)H(\tau)}{\eta^6(\tau)}, \quad H(\tau) = \sum_{m,n} (j(\tau)-1728)^{m/2} j(\tau)^{n/3} \mathcal{P}_{m,n}(j(\tau))$$

- $\eta(\tau)$:Dedekind eta function with modular weight 1/2; $j(\tau)$: Klein j-invariant
- $\Omega(S)$ is technically arbitrary, $\mathcal{P}_{m,n}(j)$ is an arbitrary polynomial function of $j(\tau)$

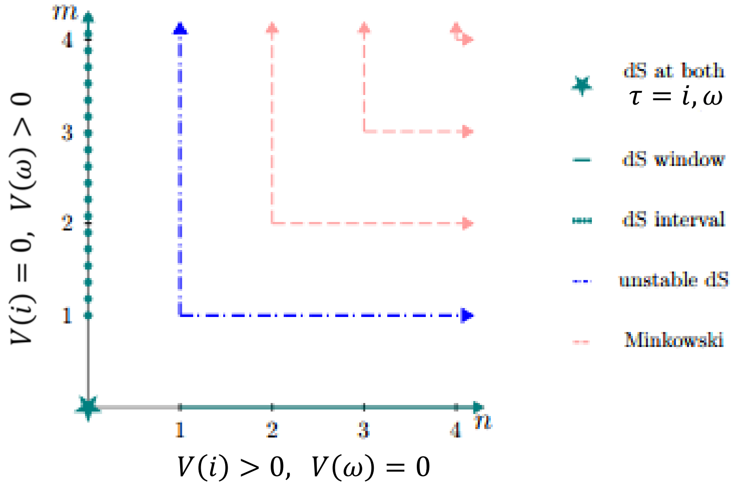
Scalar potential:

$$\boxed{V(\tau,S) = \Lambda^4 e^{K(S,\bar{S})}|\Omega(S)|^2 Z(\tau,\bar{\tau})\left[(A(S,\bar{S})-3)|H(\tau)|^2 + \hat{V}(\tau,\bar{\tau})\right]}$$

$$Z(\tau,\bar{\tau}) = \frac{1}{i(\tau-\bar{\tau})^3|\eta(\tau)|^{12}}, \quad A(S,\bar{S}) = \frac{K^{S\bar{S}}|\Omega_S + K_S\Omega|^2}{|\Omega|^2}, \quad \hat{V}(\tau,\bar{\tau}) = -\frac{(\tau-\bar{\tau})^2}{3}\left|H_\tau(\tau) - \frac{3i}{2\pi}H(\tau)\hat{G}_2(\tau,\bar{\tau})\right|$$

Free parameters: $A(S,\bar{S})$, (m,n), $\mathcal{P}_{m,n}(j)$

- Vacuum structure of modulus at fixed points $\tau = i,\ e^{2\pi i/3} \equiv \omega$



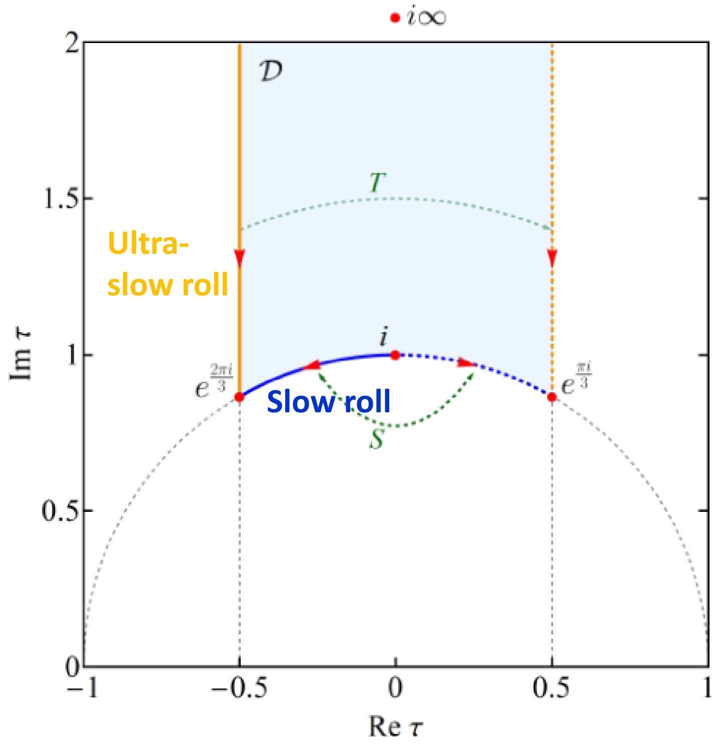
[Leedom, Righi,Westphal,2212.03876]

τ rolls toward the global minimum

- Slow roll along the boundary of fundamental domain

Boundary of fundamental domain is the **"valle"** of the scalar potential!

- $m = 0, n \geq 2$, slow roll from i (saddle point) to ω (Minkowski minimum) along the unit arc **[Ding, Jiang, Zhao, 2405.06497]**
- $m \geq 2, n \geq 2$, ultra-slow roll from $i\infty$ to ω (Minkowski minimum) along the left boundary **[Ding, Jiang, Zhao, 2405.06497]**
- $m = 0, n = 0$, slow roll from i (saddle point) to ω (dS minimum) along the unit arc **[King, Wang, 2405.08924]**



modular symmetry⇒ $\epsilon_V = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1,\ \eta_V = \frac{V''}{V} \ll 1$

Slow roll in the vicinity of τ=i

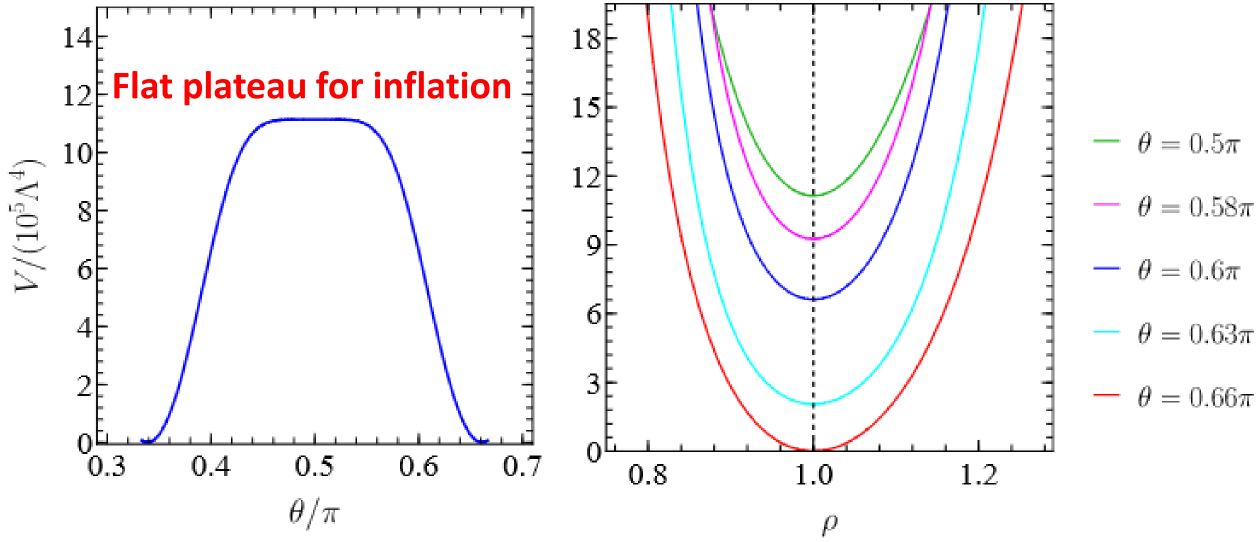
- Canonical normalization $\tau = \rho e^{i\theta}$

