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UNIVERSITY" OF THE MINISTRY OF SCIENCE AND
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A New Method for Determining the Transverse Vibration Energy of Fission Prefragments

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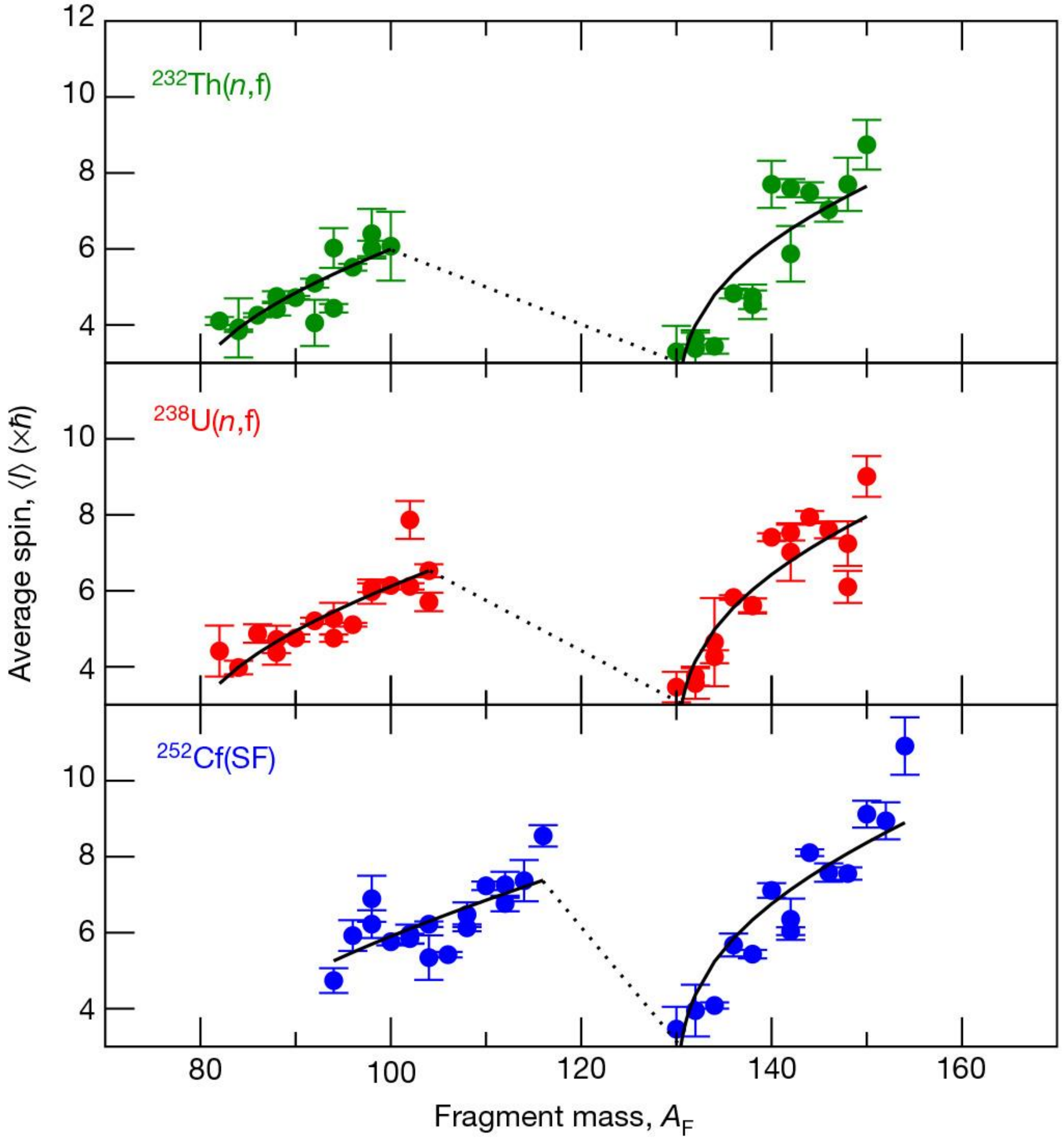


Dependences of spins on mass numbers for the fission fragments 232Th, 238U and 252Cf.

1. Wilson, J.N., Thisse, D., Lebois, M. *et al.* Angular momentum generation in nuclear fission. *Nature* **590**, 566–570 (2021).

Description of the nuclear double fission process within the framework of the ‘cold’ fissile nucleus model

$$\Psi_{K}^{JM} = b_0 \Psi_{0_K}^{JM}(\beta_\lambda) + \sum_{i \neq 0} b_i \Psi_{i_K}^{JM},$$

where the function Ψ_{ik}^{JM} describes the quasiparticle excited state of the nucleus, and $\Psi_{0k}^{JM}(\beta_\lambda)$ describes the collective deformation motion of the nucleus with excitation energy $|B_n|$.

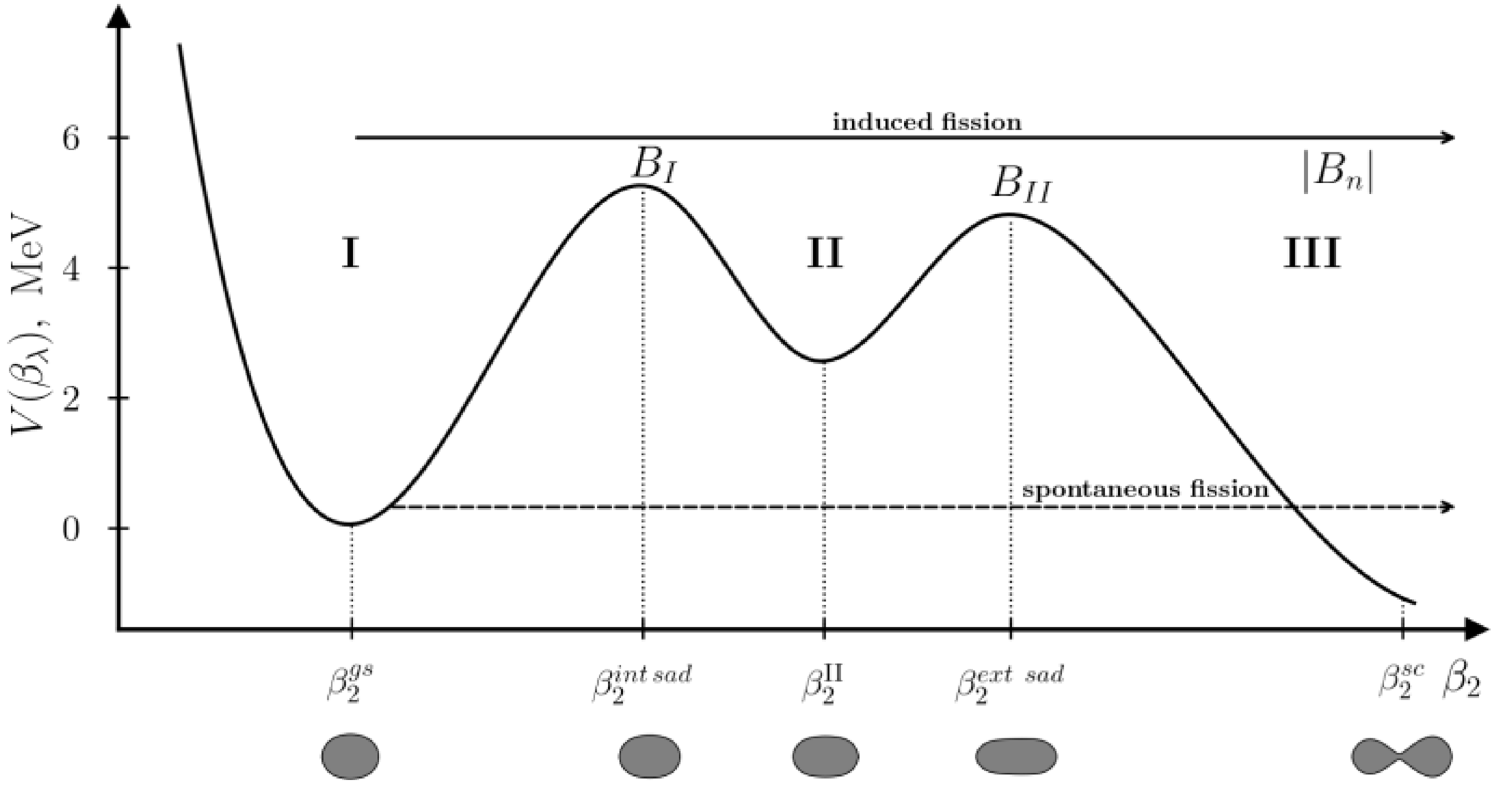
1. Kadmensky S.G. // Phys. Atom. Nucl. 2002. V. 65. P. 1390.

