High-Energy Factorization of Massive Amplitudes

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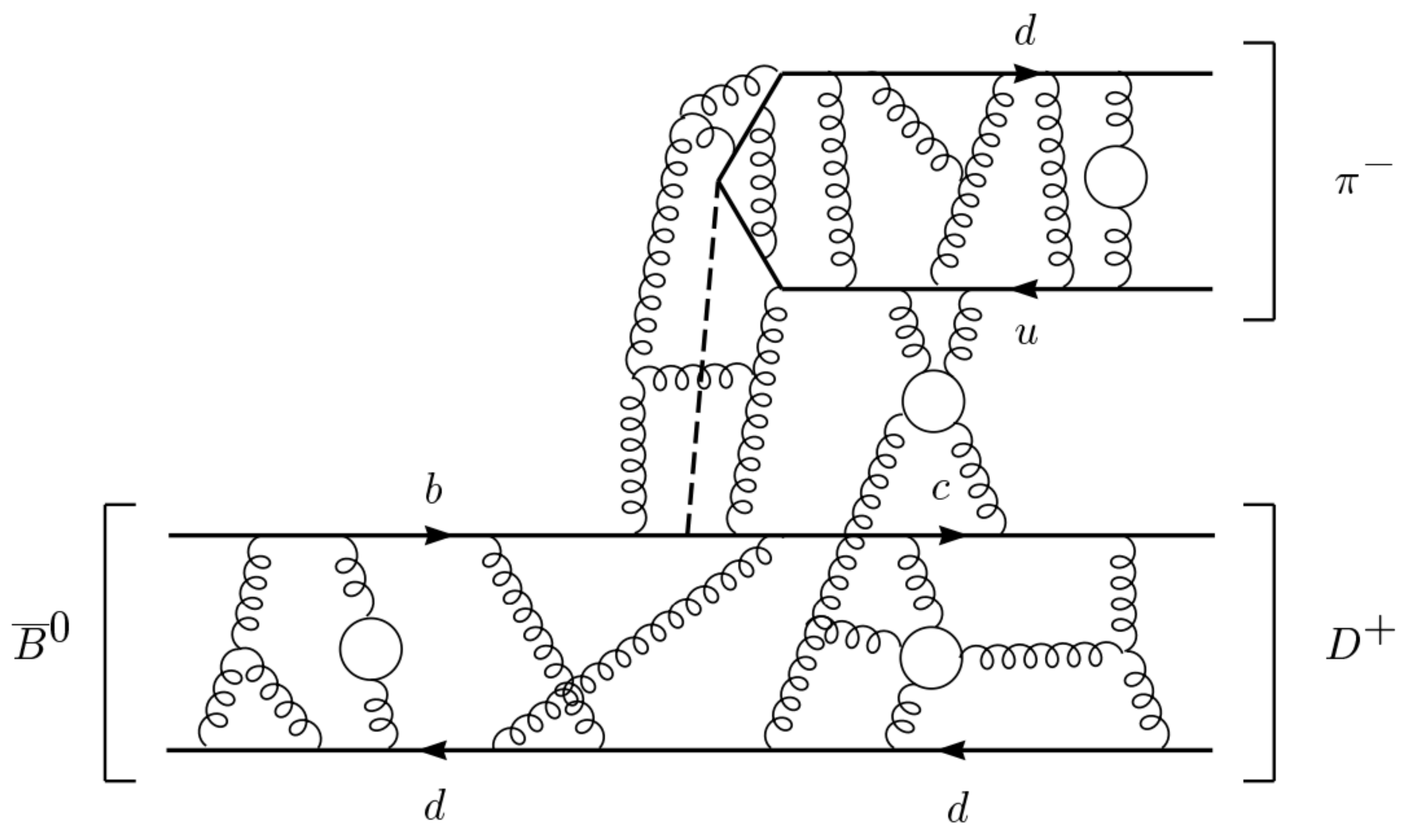
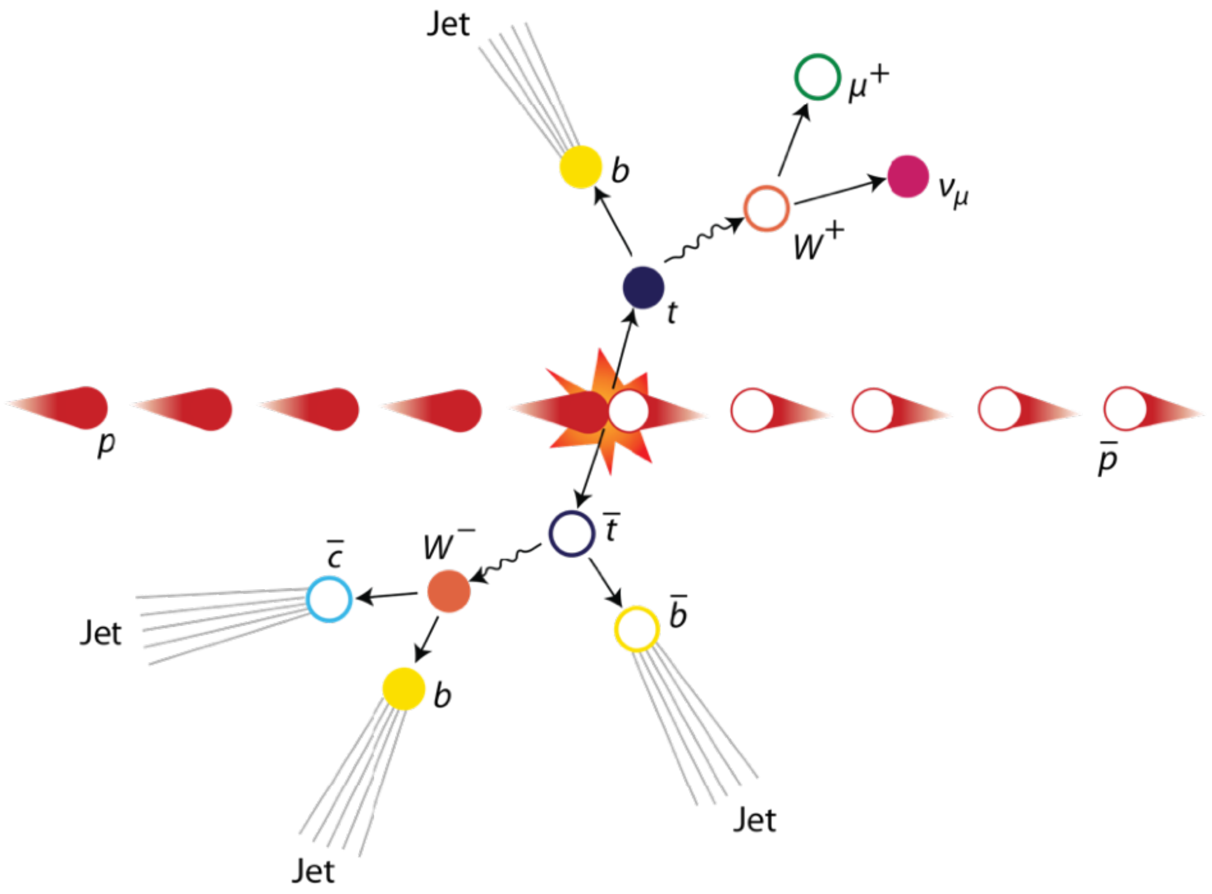
Factorization in heavy flavor physics