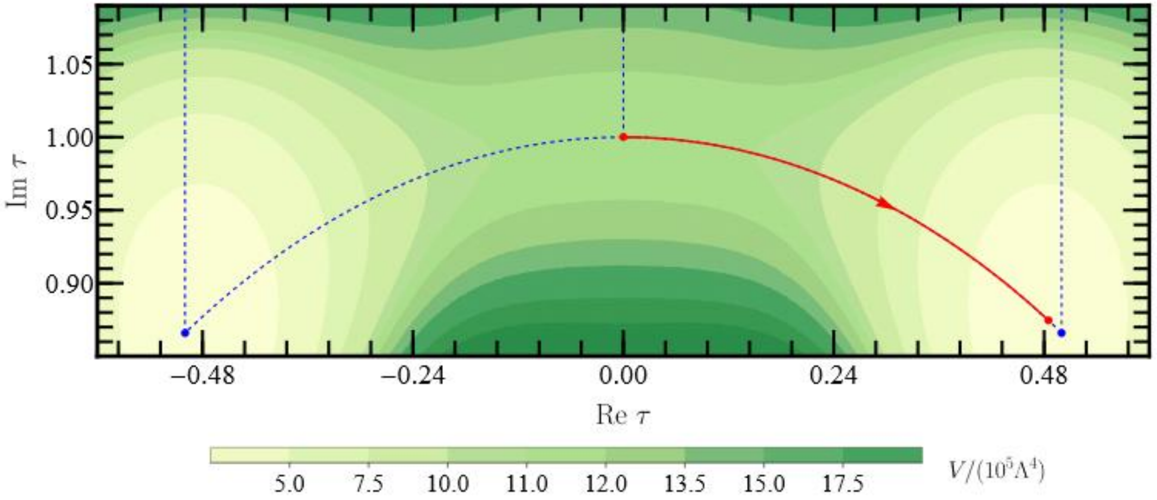
$$\boxed{\phi = \sqrt{3/2}\,\log(\tan(\theta/2))}$$

$$\mathcal{L}_{\text{kin}} = \frac{3}{(-i\tau + i\,\bar{\tau}\,)^2}\partial_\mu\tau\partial^\mu\bar{\tau} = \frac{3}{4\,\sin^2\theta}\left(\frac{1}{\rho^2}\partial_\mu\rho\partial^\mu\rho + \partial_\mu\theta\partial^\mu\theta\right) \xrightarrow{\rho = 1} \mathcal{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

- Hilltop-like potential: $V(\phi) = V_0[1 - C_2\phi^2 - C_4\phi^4 - C_6\phi^6 + \cdots]$

Invariant under $\phi \overset{S}{\rightarrow} -\phi$


Flat plateau for inflation



Example: $H(\tau) = \left(j^{1/3}(\tau) - j^{1/3}(\tau_0)\right)^2 [1 + \beta(1 - j(\tau)/1728)] \Rightarrow m = 0, n = 0{,}1{,}2$

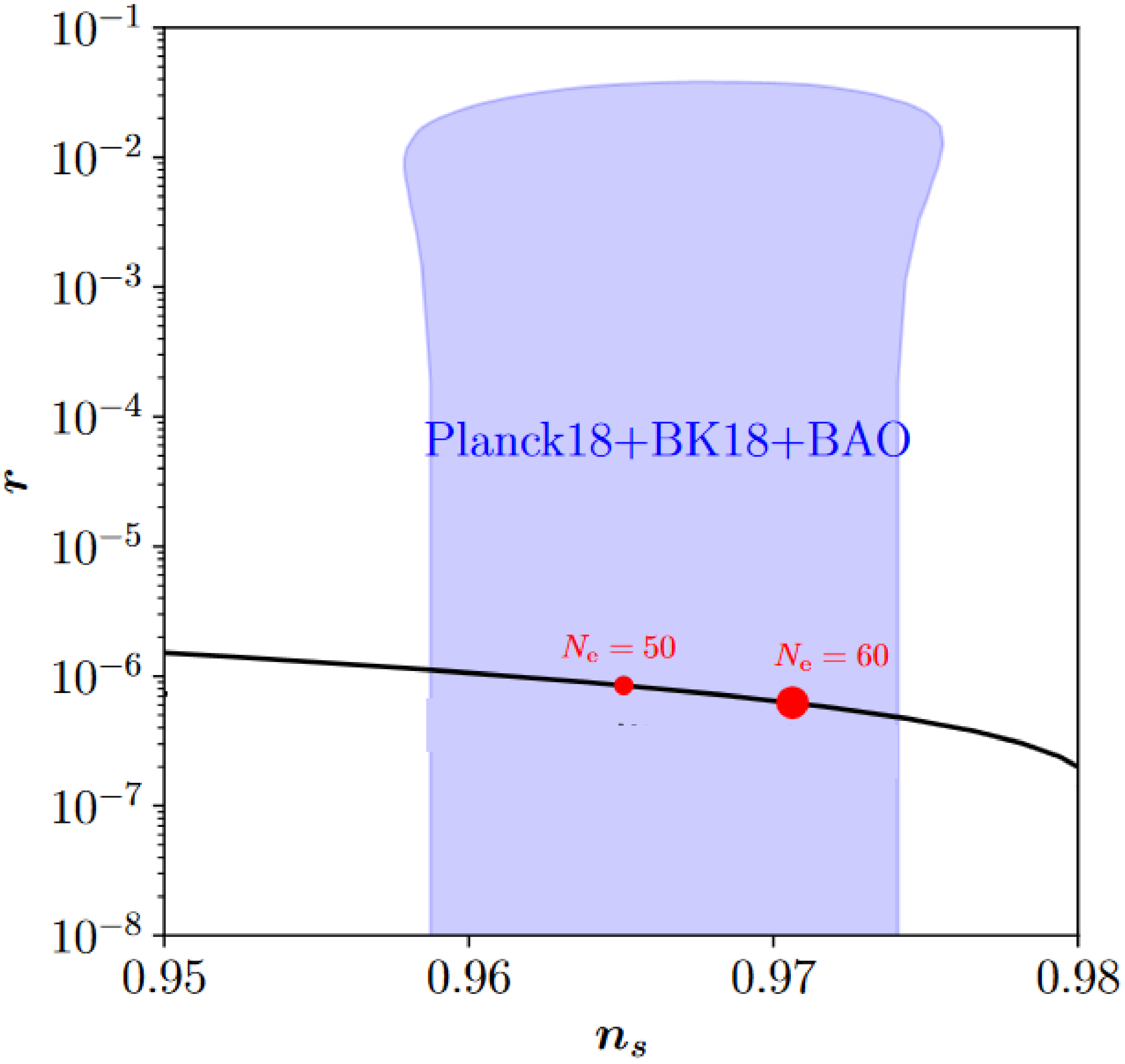
τ_0 is global minimum of the scalar potential