2. E. P. Wigner, Ann. Math. 62, 548 (1955).

3. E. P. Wigner, Ann. Math. 65, 203 (1957).

4. E. P. Wigner, Ann. Math. 67, 325 (1958)

Description of the nuclear double fission process within the framework of the ‘cold’ fissile nucleus model



Principal scheme of the potential V depending on the quadrupole deformation of the nucleus β_2. Region I corresponds to the ground state of the nucleus with β_2^{gs}. II - isomeric states, and III - the out-of-barrier region where the nucleus decays into fission fragments.

1. Kadmensky S.G. // Phys. Atom. Nucl. 2002. V. 65. P. 1390.
2. Kadmensky S.G., Titova L.V., *Physics of Atomic Nuclei*, 2009, **72**(10), 1738–1744
3. G. Danilyan, *Phys. At. Nucl*. **82**, 235 (2019);
4. A. Gagarski et al., *Phys. Rev. C* **93**, 054619 (2016)

Description of the nuclear double fission process within the framework of the ‘cold’ fissile nucleus model

$$C_{b(w)} = I_{b(w)}\hbar\omega_{b(w)} \coth\left(\frac{\hbar\omega_{b(w)}}{2T}\right) \rightarrow \begin{cases} 2I_{b(w)}T, & T \gg \hbar\omega_{b(w)} \\ I_{b(w)}\hbar\omega_{b(w)}, & T \ll \hbar\omega_{b(w)} \end{cases},$$

where $C_{b(w)}$, $I_{b(w)}$ and $\hbar\omega_{b(w)}$ are coefficient, moment of inertia and energy of pointing oscillations.

1. J.R. Nix, W.J. Swiatecki, Nucl. Phys. **71,** 1 (1965).
2. J. Randrup and R. Vogt, Phys. Rev. Lett. 127, 062502 (2021).

Spin distribution of double fission fragments

$$P\left(J_{k_x}, J_{k_y}\right) \equiv P\left(J_{k_x}\right) P\left(J_{k_y}\right) = \frac{1}{\pi I_k \hbar \omega_k} \exp\left[-\frac{J_{k_x}^2 + J_{k_y}^2}{I_k \hbar \omega_k}\right]$$

where the index k = w, b corresponds to the type of oscillations (wriggling or bending), I_k is the moment of inertia of these oscillations, the frequencies ω_k of the oscillations are determined by the classical formulae $\omega_k = \sqrt{\frac{K_k}{M_k}}$.

1. Nix J.R. and Swiatecki W.J. // Nucl. Phys. A. 1965. V. 71. P. 1.
2. S. G. Kadmensky et al., Phys. At. Nucl. 87, 359 (2024).

3. D.E. Lyubashevsky et al., arXiv preprint arXiv:2412.04410 (2024).

Spin distribution of double fission fragments

$$I_w = \frac{(I_1 + I_2) I_0}{I} \qquad I_b = \mu R^2 I_H \Big/ \left(\mu R^2 + I_H\right) \qquad I_0 = \mu\left(R_1 + R_2 + d\right)^2$$

$$I = I_0 + I_1 + I_2$$

$$J_{i\,x(y)} = \frac{I_i}{I_1 + I_2} J_{w_{x(y)}} + (-1)^{i+1} J_{b_{x(y)}}$$

$$I_{1,2} \equiv I_{i,\mathrm{rigit}} = \frac{M_i}{5} \sum R_i^2$$

$$J_{w_{x(y)}} = J_{1x(y)} + J_{2x(y)}$$

$$R_i = r_0 A^{1/3} \left[1 - \beta_i^2 / 4\pi + \beta_i \sqrt{5/4\pi}\right]$$

$$\mu = M_1 M_2 \Big/ \left(M_1 + M_2\right)$$

1. J. Randrup and R. Vogt, Phys. Rev. Lett. 127, 062502 (2021).
2. R. Vogt and J. Randrup, Phys. Rev. C 103, 014610 (2021).
3. T. Shneidman, G. Adamian, N. Antonenko, S. Ivanova, R. Jolos, and W. Scheid, Phys. Rev. C 65, 064302 (2002)
4. D. E. Lyubashevsky *et al* 2025 *Chinese Phys. C* **49** 034104

Spin distribution of double fission fragments

$$J_{b_{x(y)}} = J_{1x(y)} - \frac{I_1}{I_1 + I_2} J_{w_{x(y)}} = \frac{I_2 J_{1x(y)} - I_1 J_{2x(y)}}{I_1 + I_2}$$

$$P(J_{w_{x(y)}}, J_{b_{x(y)}}) = P(J_{w_{x(y)}}) P(J_{b_{x(y)}})$$

$$P(J_{1x.}, J_{2x}, J_{1y}, J_{2y}) =$$

$$\frac{1}{\pi^2 I_w \hbar\omega_w I_b \hbar\omega_b} \exp\left[-\left\{ \frac{J_{\omega_x}^2 + J_{\omega_y}^2}{I_w \hbar\omega_w} + \frac{J_{b_x}^2 + J_{b_y}^2}{I_b \hbar\omega_b} \right\} \right] \left| \frac{\partial(J_{w_x}, J_{b_x}, J_{w_y}, J_{b_y}}{\partial(J_{1x}, J_{2x}, J_{1y}, J_{2y})} \right| =$$

$$= \frac{1}{\pi^2 I_w \hbar\omega_w I_b \hbar\omega_b} \exp\left[\begin{array}{l} -\frac{1}{I_w \hbar\omega_w} \left\{ (J_{1x} + J_{2x})^2 + (J_{1y} + J_{2y})^2 \right\} - \\ -\frac{1}{I_b \hbar\omega_b (I_1 + I_2)^2} \left\{ (I_2 J_{1x} - I_1 J_{2x})^2 + (I_2 J_{1y} - I_1 J_{2y})^2 \right\} \end{array} \right].$$