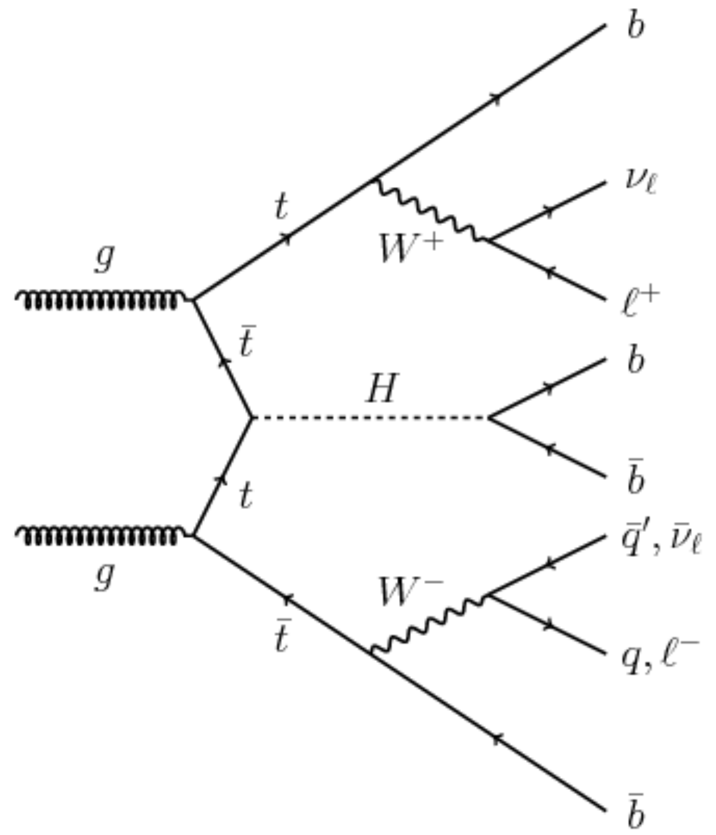
Factorization is the main theme in heavy flavor physics: natural appearance of multiple physical scales

- ➤ Disentangle dynamics at different energies
- ➤ Resum large logarithms through RGEs

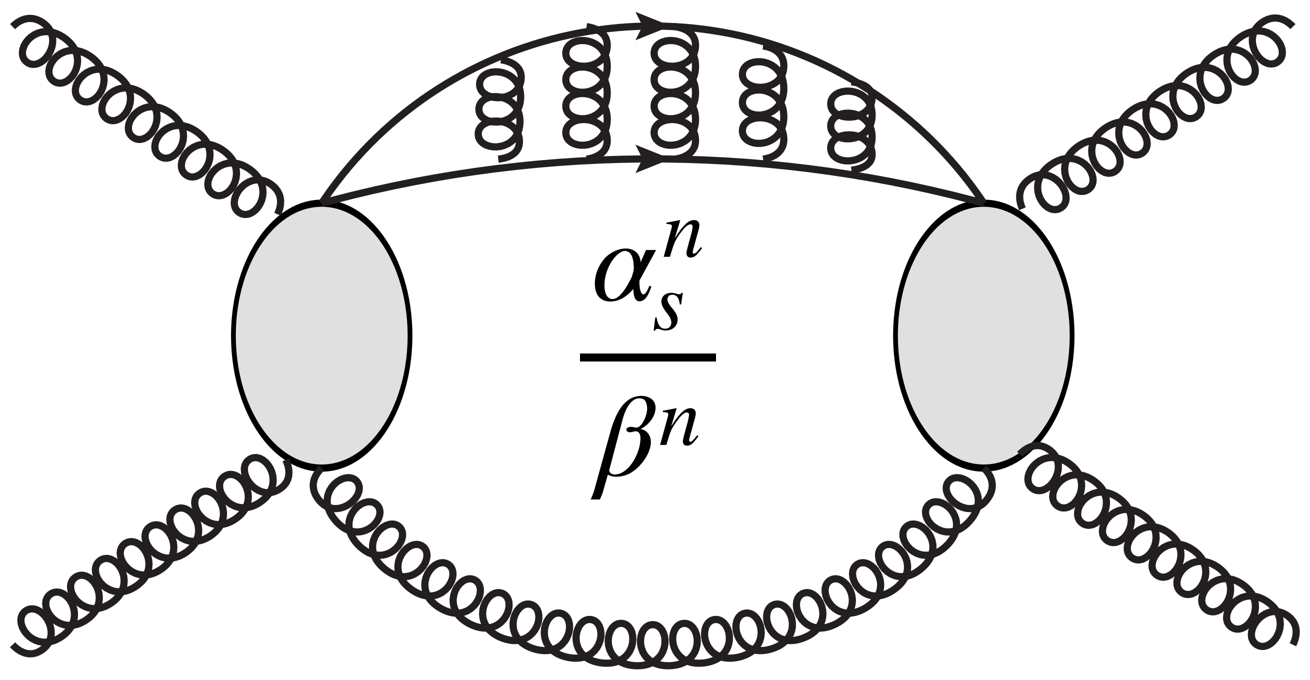


My focus will be on top quarks: the heaviest flavor in the SM
(but the results could be / have been applied to bottoms as well)





Factorization and resummation at low energies



- Non-relativistic effects in $Q\bar{Q}$ system near threshold
- Well-studied in quarkonium systems
- Relevant for threshold $t\bar{t}$ production and top quark mass measurement

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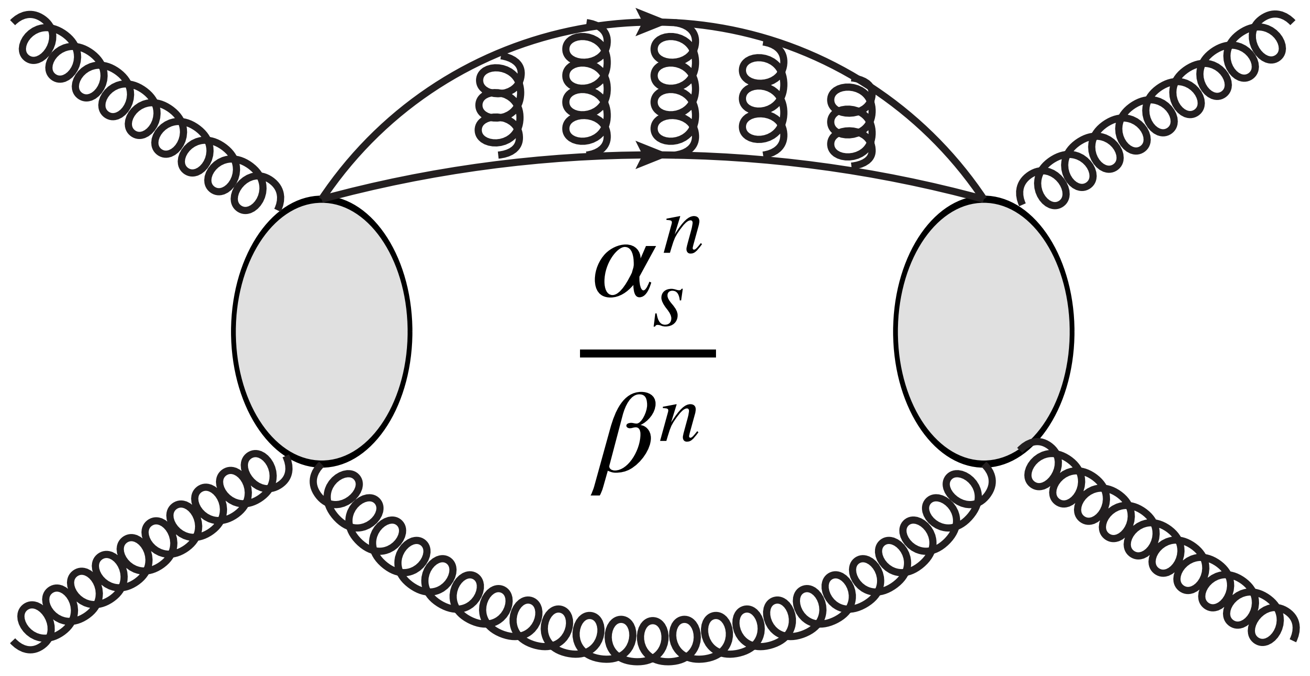