[Ding, Jiang, Xu, Zhao, 2411.xxxxx]

$$A = 24.4895,\ \ \beta = 0.1321 \Rightarrow N_e \simeq 50.807,\ \ r \simeq 8.6 \times 10^{-7},\ \ n_s \simeq 0.9649,\ \ \alpha = -7.3 \times 10^{-4},$$

inflaton mass: $m_\phi = 1.5 \times 10^{12}\text{GeV}$

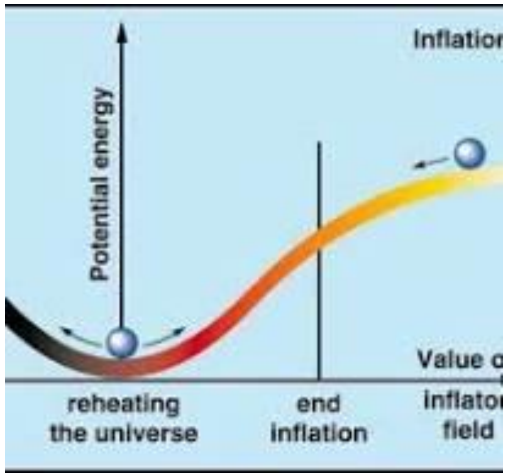
- Successful inflation can be reproduced: the tensor-to-scalar ratio $r \sim \mathcal{O}(10^{-6})$ and a negative running $\alpha \sim \mathcal{O}(-10^{-3})$ testable by future CMB measurements [Ding, Jiang, Xu, Zhao, 2411.xxxxx]



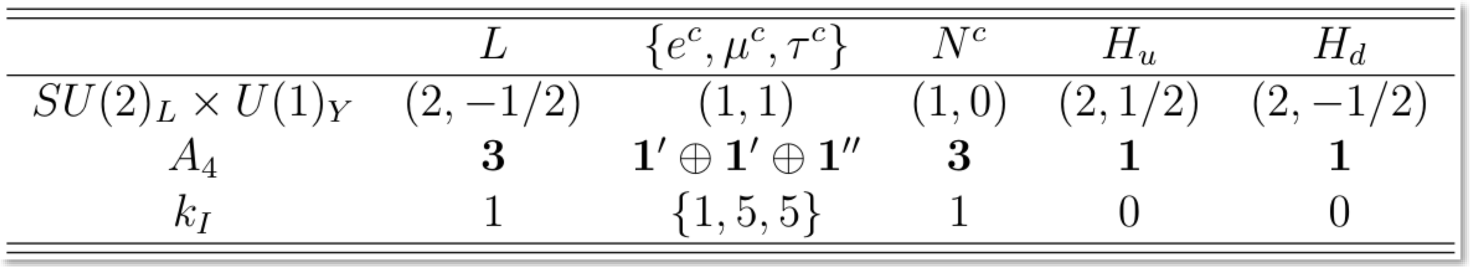
[other scenarios of modular inflation, see Gunji, Ishiwata, Yoshida, 2208.10086; Abe, Higaki, Kaneko, Kobayashi, Otsuka, 2303.02947, Casas, Ibanez, 2409.15823, Kallosh,Linde, 2408.05203;…]

Post inflation: reheating

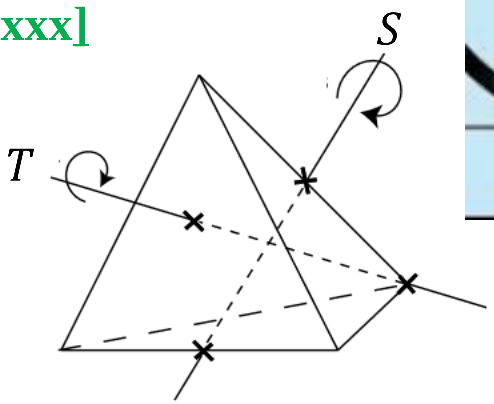
The Universe is reheated through the inflaton decays into SM particles. When modulus τ is the inflaton, its couplings with SM fields are determined by modular symmetry. Generally it has the largest coupling with RH neutrinos.



- modular model based on $\Gamma_3 \cong A_4$ [Ding, Jiang, Xu, Zhao, 2411.xxxxx]



	L	$\{e^c, \mu^c, \tau^c\}$	N^c	H_u	H_d
$SU(2)_L \times U(1)_Y$	$(2, -1/2)$	$(1, 1)$	$(1, 0)$	$(2, 1/2)$	$(2, -1/2)$
A_4	$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}''$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	$\{1, 5, 5\}$	1	0	0



- Modular invariant interactions with leptons

Charged leptons:

$$\mathcal{W}_\ell = y_1 e^c \left(LY_3^{(2)}(\tau)\right)_{1''} H_d + y_2 \mu^c \left(LY_{3I}^{(6)}(\tau)\right)_{1''} H_d + y_3 \mu^c \left(LY_{3II}^{(6)}(\tau)\right)_{1''} H_d$$
$$+ y_4\, \tau^c \left(LY_{3I}^{(6)}(\tau)\right)_{1'} H_d + y_5 \tau^c \left(LY_{3II}^{(6)}(\tau)\right)_{1'} H_d$$

Neutrinos:

$$\boxed{\mathcal{W}_\nu = g_1 \left((N^c L)_{3_S} Y_3^{(2)}(\tau)\right)_1 H_u + g_2 \left((N^c L)_{3_A} Y_3^{(2)}(\tau)\right)_1 H_u + \Lambda \left((N^c N^c)_{3_S} Y_3^{(2)}(\tau)\right)_1}$$

The interactions with RH neutrinos are strong constrained by modular symmetry

$$\mathcal{Y}_D = \begin{pmatrix} 2g_1 Y_{3,1}^{(2)}(\tau) & -g_1 Y_{\mathbf{3,3}}^{(2)}(\tau) - g_2 Y_{3,3}^{(2)}(\tau) & -g_1 Y_{3,2}^{(2)}(\tau) + g_2 Y_{3,2}^{(2)}(\tau) \\ -g_1 Y_{3,3}^{(2)}(\tau) + g_2 Y_{3,3}^{(2)}(\tau) & 2g_1 Y_{3,2}^{(2)}(\tau) & -g_1 Y_{3,1}^{(2)}(\tau) - g_2 Y_{3,1}^{(2)}(\tau) \\ -g_1 Y_{\mathbf{3},2}^{(2)}(\tau) - g_2 Y_{3,2}^{(2)}(\tau) & -g_1 Y_{3,1}^{(2)}(\tau) + g_2 Y_{3,1}^{(2)}(\tau) & 2g_1 Y_{3,3}^{(2)}(\tau) \end{pmatrix}$$

$$\mathcal{Y}_N = \begin{pmatrix} 2Y_{3,1}^{(2)}(\tau) & -Y_{\mathbf{3,3}}^{(2)}(\tau) & -Y_{3,2}^{(2)}(\tau) \\ -Y_{3,3}^{(2)}(\tau) & 2Y_{3,2}^{(2)}(\tau) & -Y_{3,1}^{(2)}(\tau) \\ -Y_{\mathbf{3},2}^{(2)}(\tau) & -Y_{3,1}^{(2)}(\tau) & 2Y_{3,3}^{(2)}(\tau) \end{pmatrix}$$