1. D.E. Lyubashevsky et al., arXiv preprint arXiv:2412.04410 (2024).

Spin distribution of double fission fragments

$$P(J_1, J_2, \varphi) = \frac{2\,J_1 J_2}{\pi I_w \hbar\omega_w I_b \hbar\omega_b} \exp\left[\begin{array}{c} -J_1^2(\alpha I_2^2 + \beta) - J_2^2(\alpha I_1^2 + \beta) + \\ + 2J_1 J_2 \cos\varphi(\alpha J_1 J_2 - \beta) \end{array}\right],$$

$$\alpha = \frac{1}{I_b \hbar\omega_b \left(I_1 + I_2\right)^2}, \beta = \frac{1}{I_w \hbar\omega_w}$$

1. D.E. Lyubashevsky et al., arXiv preprint arXiv:2412.04410 (2024).

Spin distribution of double fission fragments

$$P(J_i) = \frac{2J_i}{d_i} \exp\left[-\frac{J_i^2}{d_i} \right] \qquad d_i = \frac{I_i^2 I_w \hbar\omega_w}{(I_1 + I_2)^2} + I_b \hbar\omega_b$$

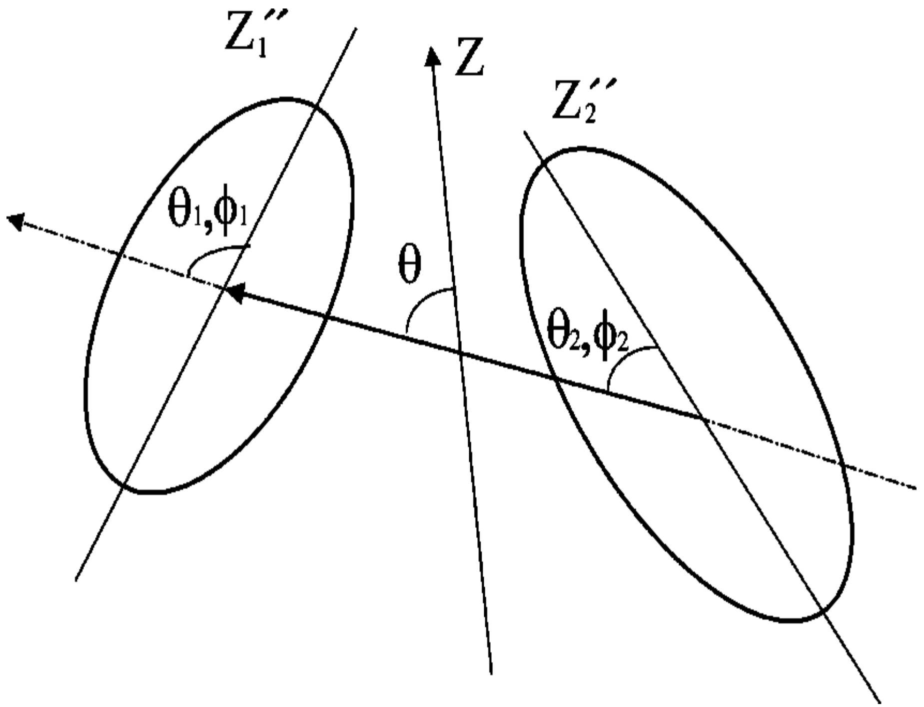
$$\overline{J}_i = \int_0^{\propto} P(J_i)\, J_i\, dJ_i = \int_0^{\propto} \frac{2J_i^2}{d_i} \exp\left[-\frac{J_i^2}{d_i} \right] dJ_i = \frac{1}{2}\sqrt{\pi d_i}$$

1. S. G. Kadmensky et al., Phys. At. Nucl. 87, 359 (2024).
2. D.E. Lyubashevsky et al., arXiv preprint arXiv:2412.04410 (2024).

Axially symmetric compound fissile systems

$$R = R_m \approx R_1\left(1 - \frac{\beta_1^2}{4\pi} + \sqrt{\frac{5}{4\pi}}\beta_1\right) + R_2\left(1 - \frac{\beta_2^2}{4\pi} + \sqrt{\frac{5}{4\pi}}\beta_2\right) + d$$

$$U(R,\beta_i,\ \Omega_i)=U_C(R,\beta_i\ ,\Omega_i)+U_N(R,\beta_i\ ,\Omega_i).$$



Schematic representation and definitions of various configuration coordinates of constituent fissile systems.

1. T. Shneidman, G. Adamian, N. Antonenko, S. Ivanova, R. Jolos, and W. Scheid, Phys. Rev. C 65, 064302 (2002);
2. G. G. Adamian et al., Int. J. Mod. Phys. E 5, 191 (1996)

Axially symmetric compound fissile systems

$$U_\epsilon = \frac{1}{2} C_\epsilon \epsilon^2 \qquad C_b = C_{11} + 2\left(\frac{\widetilde{R_1}}{\widetilde{R_2}}\right) C_{12} + \left(\frac{\tilde{R}_1}{\widetilde{R_2}}\right)^2 C_{22}$$

Schrödinger equation for bending - oscillations:

$$-\frac{\hbar^2}{2J_\epsilon}\frac{1}{\epsilon}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\epsilon\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right)\psi_n + \frac{1}{2}C_\epsilon \epsilon^2 \psi_n = E_n \psi_n$$

Its solutions:

$$\Psi\left(J_{b_x}, J_{b_y}\right) \equiv \Psi\left(J_{b_x}\right)\Psi\left(J_{b_y}\right) = \frac{1}{\pi I_b \hbar \omega_b} \exp\left[-\frac{J_{b_x}^2 + J_{b_y}^2}{I_b \hbar \omega_b}\right],$$

$$E_b = \hbar w_b\left(2n+1\right), \; n = 0,1,2,... \qquad w_b = \sqrt{C_b / I_b}$$