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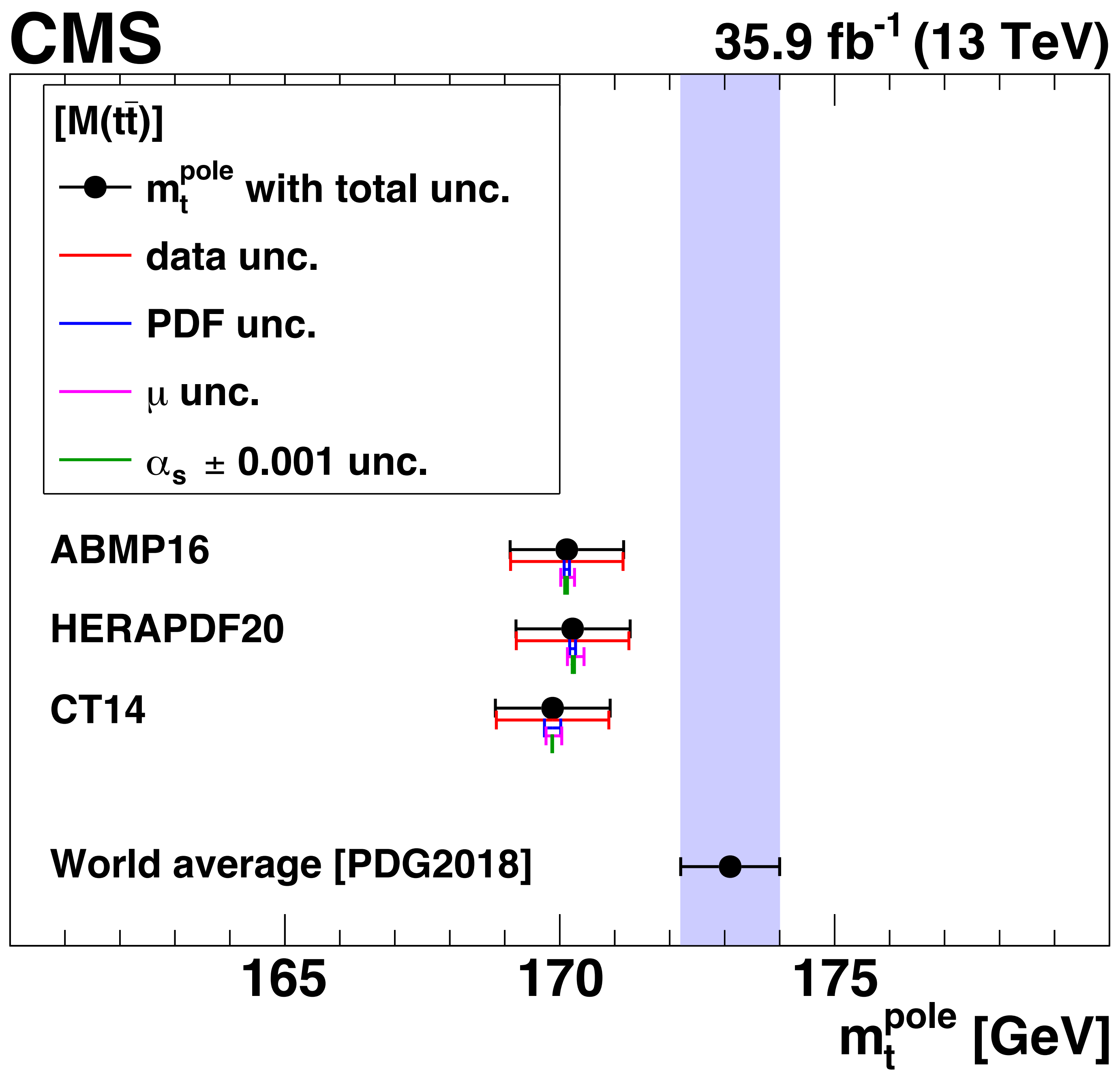
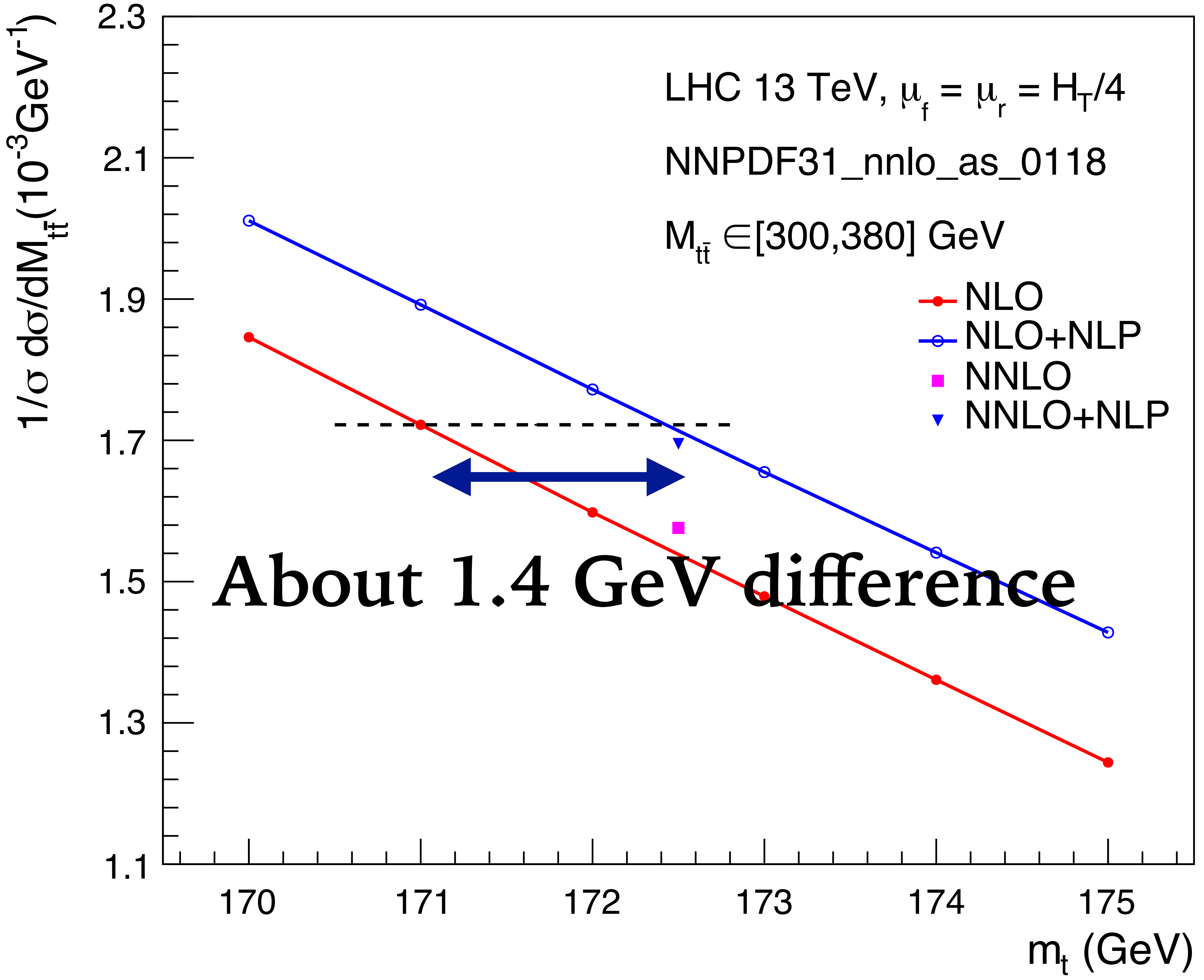


ATLAS collaboration: 1905.02302

CMS collaboration: 1904.05237

The measured value would deviate if such effects were not considered!