- At the global minimum $\tau = \tau_0 = 0.485 + 0.875i$ ⇒lepton masses and mixing

$$\left.\begin{matrix} \sin^2\theta_{12} = 0.307,\ \ \sin^2\theta_{13} = 0.02224,\ \ \sin^2\theta_{23} = 0.454, \delta_{CP} = 1.145\pi,\ \alpha_{21} = 1.062\pi, \\ \alpha_{31} = 1.729\pi, m_1 = 25.725\ \text{meV}, m_2 = 27.127\ \text{meV}\ , m_3 = 56.274\ \text{meV}\ , m_{\beta\beta} = 9.615\ \text{meV} \end{matrix}\right\} \text{within } 1\sigma$$

↪Quasi-degenerate heavy neutrino masses: $(M_1, M_2, M_3) = (1.37, 1.45, 2.82)\Lambda$, seesaw scale Λ is free

↪ CP violation source: complex couplings y_3, y_5, g_2

- Quantum fluctuations around global minimum $\tau = \tau_0 + \delta\tau \Rightarrow$ reheating

Expand the modular invariant interactions around τ_0:

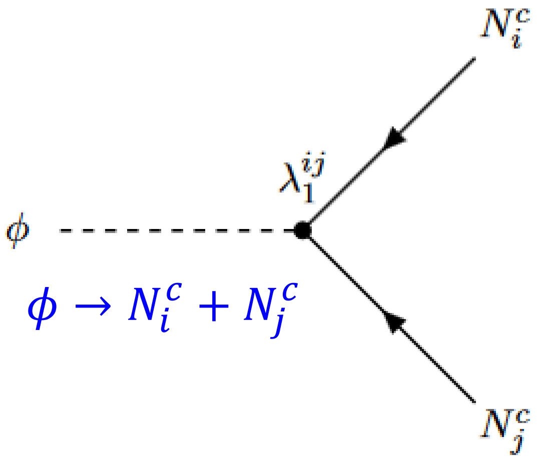
Inflaton-(s)neutrino-(s)neutrino interaction:

$$\mathcal{L} = \frac{\Lambda}{2M_{\rm pl}}\lambda_1^{ij}\phi N_i^c N_j^c + \frac{\Lambda^2}{2M_{\rm pl}}\lambda_2^{ij}\widetilde{N}_i^{c*}\widetilde{N}_j^c + \text{h.c.}$$

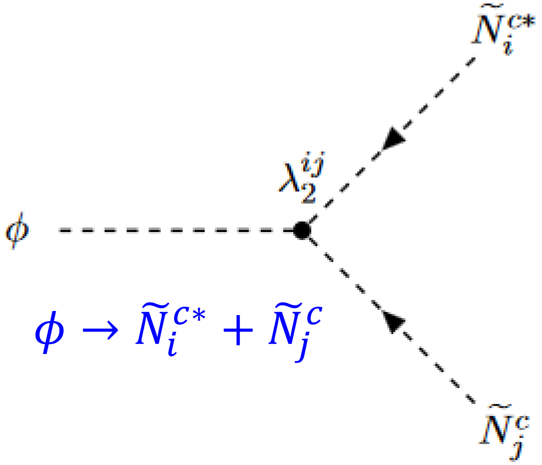
$$\lambda_1^{ij} = \frac{d\mathcal{Y}_N^{ij}(\tau)}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0}$$

$$\lambda_2^{ij} = \frac{d\left(\mathcal{Y}_N^T(\tau)\times\mathcal{Y}_N^*(\tau)\right)^{ij}}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0}$$

2-body decays:



$\phi \to N_i^c + N_j^c$



$\phi \to \widetilde{N}_i^{c*} + \widetilde{N}_j^c$

$$\Gamma(\phi \to N_i^c N_j^c) = \frac{m_\phi \Lambda^2}{8(1+\delta_{ij})\pi M_{\rm pl}^2}\left[|\lambda_1^{ij}|^2\left(1-\frac{M_i^2+M_j^2}{m_\phi^2}\right) - 2{\rm Re}[(\lambda_1^{ij})^2]\frac{M_iM_j}{m_\phi^2}\right] \times \sqrt{\left(1-\frac{(M_i-M_j)^2}{m_\phi^2}\right)\left(1-\frac{(M_i+M_j)^2}{m_\phi^2}\right)}$$

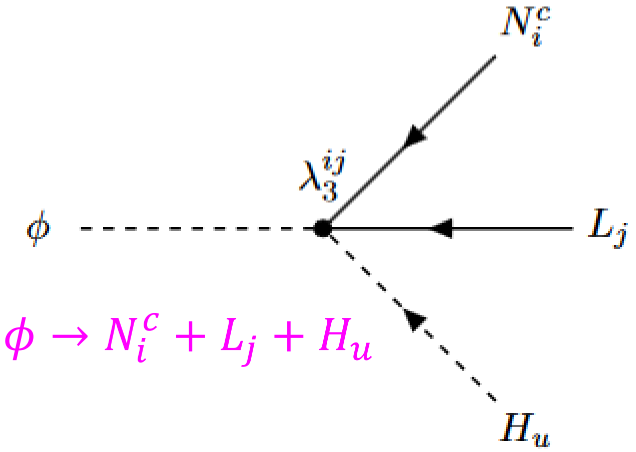
$$\Gamma(\phi \to \tilde{N}_i^c \tilde{N}_j^{c*}) = \left|\frac{\Lambda^2\lambda_2^{ij}}{M_{\rm pl}}\right|^2 \frac{1}{16\pi m_\phi}\sqrt{\left(1-\frac{(M_i-M_j)^2}{m_\phi^2}\right)\left(1-\frac{(M_i+M_j)^2}{m_\phi^2}\right)}$$

Inflaton-lepton-neutrino-Higgs interactions:

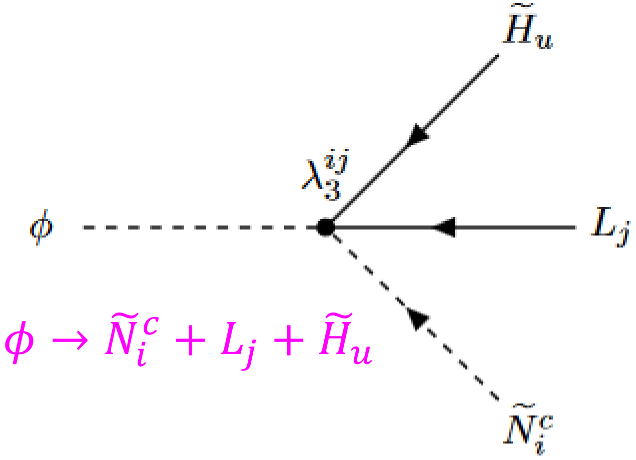
$$\lambda_3^{ij} = \frac{d\mathcal{Y}_D^{ij}(\tau)}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0} \qquad \lambda_4^{ij} = \frac{d\left(\mathcal{Y}_N^\dagger(\tau)\times\mathcal{Y}_D(\tau)\right)^{ij}}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0}$$

$$\mathcal{L} = \frac{1}{M_{\rm pl}}\lambda_3^{ij}\phi N_i^c\left(L_j\cdot H_u\right) + \frac{1}{M_{\rm pl}}\lambda_3^{ij}\phi\,\widetilde{N}_i^c\,\left(L_j\cdot\widetilde{H}_u\right) + \frac{1}{M_{\rm pl}}\lambda_3^{ij}\phi\,N_i^c\,\left(\tilde{L}_j\cdot\widetilde{H}_u\right) + \frac{\Lambda}{M_{\rm pl}}\lambda_4^{ij}\phi\,\widetilde{N}_i^{c*}\,\left(\tilde{L}_j\cdot H_u\right) + \text{h. c.}$$