Axially symmetric compound fissile systems

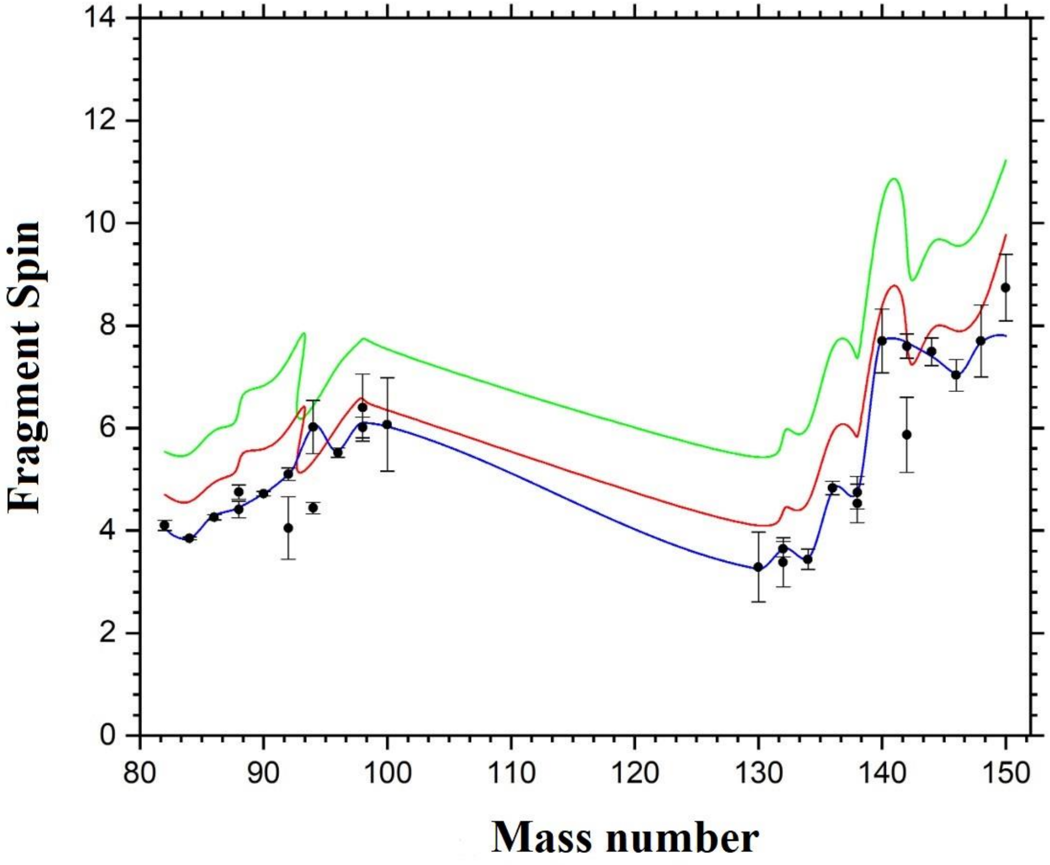
$$C_w = C_{11} - 2\frac{R_1}{R_2}C_{12} + \left(\frac{R_1}{R_2}\right)^2 C_{22}$$

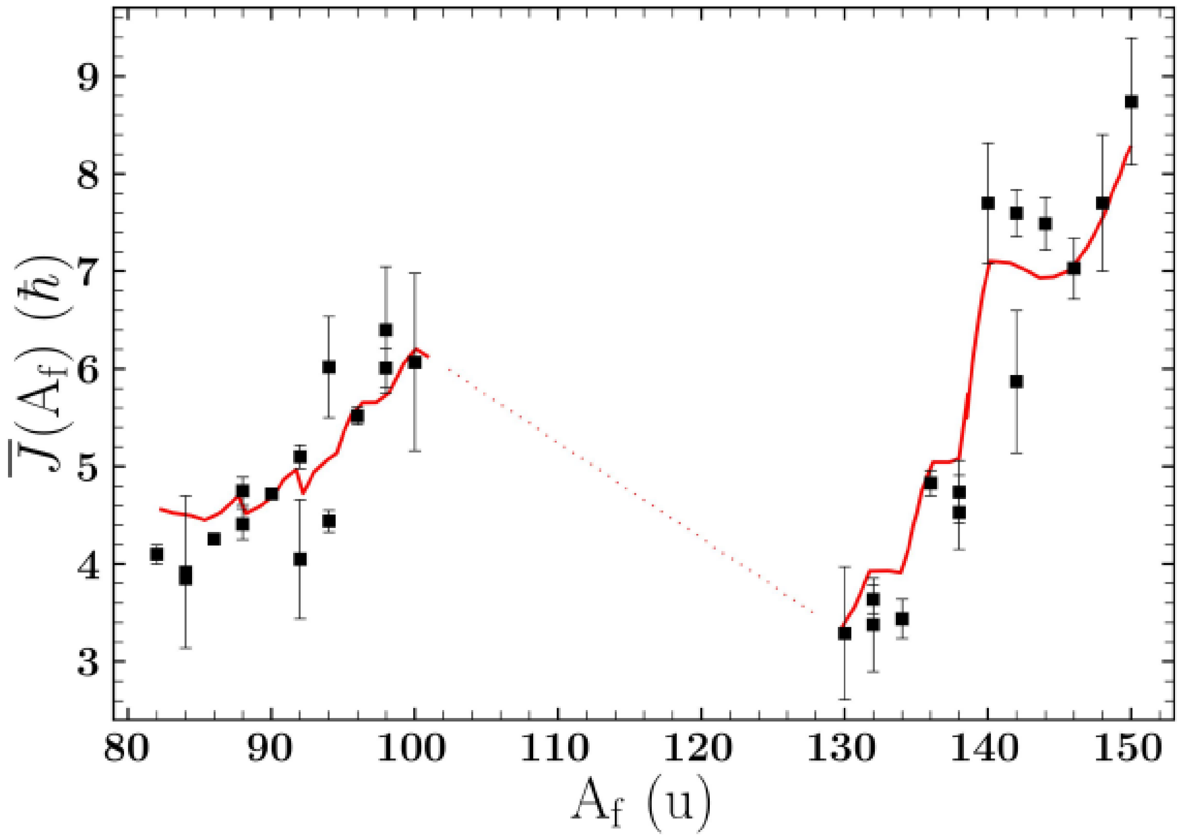
$$\Psi\left(J_{w_x}, J_{w_y}\right) \equiv \Psi\left(J_{w_x}\right)\Psi\left(J_{w_y}\right) = \frac{1}{\pi I_w \hbar\omega_w}\exp\left[-\frac{J_{w_x}^2 + J_{w_y}^2}{I_w \hbar\omega_w}\right],$$