Bound-state effects in top quark pair production



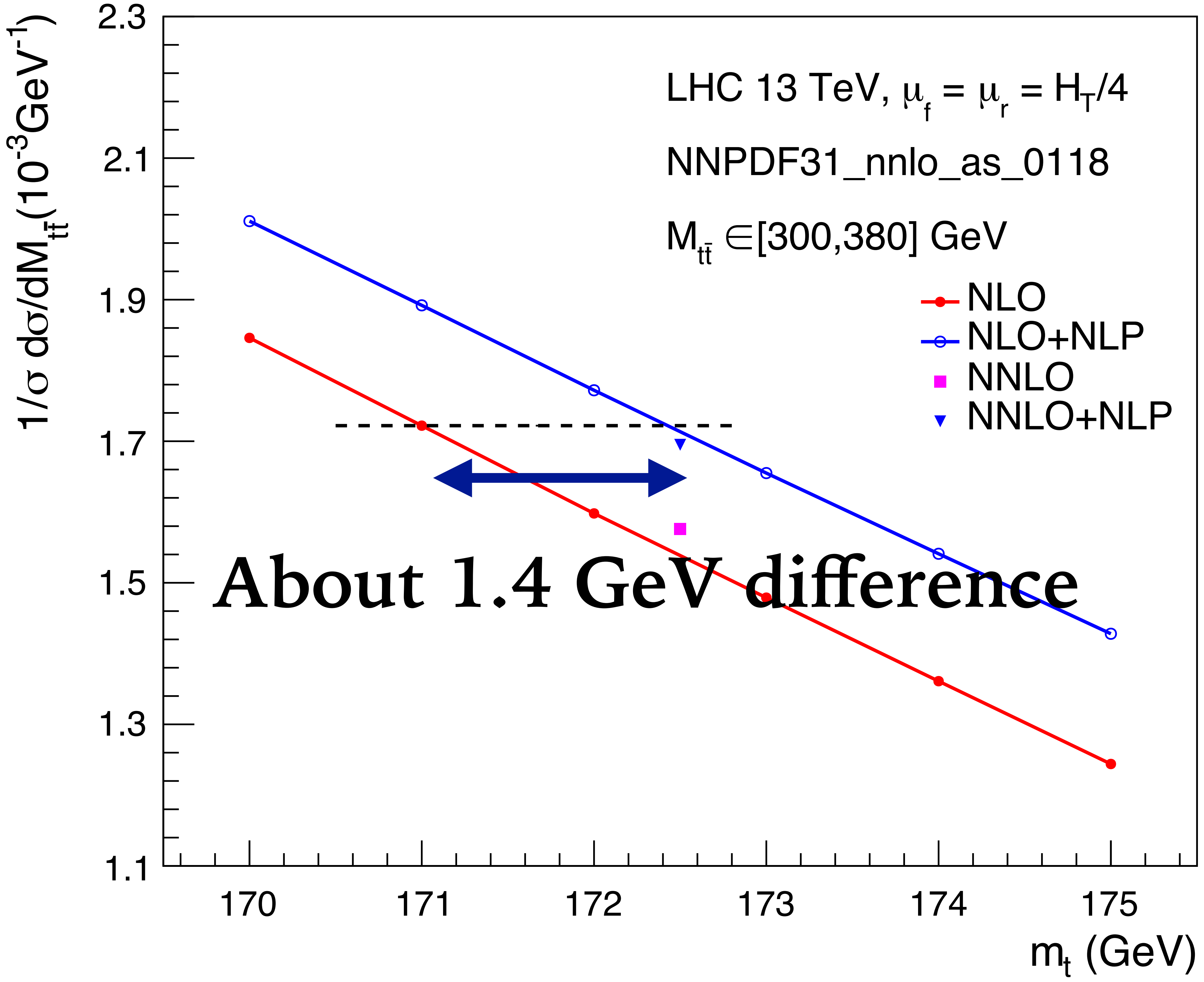
Ju, Wang, Wang, Xu, Xu, LLY: 1908.02179, 2004.03088

$$\frac{d\sigma}{dM_{t\bar{t}}d\Theta} \sim \int H \times J \times f \times f$$

We demonstrated that bound-state effects can account for most of the deviation

See also:
Kiyo et al.: 0812.0919
Sumino and Yokoya: 1007.0075
Fuks et al.: 2102.11281
Garzelli et al.:2412.16685

Bound-state effects in top quark pair production



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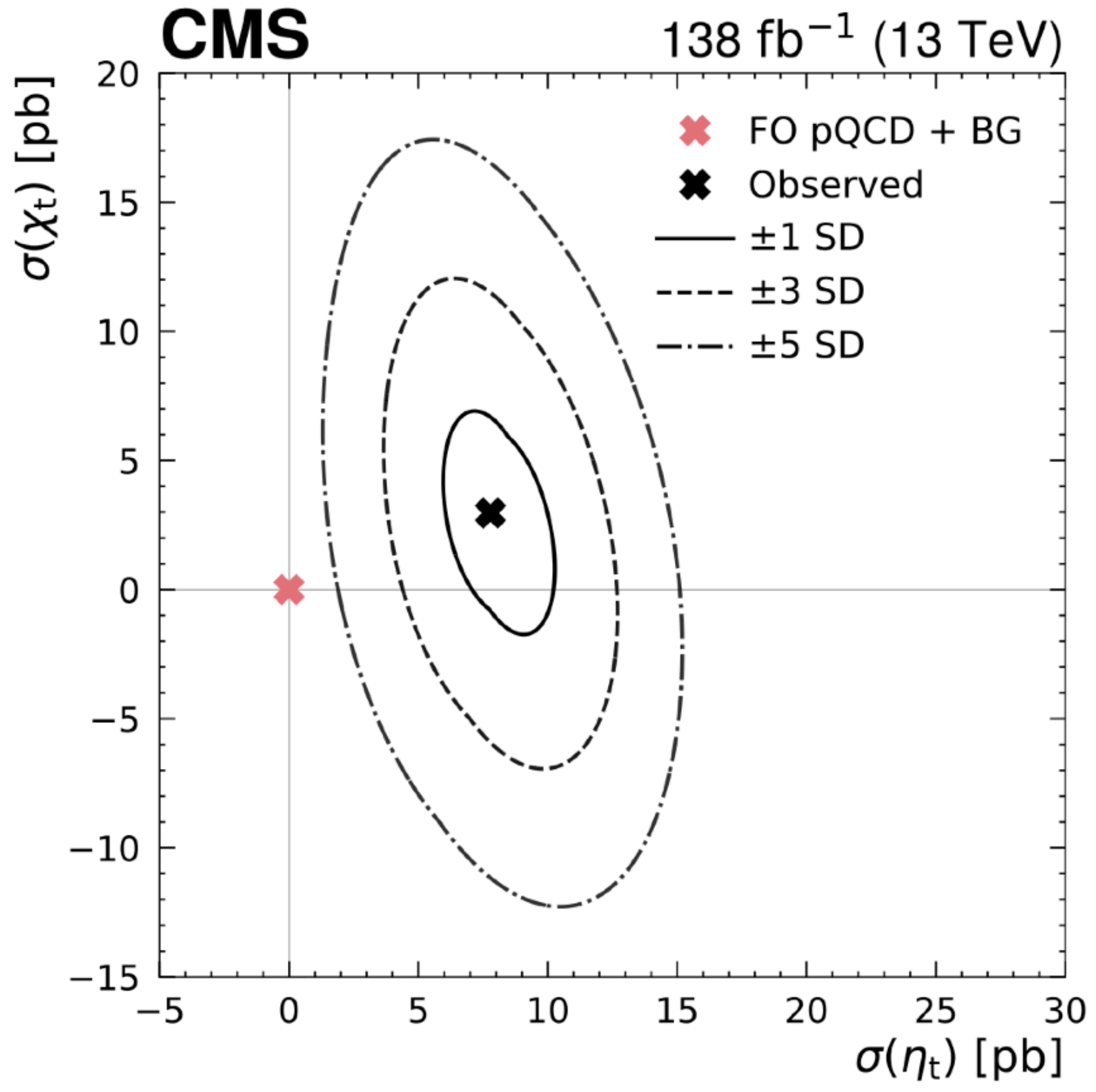
Recently, such bound-state effects, or the "toponium", has been **confirmed by the CMS experiment**