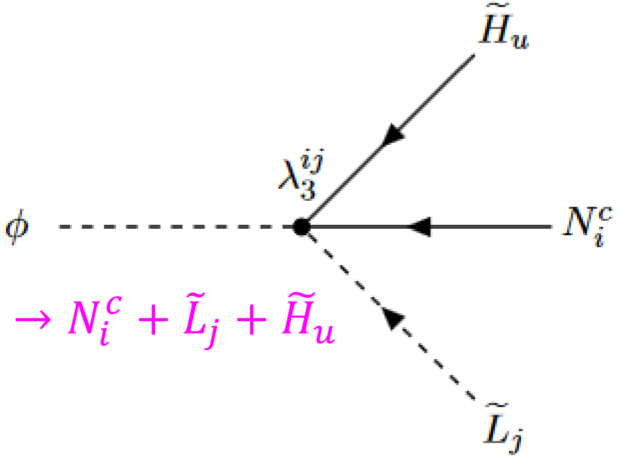
3-body decays:



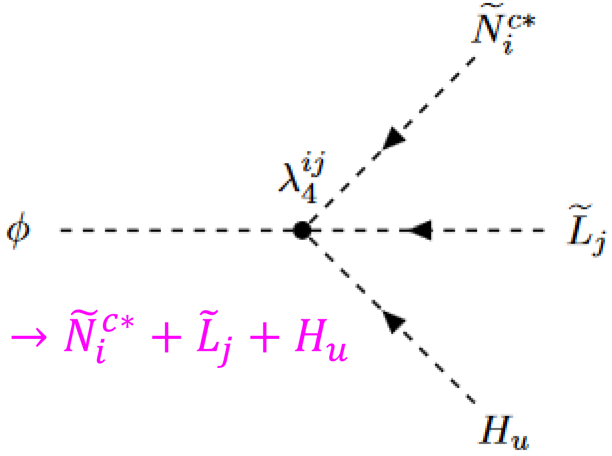
$\phi \to N_i^c + L_j + H_u$



$\phi \to \widetilde{N}_i^c + L_j + \widetilde{H}_u$



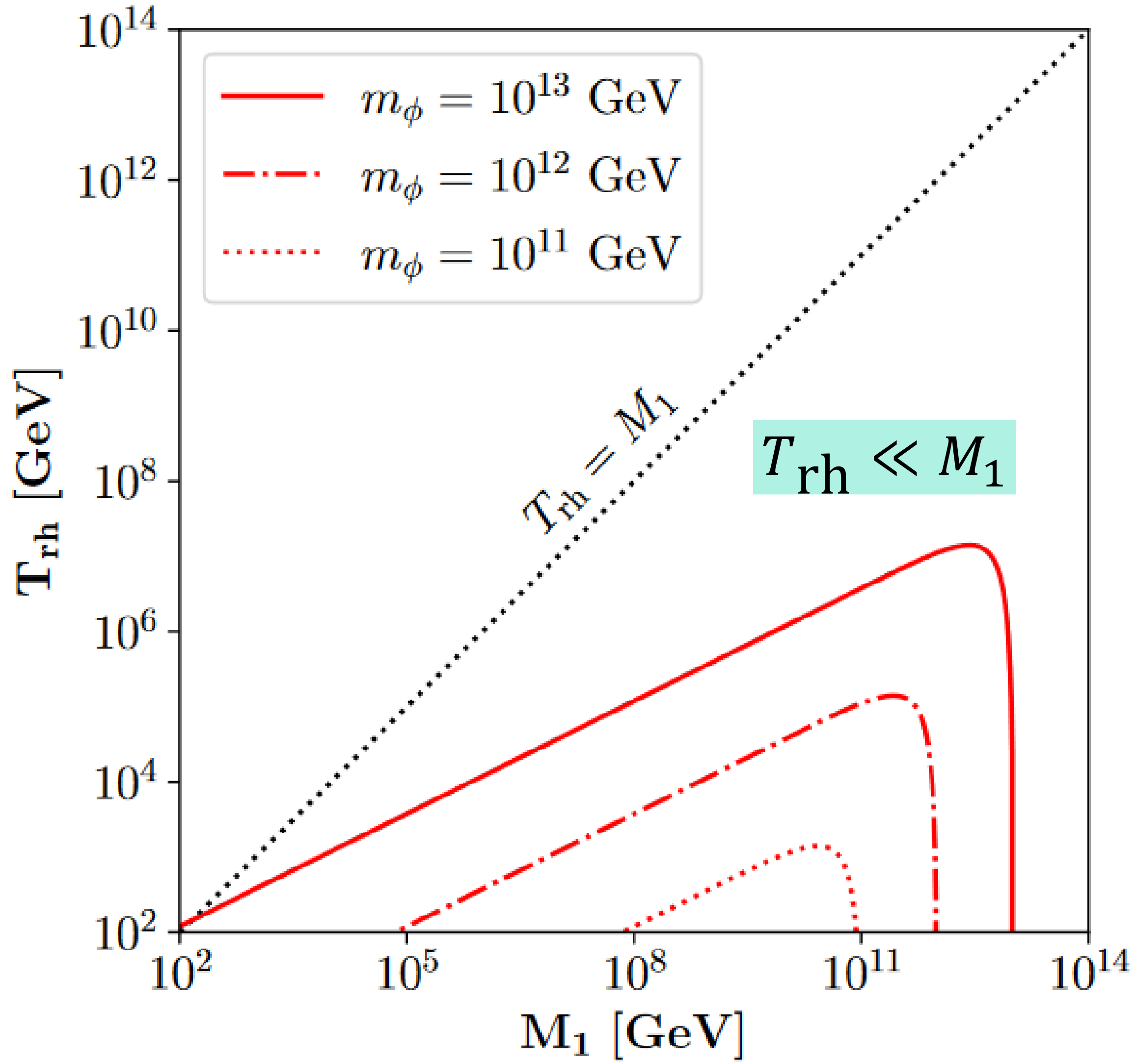
$\phi \to N_i^c + \tilde{L}_j + \widetilde{H}_u$



$\phi \to \widetilde{N}_i^{c*} + \tilde{L}_j + H_u$

- Reheating temperature

$$\boxed{T_{\mathrm{rh}} = \sqrt{\frac{2}{\pi}\left(\frac{10}{g_*}\right)^{1/4}}\sqrt{M_{\mathrm{pl}}\Gamma_\phi}}$$