$$E_w = \hbar w_w\left(2n+1\right),\ \ n = 0,1,2,\ldots$$

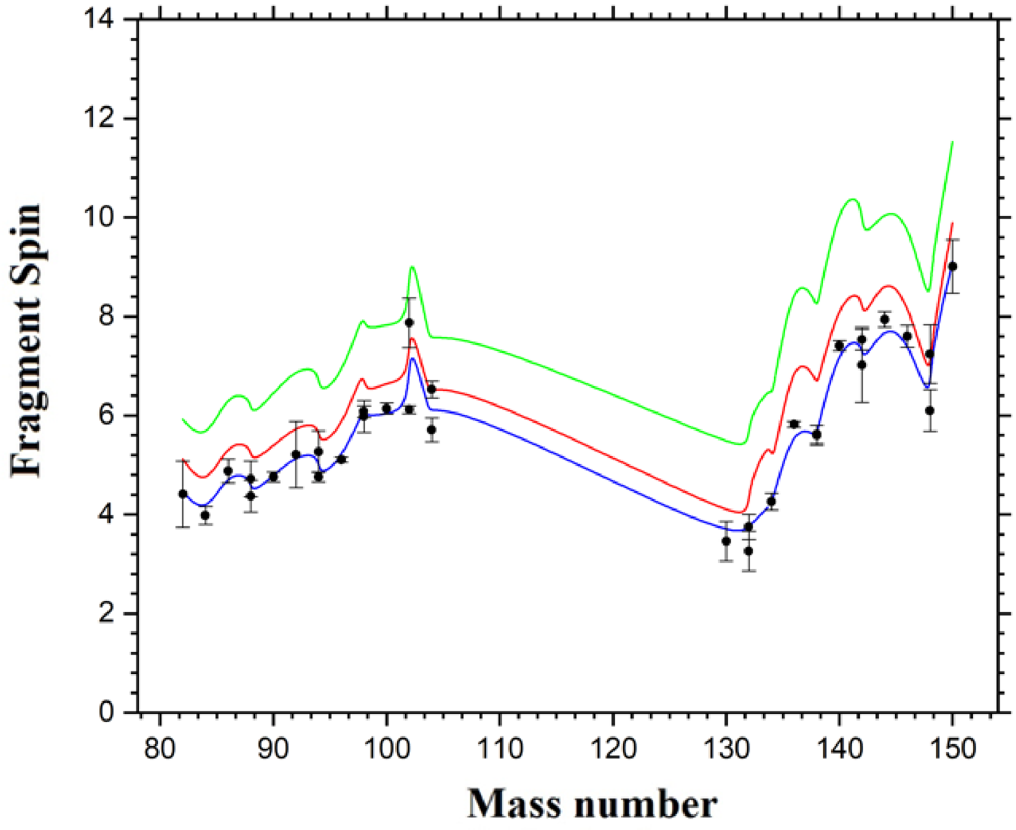
$$w_w = \sqrt{C_w/I_w}$$

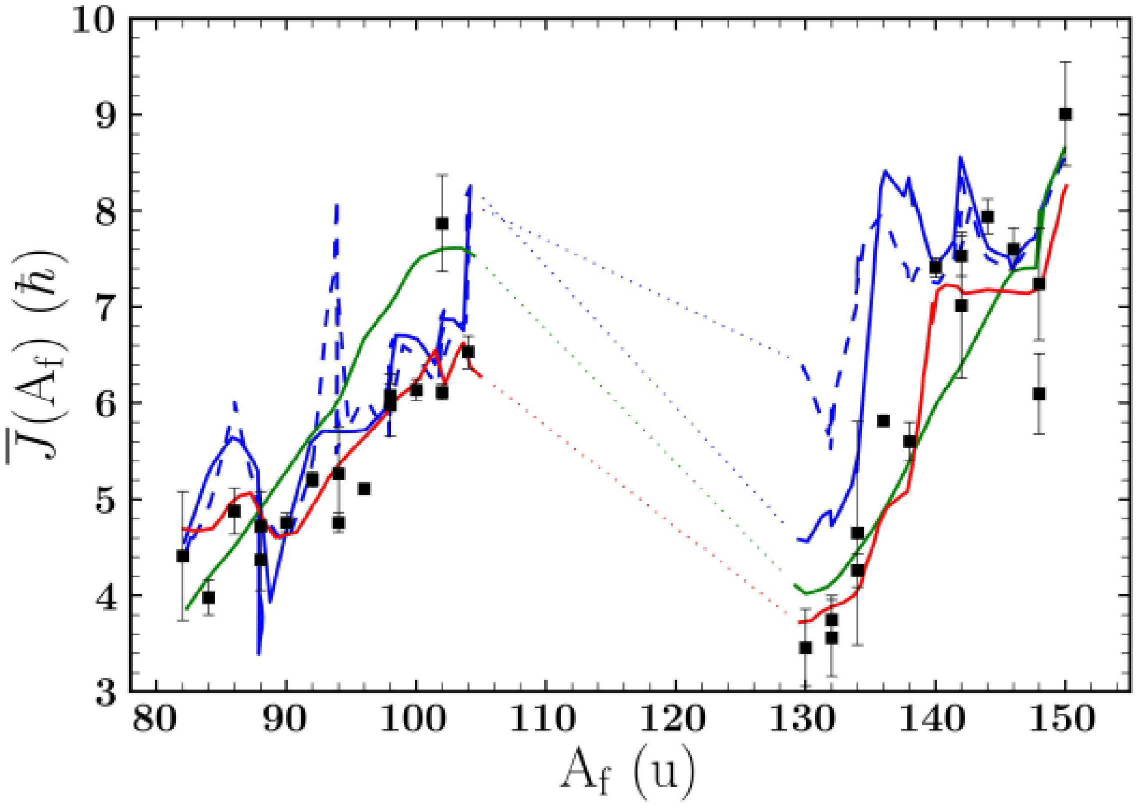
1. S. Misicu, A. Săndulescu, G. M. Ter-Akopian, W. Greiner, *Phys. Rev. C*, **60**, 034613