CMS Collaboration: 2503.22382

See also:
Kiyo et al.: 0812.0919
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Factorization and resummation at high energies

At high energies, two kinds of scale hierarchy:

$$\frac{\hat{s} - M^2}{\hat{s}} \ll 1$$

$$\frac{m_t^2}{M^2} \ll 1$$

A universal framework to resum both kinds of large logarithms

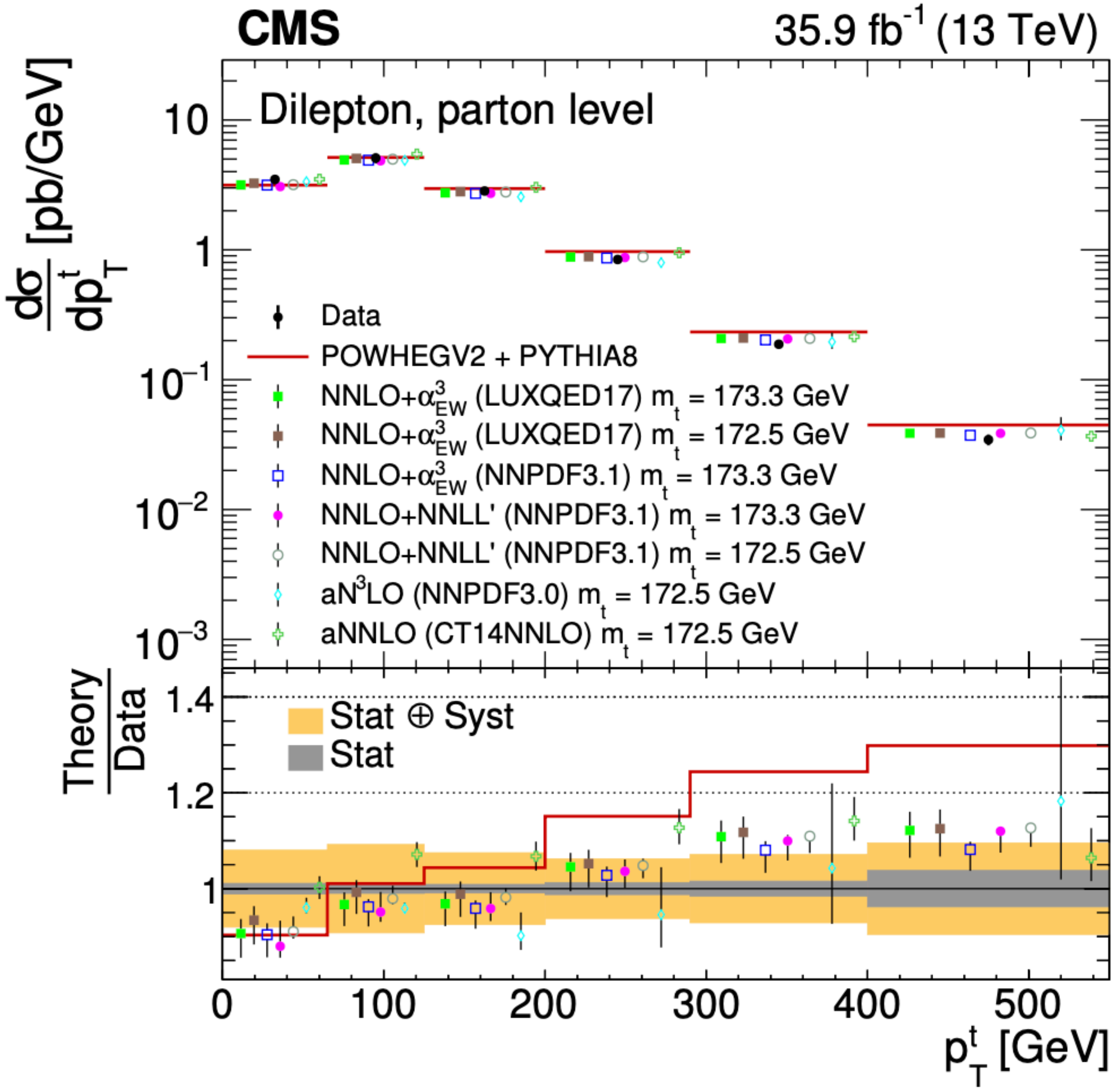
Ferroglia, Pejack, LLY: 1205.3662

$$C_{ij}(z, M, m_t, \cos\theta, \mu_f) = C_D^2(m_t, \mu_f) \, \mathrm{Tr}\left[\boldsymbol{H}_{ij}(M, t_1, \mu_f) \, \boldsymbol{S}_{ij}(\sqrt{\hat{s}}(1-z), t_1, \mu_f) \right]$$

$$\otimes C_{ff}^{ij}(z, m_t, \mu_f) \otimes C_{t/t}(z, m_t, \mu_f) \otimes C_{t/t}(z, m_t, \mu_f)$$

$$\otimes S_D(m_t(1-z), \mu_f) \otimes S_D(m_t(1-z), \mu_f) + \mathcal{O}(1-z) + \mathcal{O}\left(\frac{m_t}{M}\right)$$

Application to top quark pair production



Pecjak, Scott, Wang, LLY: 1601.07020
Czakon et al.: 1803.07623, 1901.08281

State-of-the-art theoretical prediction

NNLO+NNLL' in QCD + NLO in EW

Factorization in the high energy limit

It was suggested that a massive amplitude can be factorized in the high-energy limit into a massless amplitude and a collinear factor for each leg

Mitov, Moch: hep-ph/0612149

$$\mathcal{M}^{[p],(m)}\left(\{k_i\},\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\varepsilon\right) = \prod_{i\in\,\{\text{all legs}\}}\left(Z_{[i]}^{(m|0)}\left(\frac{m^2}{\mu^2},\alpha_s(\mu^2),\varepsilon\right)\right)^{\frac{1}{2}} \times \mathcal{M}^{[p],(m=0)}\left(\{k_i\},\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\varepsilon\right)$$