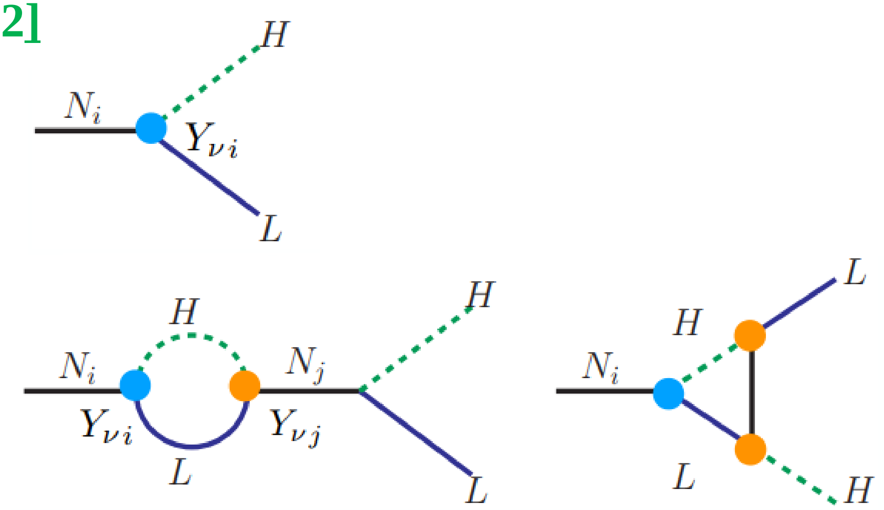
The thermal production of right-handed neutrinos is Boltzmann suppressed.

Baryon asymmetry from non-thermal leptogenesis

- The first two RHNs are quasi-degenerate $M_1 : M_2 : M_3 = 1 : 1.05 : 2.05$, the CP asymmetry of RHNs decays is **enhanced** [Pilaftsis,Underwood,hep-ph/0309342]

$$\epsilon_i = \frac{\Gamma(N_i \to L + H_u) - \Gamma(N_i \to \bar{L} + \bar{H}_u)}{\Gamma(N_i \to L + H_u) - \Gamma(N_i \to \bar{L} + \bar{H}_u)}$$

$$= \frac{\mathrm{Im}\left\{\left(\mathcal{Y}_D \mathcal{Y}_D^\dagger\right)_{ij}^2\right\}}{\left(\mathcal{Y}_D \mathcal{Y}_D^\dagger\right)_{ii}\left(\mathcal{Y}_D \mathcal{Y}_D^\dagger\right)_{jj}} \frac{\left(M_i^2 - M_j^2\right) M_i \Gamma_j}{\left(M_i^2 - M_j^2\right)^2 + M_i^2 \Gamma_j^2}$$

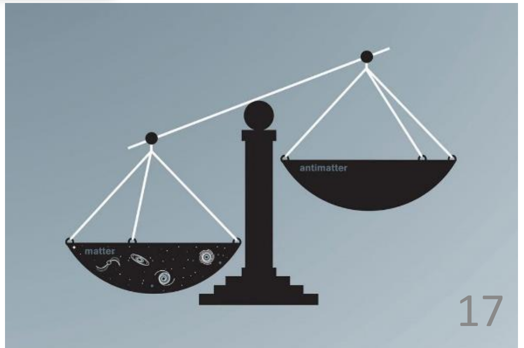


- The lepton asymmetry is converted to the Baryon asymmetry via SM sphalerons

$$\boxed{Y_B \equiv \frac{n_B}{s} \simeq c_{\rm sph} \frac{3}{4} \frac{T_{\rm rh}}{m_\phi} \sum_i \epsilon_i \times \left[2\mathrm{Br}(\phi \to \overset{(\sim)}{N}_i + \overset{(\sim)}{N}_i) + \mathrm{Br}(\phi \to \overset{(\sim)}{N}_i + \mathrm{others}) \right]}$$

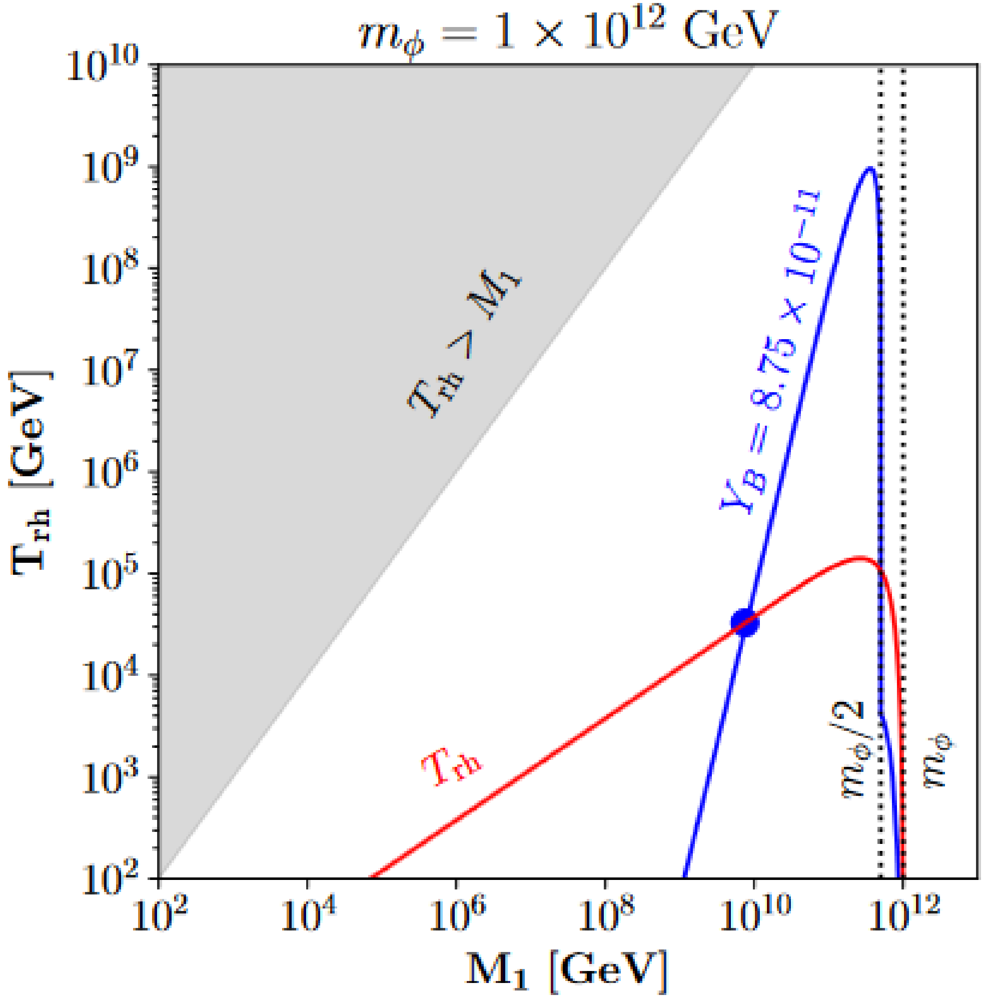
$c_{\rm sph} = -8/23$ in minimal SUSY standard model