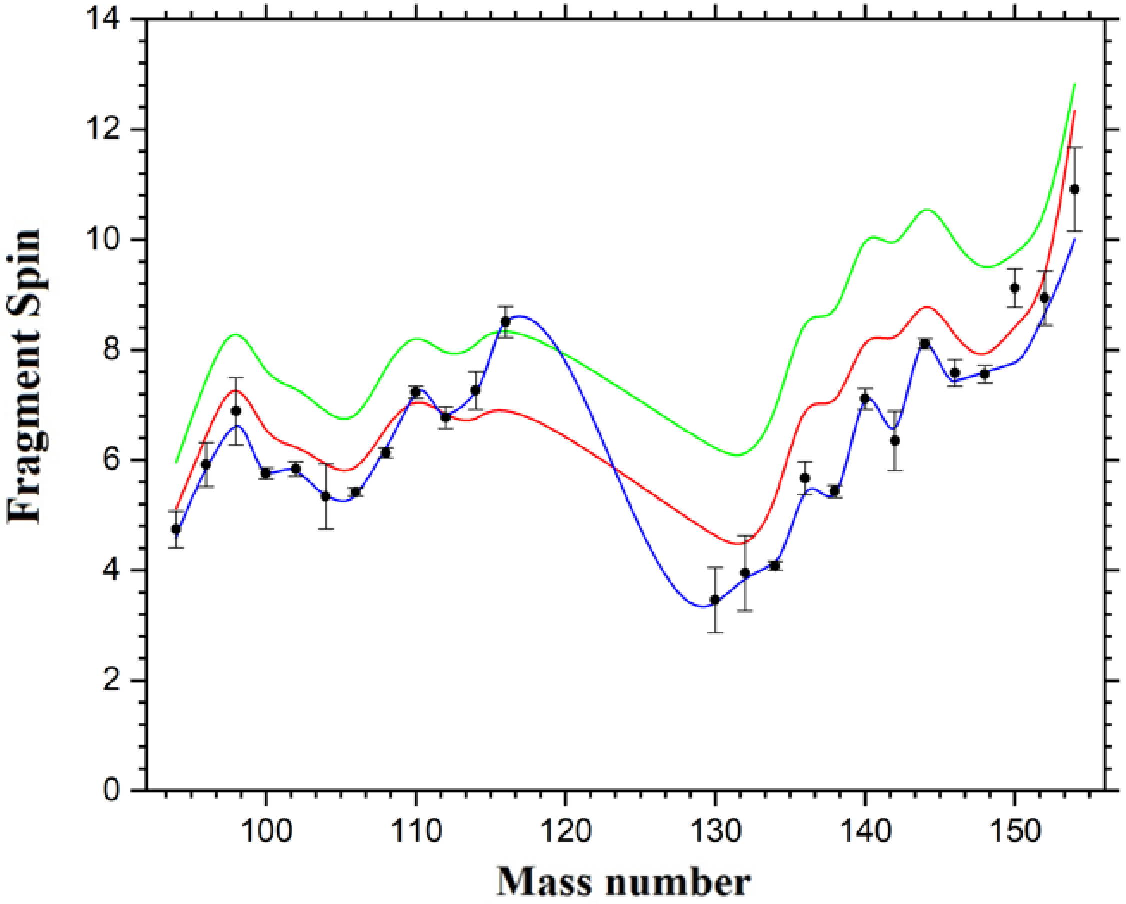


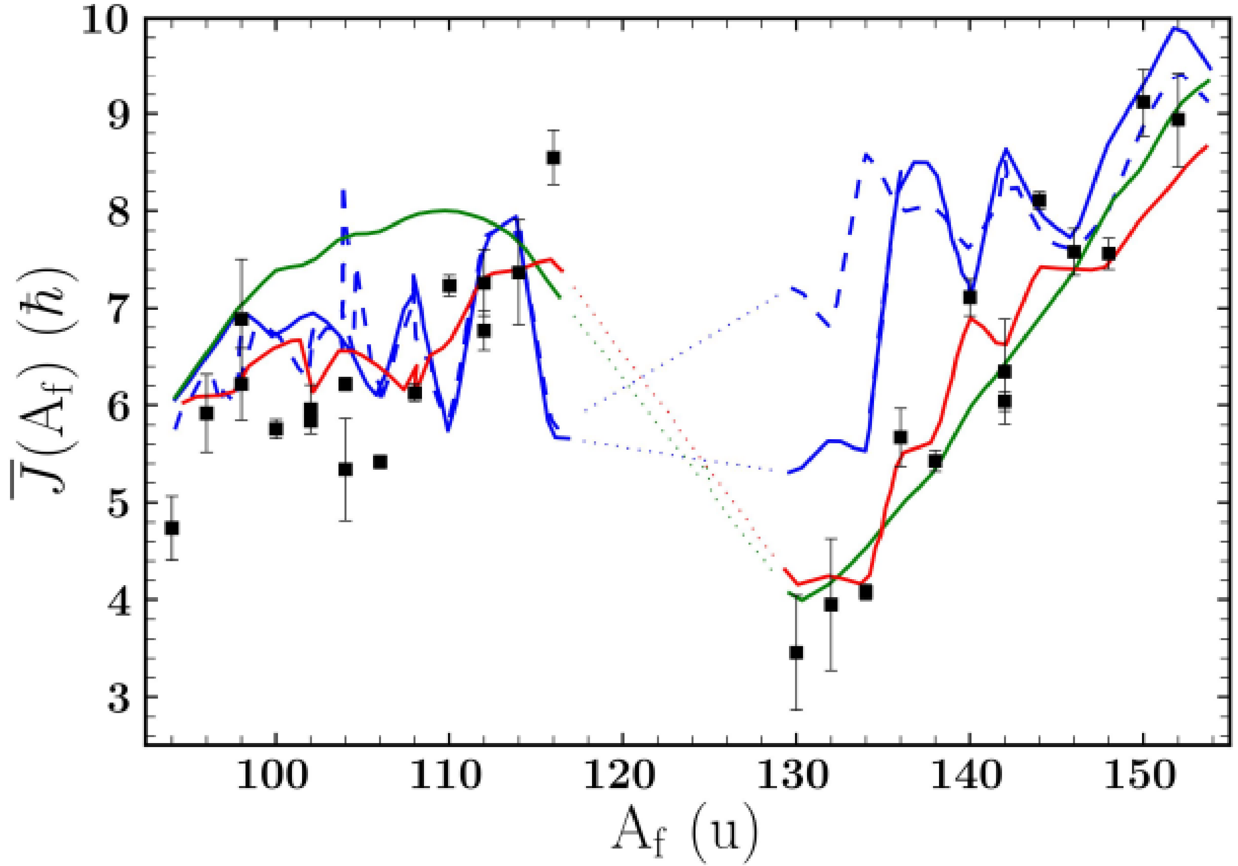
1. Wilson, J.N., Thisse, D., Lebois, M. *et al.* Angular momentum generation in nuclear fission. *Nature* **590**, 566–570 (2021).





1. Wilson, J.N., Thisse, D., Lebois, M. *et al*. Angular momentum generation in nuclear fission. *Nature* **590**, 566–570 (2021).
2. J. Randrup and R. Vogt PHYSICAL REVIEW LETTERS 127, 062502 (2021).
3. I. Stetcu , A. E. Lovell , P. Talou , T. Kawano , S. Marin , S. A. Pozzi , and A. Bulgac PHYSICAL REVIEW LETTERS 127, 222502 (2021).





1. Wilson, J.N., Thisse, D., Lebois, M. *et al.* Angular momentum generation in nuclear fission. *Nature* **590**, 566–570 (2021).
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3. I. Stetcu , A. E. Lovell , P. Talou , T. Kawano , S. Marin , S. A. Pozzi , and A. Bulgac PHYSICAL REVIEW LETTERS 127, 222502 (2021).

Correlation moment and correlation coefficient

$$\mu_{J_1J_2} = \int_0^\infty \int_0^\infty \left(J_1 - \langle J_1 \rangle\right)\left(J_2 - \langle J_2 \rangle\right) \times P\left(J_1, J_2\right) dJ_1 dJ_2,$$

$$\int_0^\infty e^{x\cos\varphi} d\varphi = \pi \tilde{j}_0\left(t\right) = \pi j\left(it\right) = \pi \sum_{k=0}^{\infty} \frac{x^{2k}}{\left(2^k k!\right)^2},$$

$$P\left(J_1, J_2\right) = \frac{2J_1J_2}{C_w C_b} \sum_{n=0}^{\infty} \left[\frac{J_1J_2}{n!}\left(\alpha I_1 I_2 - \beta\right)^n\right]^2 \times \left[-J_1^2\left(\alpha I_2^2 + \beta\right) - J_2^2\left(\alpha I_1^2 + \beta\right)\right].$$

$$c_{J_1J_2}\left(A_1, A_2\right) = \frac{\mu J_1 J_2}{\sigma_{J_1}\sigma_{J_2}},$$

$$\sigma_{J_i} = \sqrt{\left\langle J_i^2 \right\rangle - \left\langle J_i \right\rangle^2}$$

$$\tilde{c}_{J_1J_2} = \frac{\sum c_{J_1J_2}\left(A_1, A_2\right) Y\left(A_1, A_2\right)}{\sum Y\left(A_1, A_2\right)},$$