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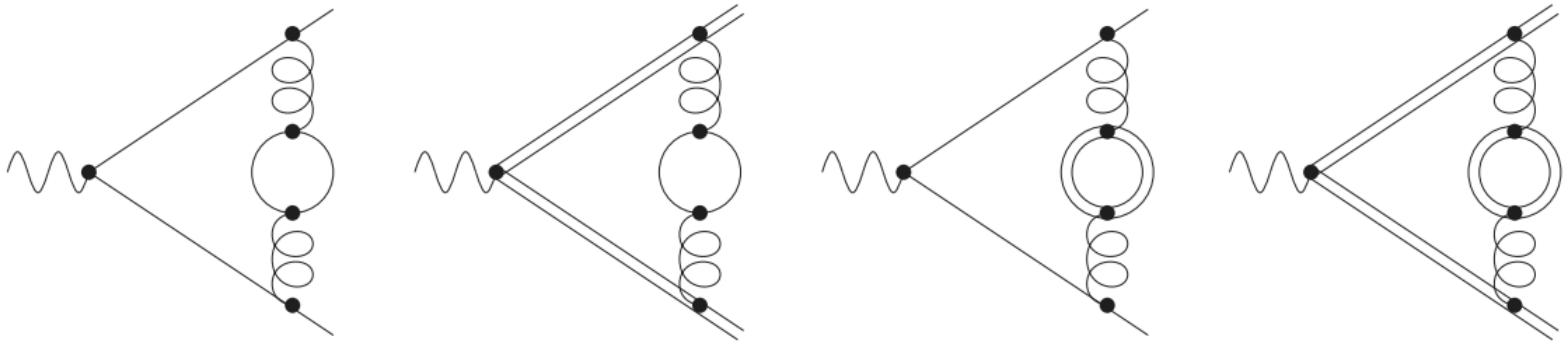


ll hl lh hh

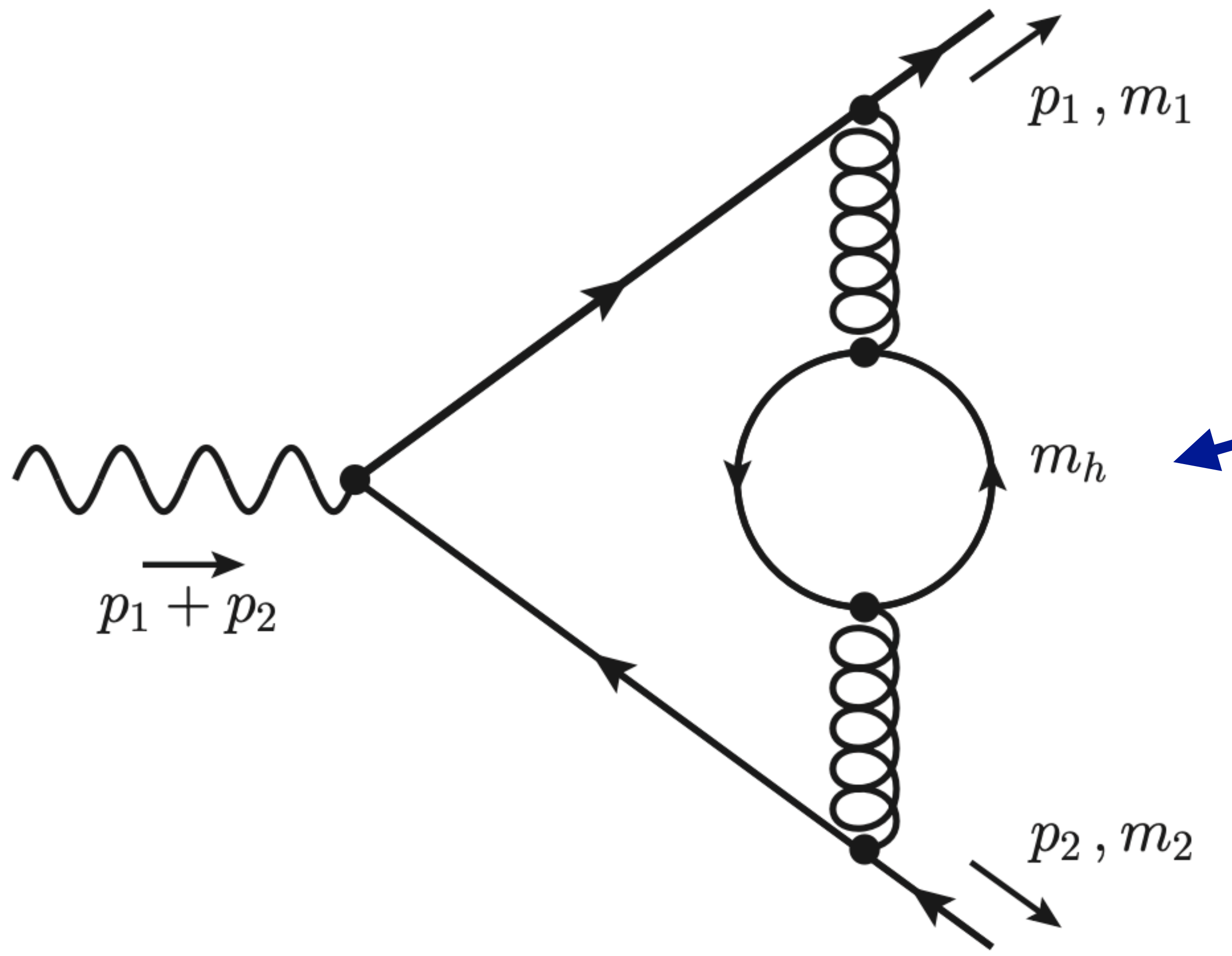
But the heavy-quark bubbles were not included!