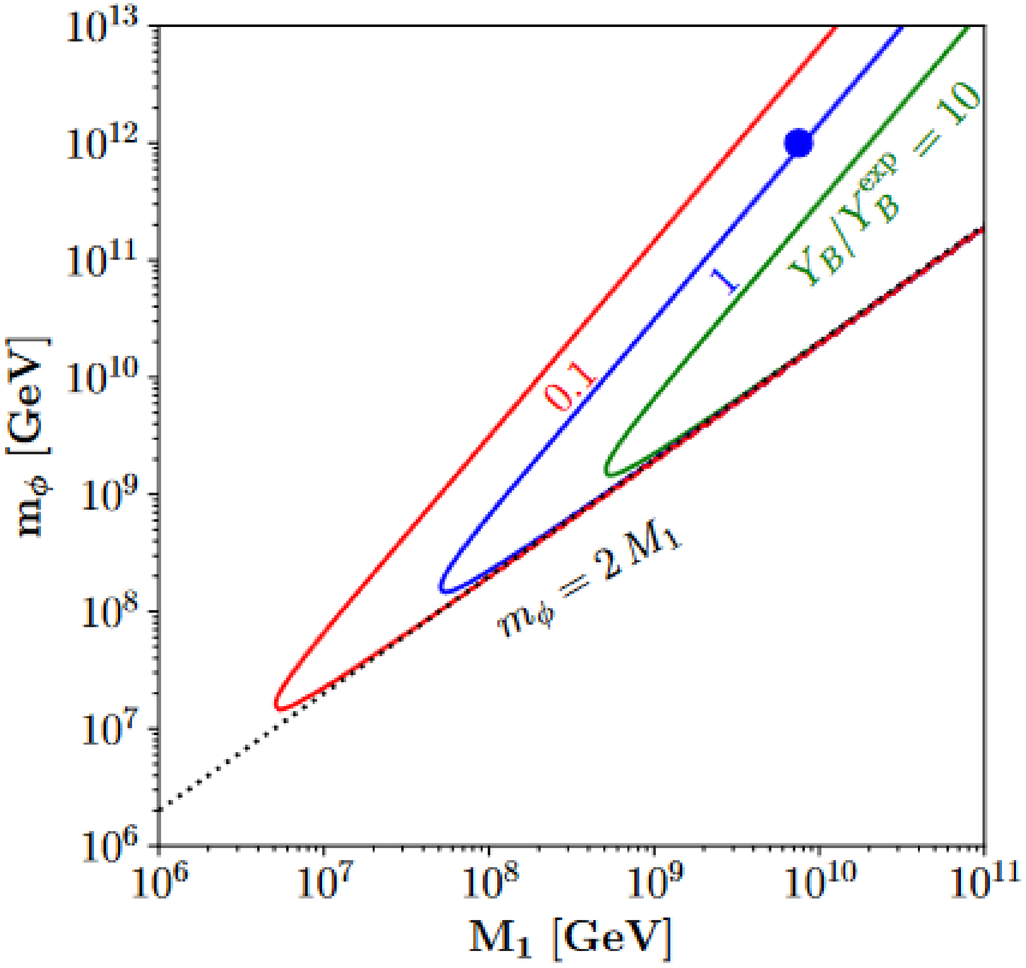
[Asaka, Hamaguchi,Kawasaki,Yanagida, hep-ph/9906366]



➢ Numerical results: Baryon asymmetry

[Ding, Jiang, Xu, Zhao, 2411.xxxxx]



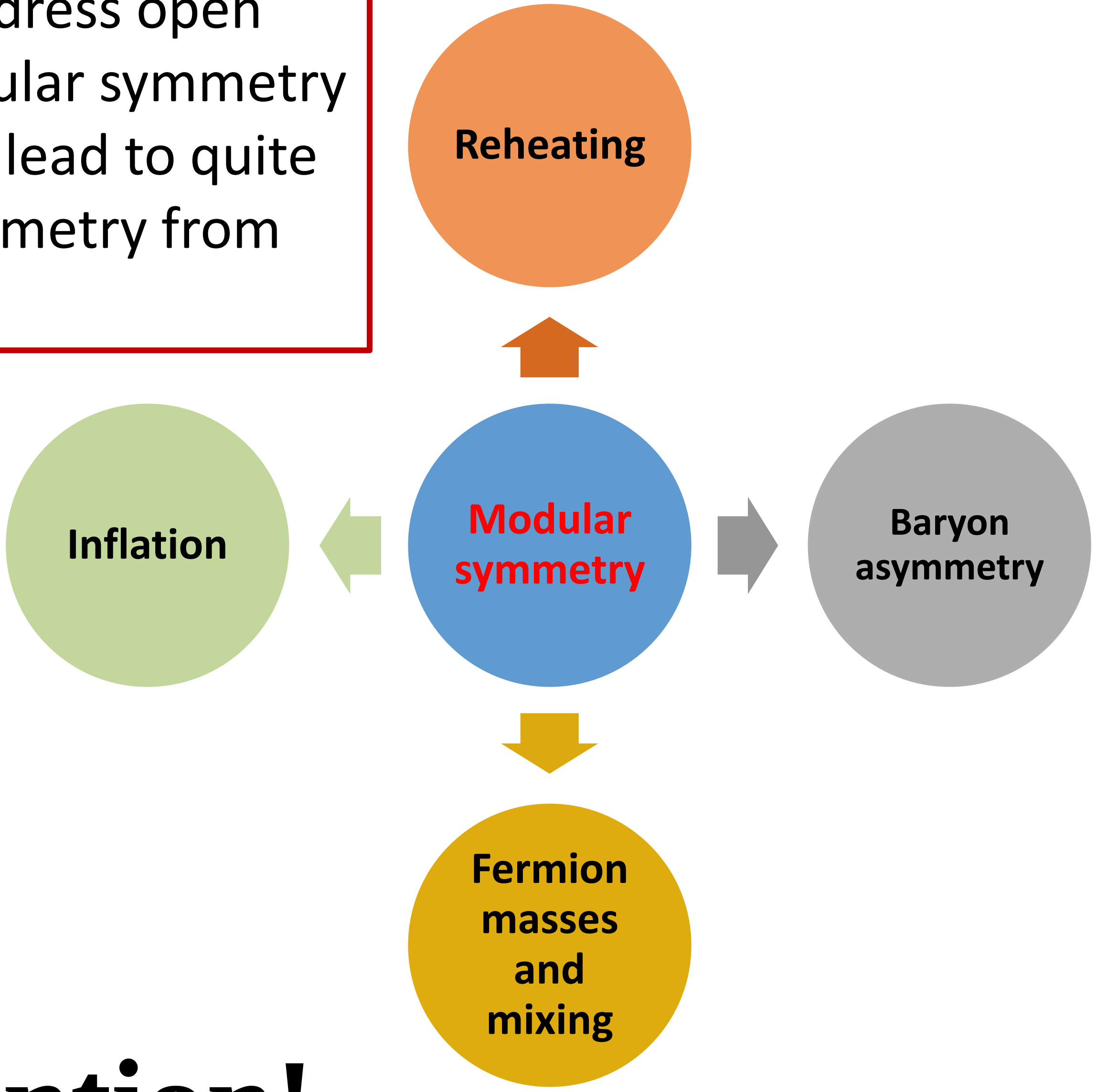


The observed baryon asymmetry can be produced the RH neutrino mass $M_1 \simeq 10^{10}$GeV

Summary

Modular symmetry is a promising approach to address open problems in particle physics and cosmology. Modular symmetry strongly constrains the interactions so that it can lead to quite predictive scenarios. One can probe modular symmetry from multiple perspectives.