1. D. E. Lyubashevsky, A. A. Pisklyukov, S. V. Klyuchnikov, and P. V. Kostryukov, Phys. Rev. C 111. 054601. (2025).

Correlation moment and correlation coefficient

$$c(S_L, S_H) = \frac{\sigma(S_L, S_H)}{\sigma(S_L)\sigma(S_H)},$$

$$c(S_L, S_H) = \frac{\langle S_L \cdot S_H \rangle - \langle S_L \rangle \langle S_H \rangle}{\sigma_L \sigma_H} = -\sqrt{\frac{I_L I_H}{(I_R + I_L)(I_R + I_H)}},$$

	^{235}U (n, f)	^{238}U (n, f)	^{239}Pu (n, f)	^{252}Cf (sf)
$\langle S_L \rangle$	4.27	4.43	4.58	5.08
$\langle S_H \rangle$	5.66	5.80	5.93	6.33
$c(S_L, S_H)$	0.002	0.002	0.001	0.001

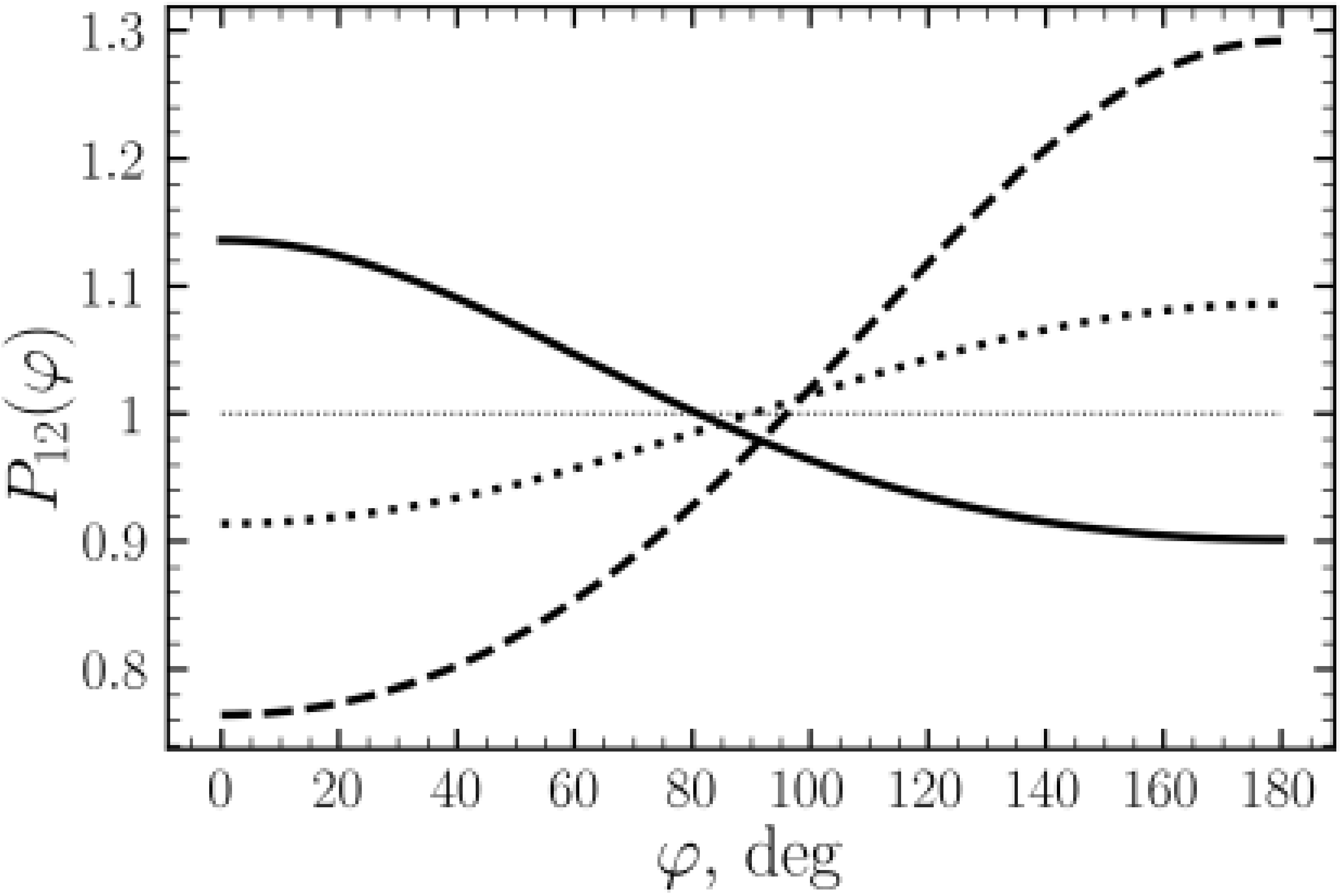
Mean values of the primary spins of the fission fragments $\langle S_L \rangle$ and $\langle S_H \rangle$ and the corresponding correlation coefficients $c(s_L, s_H)$

Coefficients $C_{J_1 J_2}$ and $\tilde{c}_{J_1 J_2}$, and yields $Y(A_f)$ for the studied reactions