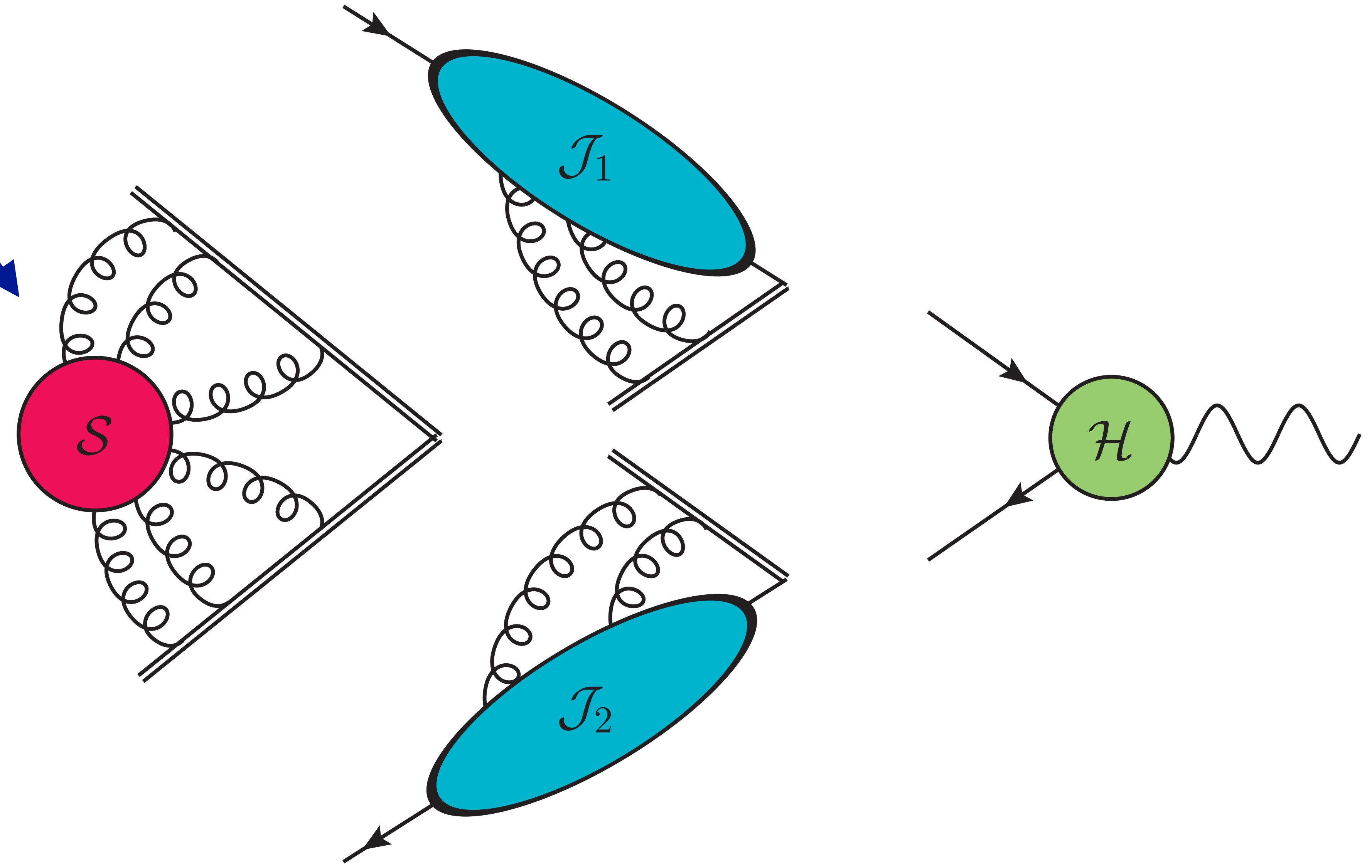
Improved factorization formula

Wang, Xia, LLY, Ye: 2312.12242

$$\left|\mathcal{M}^{\text{massive}}(\{p\},\{m\})\right\rangle = \prod_i \left(\mathcal{Z}_{[i]}^{(m|0)}(\{m\})\right)^{1/2} \boldsymbol{\mathcal{S}}(\{p\},\{m\}) \left|\mathcal{M}^{\text{massless}}(\{p\})\right\rangle$$

A new soft function



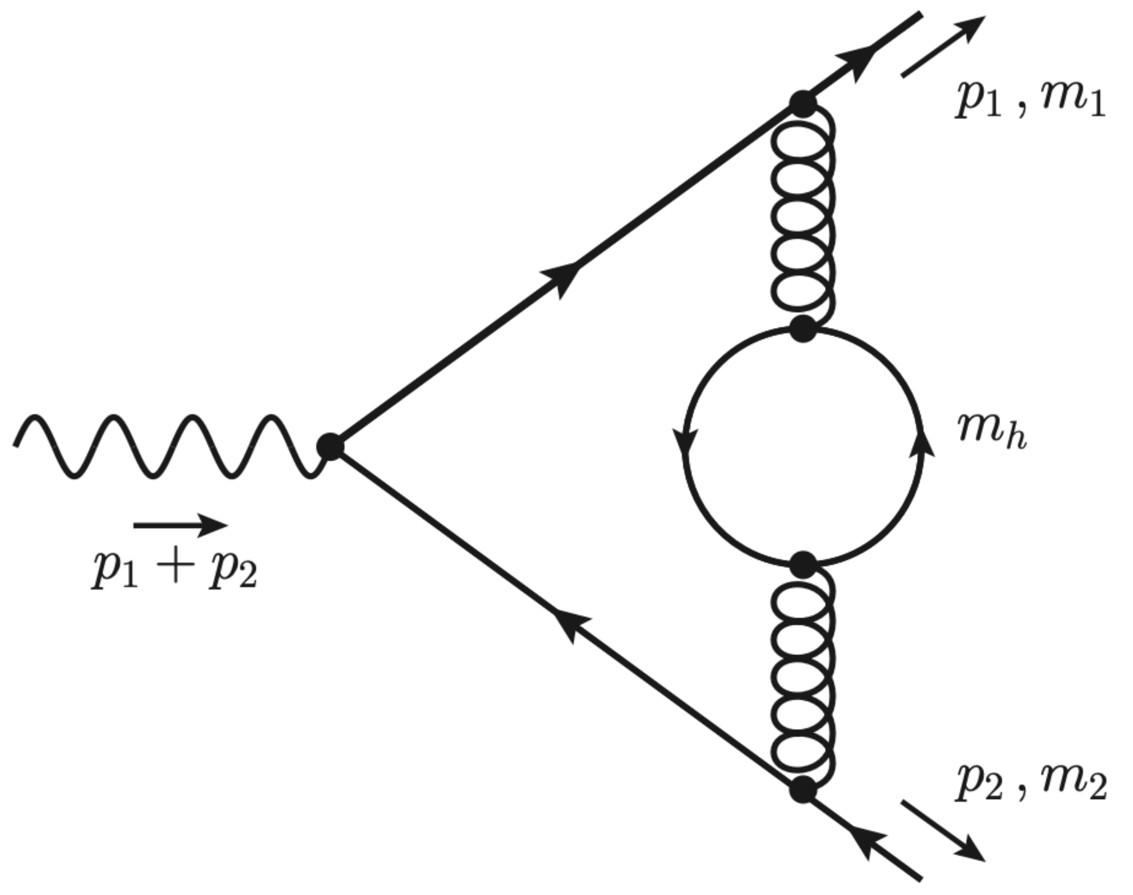


Applied to bottom quark production in, e.g.:

Mazzitelli et al.: 2404.08598
Biello et al.: 2412.09510

The new soft function

Wang, Xia, LLY, Ye: 2312.12242



$$\begin{aligned}\text{hard} : k^\mu &\sim \sqrt{|s|}\,,\\ n_i\text{-collinear} : (n_i\cdot k,\, \bar{n}_i\cdot k,\, k_\perp) &\sim \sqrt{|s|}\,(\lambda^2,\,1,\,\lambda)\\ \text{soft} : k^\mu &\sim \sqrt{|s|}\,\lambda\,.\end{aligned}$$

Rapidity divergence: analytic regulator

$$\begin{aligned}I_{\{a_i\}} \equiv \mu^{4\epsilon}\int \frac{dk_1}{(2\pi)^d}\frac{dk_2}{(2\pi)^d}\frac{1}{[k_1^2-m_h^2]^{a_1}}\frac{1}{[k_2^2-m_h^2]^{a_2}}\frac{1}{[(k_1+k_2)^2]^{a_3}}\frac{1}{[(k_1+k_2-p_1)^2-m_1^2]^{a_4}}\\ \times\frac{(-\tilde{\mu}^2)^\nu}{[(k_1+k_2+p_2)^2-m_2^2]^{a_5+\nu}}\frac{1}{[(k_1-p_1)^2]^{a_6}}\frac{1}{[(k_1+p_2)^2]^{a_7}}\,, \quad (3.4)\end{aligned}$$