Kill four birds with one stone!



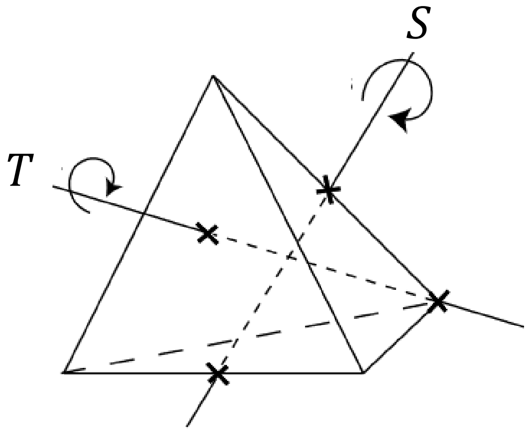
Thank you for your attention!

Backup

A_4 modular symmetry

- $\boldsymbol{A_4} \cong \boldsymbol{\Gamma_3}$ is the symmetry group of a tetrahedron, it is the smallest non-abelian finite with 3-dim irreducible representation.

$$A_4 : S^2 = T^3 = (ST)^3 = 1$$



(promising for 3 generations!)

A_4 has only 4 irreducible inequivalent representations: **1, 1', 1'',3**

singlets $$\begin{cases} \mathbf{1}: & S=1, \quad T=1 \\ \mathbf{1}': & S=1, \quad T=\omega \\ \mathbf{1}'': & S=1, \quad T=\omega^2 \end{cases}$$

triplets $$\mathbf{3}: \; S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

- Tensor product:

For two triplets $(\alpha_1, \alpha_2, \alpha_3) \sim 3, \quad (\beta_1, \beta_2, \beta_3) \sim 3$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{\mathbf{3}} = (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)_{\mathbf{1}} \oplus (\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)_{\mathbf{1}'} \oplus (\alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1)_{\mathbf{1}''}$$

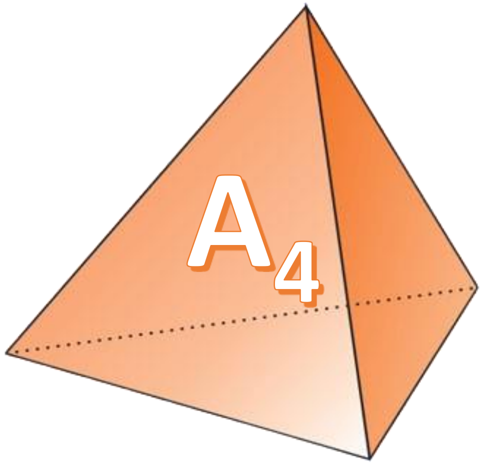
$$\oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \end{pmatrix}_{\mathbf{3}_S} \oplus \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}_{\mathbf{3}_A} .$$

Modular forms of level 3

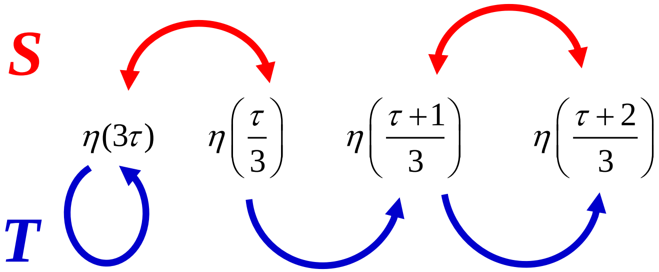
- Three weight 2 and level 3 modular forms transforming as a triplet 3 of A_4

$$Y(\tau)=\left(Y_1(\tau),\quad Y_2(\tau),\quad Y_3(\tau)\right)^T$$

A_4 triplet **[Feruglio, 1706.08749]**



$$
\boxed{
\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right]\\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]\\
Y_3(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]
\end{aligned}
}
$$



Dedekind eta function: $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,,\;\; q\equiv e^{2\pi i\tau}$

$$Y_1(\tau)=1+12q+36q^2+12q^3+84q^4+\ldots,$$
$$Y_2(\tau)=-6q^{1/3}(1+7q+8q^2+18q^3+14q^4+\ldots),$$
$$Y_3(\tau)=-18q^{2/3}(1+2q+5q^2+4q^3+8q^4+\ldots)$$

Tensor products of $Y_{1,2,3}$ generate higher weight modular forms

➢ Klein j-invariant

The Klein j-invariant function is a modular form of weight zero, defined in terms of Dedekind eta function and Eisenstein series as follows:

$$j(\tau) \equiv \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\eta^{24}(\tau)} = \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\Delta(\tau)}, \quad \Delta(\tau) \equiv \eta^{24}(\tau)$$

The q-expansion of j-function is given by

$$\begin{aligned} j(\tau) = 744 &+ \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 \\ &+ 333202640600q^5 + 4252023300096q^6 + 44656994071935q^7 + \mathcal{O}(q^8) \end{aligned}$$

- The best fit values of coupling constants

$$\langle\tau\rangle = 0.485 + 0.875i \equiv \tau_0,\ \ \frac{y_2}{y_1} = 1582.5,\ \ \frac{y_3}{y_1} = 554.4e^{3.712i},\ \ \frac{y_4}{y_1} = 60644.8,\ \ \frac{y_5}{y_1} = 38717.9e^{3.195i},$$

$$\frac{g_2}{g_1} = 0.246,\ \ y_1 v_d = 0.250\ \mathrm{MeV},\ \ \frac{(g_1 v_u)^2}{\Lambda} = 19.967\ \mathrm{meV}$$

- The couplings between inflaton and RH neutrinos

$$\lambda_1^{ij} = \begin{pmatrix} -2.078 - 0.980i & 1.420i & 1.017 \\ 1.420i & 0.987 - 1.033i & 1.079i \\ 1.017 & 1.080 & -1.091 - 2.013i \end{pmatrix}$$

$$\lambda_2^{ij} = \begin{pmatrix} -5.701 & -0.002 + 0.106i & -0.184 - 4.258i \\ -0.002 - 0.106i & 2.856 & 1.478 - 0.097i \\ -0.184 + 4.258i & 1.478 + 0.097i & -6.149 \end{pmatrix}$$

$$\lambda_3^{ij} = \begin{pmatrix} -0.980 + 2.078i & 1.557 + 0.248i & -0.006 + 0.753i \\ 1.282 - 0.248i & -1.033 - 0.988i & -0.831 - 0.001i \\ -0.006 - 1.282i & 1.329 + 0.001i & 2.013 - 1.090i \end{pmatrix} \qquad \lambda_4^{ij} = \begin{pmatrix} 5.120i & 0.802 & 3.917i \\ 0.240 & -2.362i & 1.830 \\ -5.375i & 1.832 & -6.156i \end{pmatrix}$$