Fragment	^{232}Th (n, f)		^{238}U (n, f)		^{252}Cf (sf)	
	$c_{J_1 J_2}$	$Y(A_f)$	$c_{J_1 J_2}$	$Y(A_f)$	$c_{J_1 J_2}$	$Y(A_f)$
^{82}Ge	0.203	0.64	0.207	0.12		
^{84}Ge	0.196	0.32				
^{84}Se	0.198	1.09	0.202	0.17		
^{86}Se	0.085	4.68	0.121	0.84		
^{88}Se	0.042	2.21	0.064	0.54		
^{88}Kr	0.114	0.85	0.133	0.37		
^{90}Kr	0.027	5.34	0.017	1.85		
^{92}Kr	0.012	3.92	0.005	2.50		
^{94}Kr	0.001	0.60				
^{92}Sr	0.062	0.20	0.006	0.73		
^{94}Sr	0.007	2.04	0.006	1.51	0.055	0.55
^{96}Sr	0.020	3.54	0.002	4.13	0.018	0.89
^{98}Sr	0.019	1.32	0	2.27	0.015	0.37
^{98}Zr	0.020	0.37	0	0.49	0.014	0.59
^{100}Zr	0.047	0.88	0.016	3.30	0.001	2.06
^{102}Zr			0.033	4.09	0	1.45
^{104}Zr			0.037	1.01	0.002	0.22
^{102}Mo			0.025	0.08	0.002	0.46
^{104}Mo			0.031	1.08	0.001	2.83
^{106}Mo					0.001	3.47
^{108}Mo					0	0.67
^{108}Ru					0	1.98
^{110}Ru					0.011	3.62
^{112}Ru					0.014	0.94
^{112}Pd					0.014	0.75
^{114}Pd					0.035	1.82
^{116}Pd					0.038	0.82
^{130}Sn	0.031	0.84	0.028	1.65	0.050	0.36
^{132}Sn	0.023	1.54	0.027	1.88	0.037	0.14
^{134}Sn			0.026	0.18		
^{132}Te	0.008	0.35	0.029	0.47		
^{134}Te	0.023	3.11	0.024	3.95	0.059	2.35
^{136}Te	0.006	3.44	0.013	3.53	0.050	0.91
^{138}Te	0.005	0.76	0.007	0.55	0.047	3.63
^{138}Xe	0.006	2.08	0.007	2.04	0.026	2.55
^{140}Xe	0.011	5.73	0.004	4.04	0.023	0.37
^{142}Xe	0.014	2.25	0.009	1.53	0.019	2.70
^{142}Ba	0.014	0.64	0.011	0.69	0.001	3.37
^{144}Ba	0.035	4.49	0.012	2.46	0	0.98
^{146}Ba	0.040	2.76	0.014	1.98		
^{148}Ba			0.035	0.25		
^{148}Ce	0.056	0.56	0.036	0.75	0	2.35
^{150}Ce	0.079	0.41	0.067	0.86	0.004	0.94
^{152}Nd					0.009	0.83
^{154}Nd					0.020	0.42
$\tilde{c}_{J_1 J_2}$	0.034		0.020		0.017	

1. J. Randrup and R. Vogt, Phys. Rev. Lett. 127, 062502 (2021).
2. D. E. Lyubashevsky, A. A. Pisklyukov, S. V. Klyuchnikov, and P. V. Kostryukov, Phys. Rev. C 111. 054601. (2025).

Correlation moment and correlation coefficient



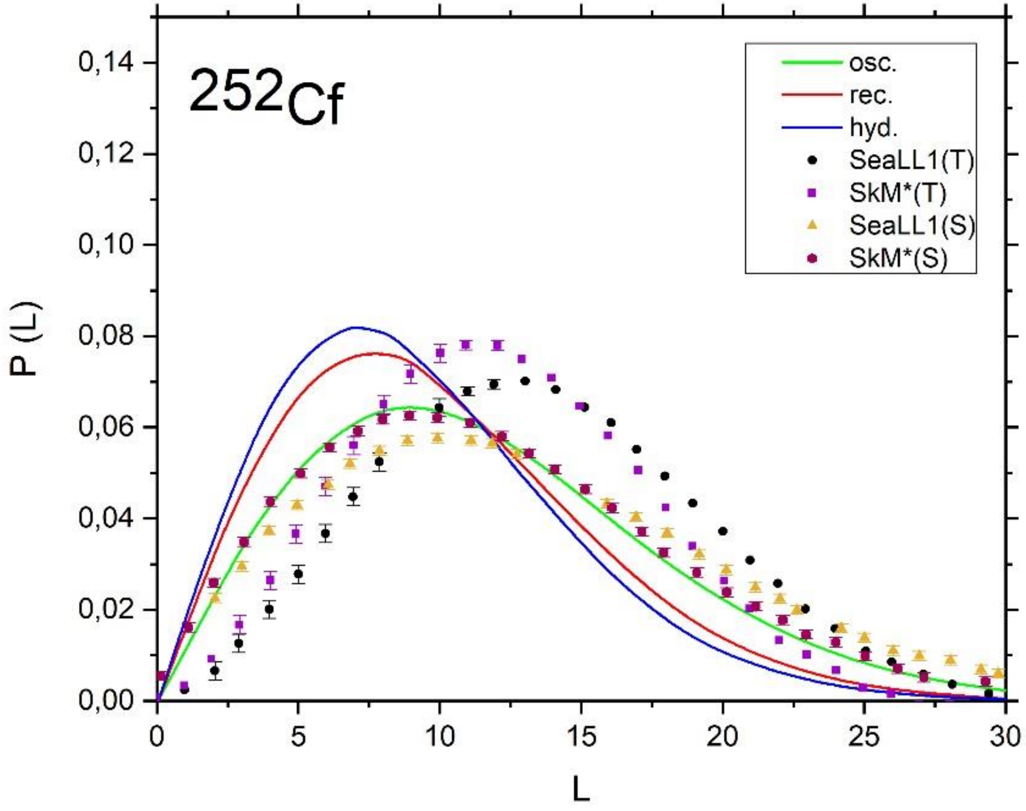
Comparison of the angle dependence of the spin distribution for the 252Cf (sf) response. The solid line is the result of the present study, the short and long dashed lines are the first and second limiting cases of the Randrup [2] approach

1. G. Scamps; I. Abdurrahman; M. Kafker; A. Bulgac; I. Stetcu PhysRev C. 108L061602(2023).
2. J. Randrup, T. Døssing, and R. Vogt, Phys. Rev. C 106, 014609 (2022).

Orbital momentum of double fission fragments

$$P(L) = \int_0^{\pi} \frac{L}{\pi I_w \hbar \omega_w} \exp\left[-\frac{L^2}{I_w \hbar \omega_w}\right] d\varphi_L = \frac{2L}{I_w \hbar \omega_w} \exp\left[-\frac{L^2}{I_w \hbar \omega_w}\right].$$

$$\bar{L} = \int_0^{\infty} \frac{2L^2}{I_w \hbar \omega_w} \exp\left[-\frac{L^2}{I_w \hbar \omega_w}\right] dL = \frac{\sqrt{I_w \hbar \omega_w \pi}}{2}.$$



Comparison of the distribution of orbital momentum of nuclear double fission fragments calculated by formula (5) (solid line) with similar values obtained with the nuclear matter density functional (NEDF), SkM in the case of the 252Cf nucleus

1. Bulgac A., SnAbdurrahman SnI., Godbey K., and SnplaceStetcu SnI. // Phys. Rev. Lett. 2022. **128**, 022501

Conclusion

1. Within the framework of the «cold» nucleus model, the projection on the Z axis vanishes and a two-dimensional spin model is implemented.
2. Using the approach developed by our group, taking into account the moments of inertia of the fission prefragments obtained in the work of D. E. Lyubashevsky et al 2025 Chinese Phys. C49 034104 the stiffness coefficients of bending and wriggling vibrations, as well as the frequencies of these vibrations, were evaluated. There is a good agreement with the experimental data of Wilson, J.N., Thisse, D., et al. Nature 590, 566-570 (2021), surpassing the accuracy of descriptions of other theoretical groups.
3. The correlation coefficients of the spins of the fission fragments were calculated, and the estimates found reasonable agreement with experimental data from Wilson, J.N., and Thisse, D., et al. Nature 590, 566-570 (2021), as well as with the theoretical predictions of the theoretical group J. Randrup and R. Vogt, Phys. Rev. Lett. 127, 062502 (2021).

Thank you for your attention!