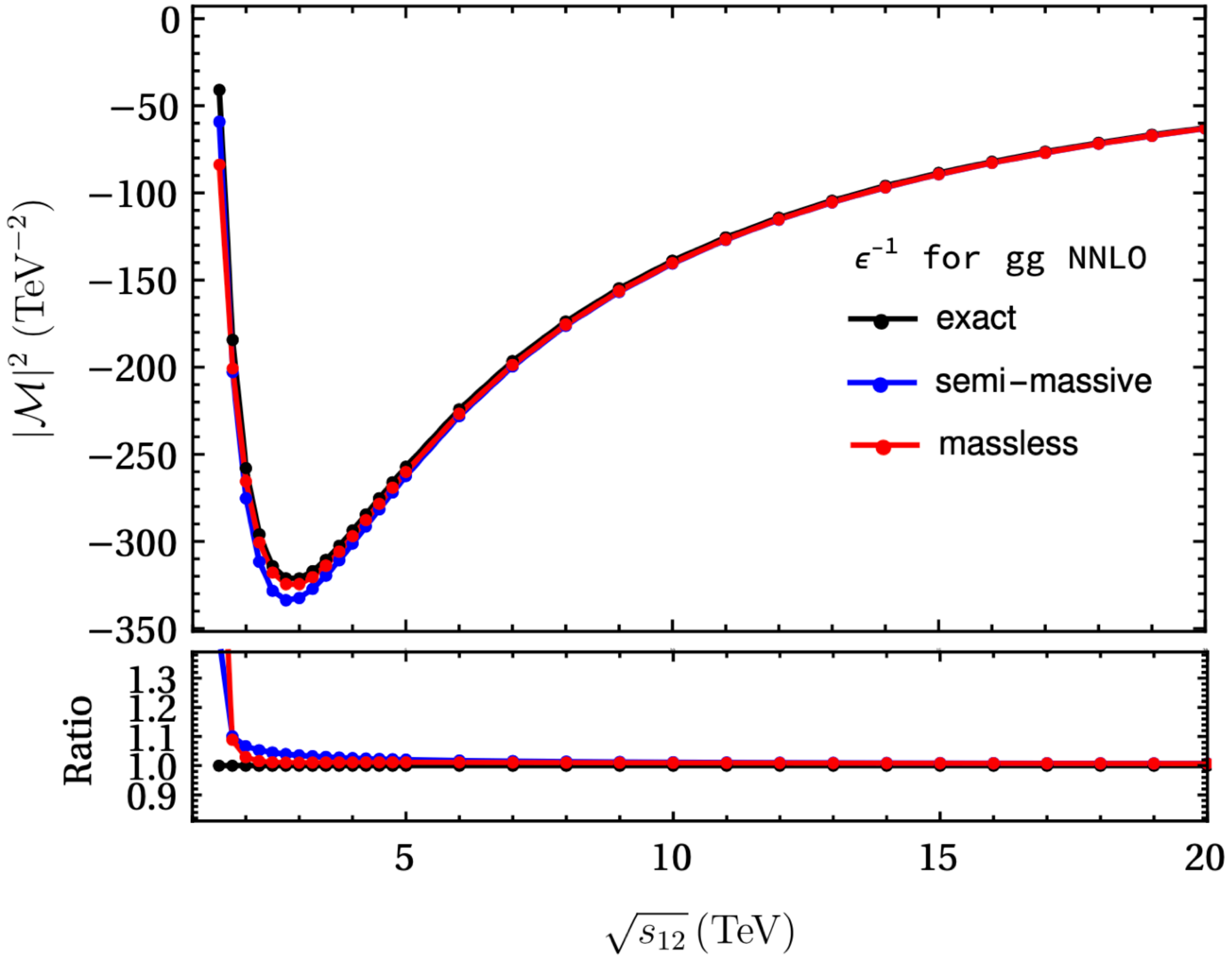
$$\boldsymbol{\mathcal{S}}(\{p\},\{m\}) = 1+\left(\frac{\alpha_s}{4\pi}\right)^2\sum_{\substack{i,j\\ i\neq j}}(-\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\sum_h \mathcal{S}^{(2)}(s_{ij},m_h^2)+\mathcal{O}(\alpha_s^3)$$

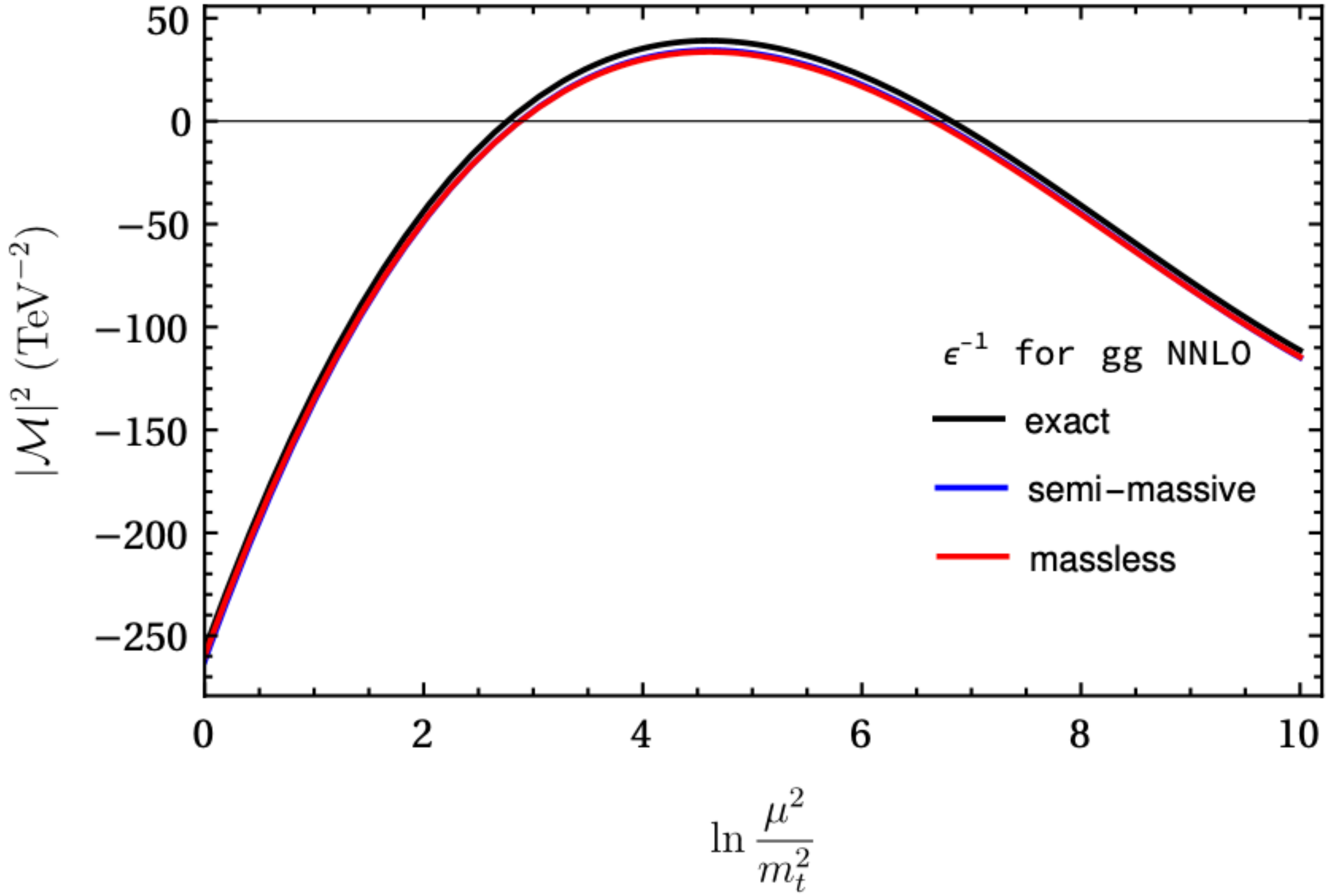
$$\mathcal{S}^{(2)}(s_{ij},m_h^2) = T_F\left(\frac{\mu^2}{m_h^2}\right)^{2\epsilon}\left(-\frac{4}{3\epsilon^2}+\frac{20}{9\epsilon}-\frac{112}{27}-\frac{4\zeta_2}{3}\right)\ln\frac{-s_{ij}}{m_h^2}$$

Application: two-loop amplitudes for tTH production

$$\left|\mathcal{M}^{\text{massive}}(\{p\},\{m\})\right\rangle = \prod_i \left(\mathcal{Z}_{[i]}^{(m|0)}(\{m\})\right)^{1/2} \boldsymbol{\mathcal{S}}(\{p\},\{m\}) \left|\mathcal{M}^{\text{massless}}(\{p\})\right\rangle$$

Wang, Xia, LLY, Ye: 2402.00431





IR poles validated against exact results in Chen, Ma, Wang, LLY, Ye: 2202.02913

Note: without our new factorization formula, the scale-dependence would be wrong!

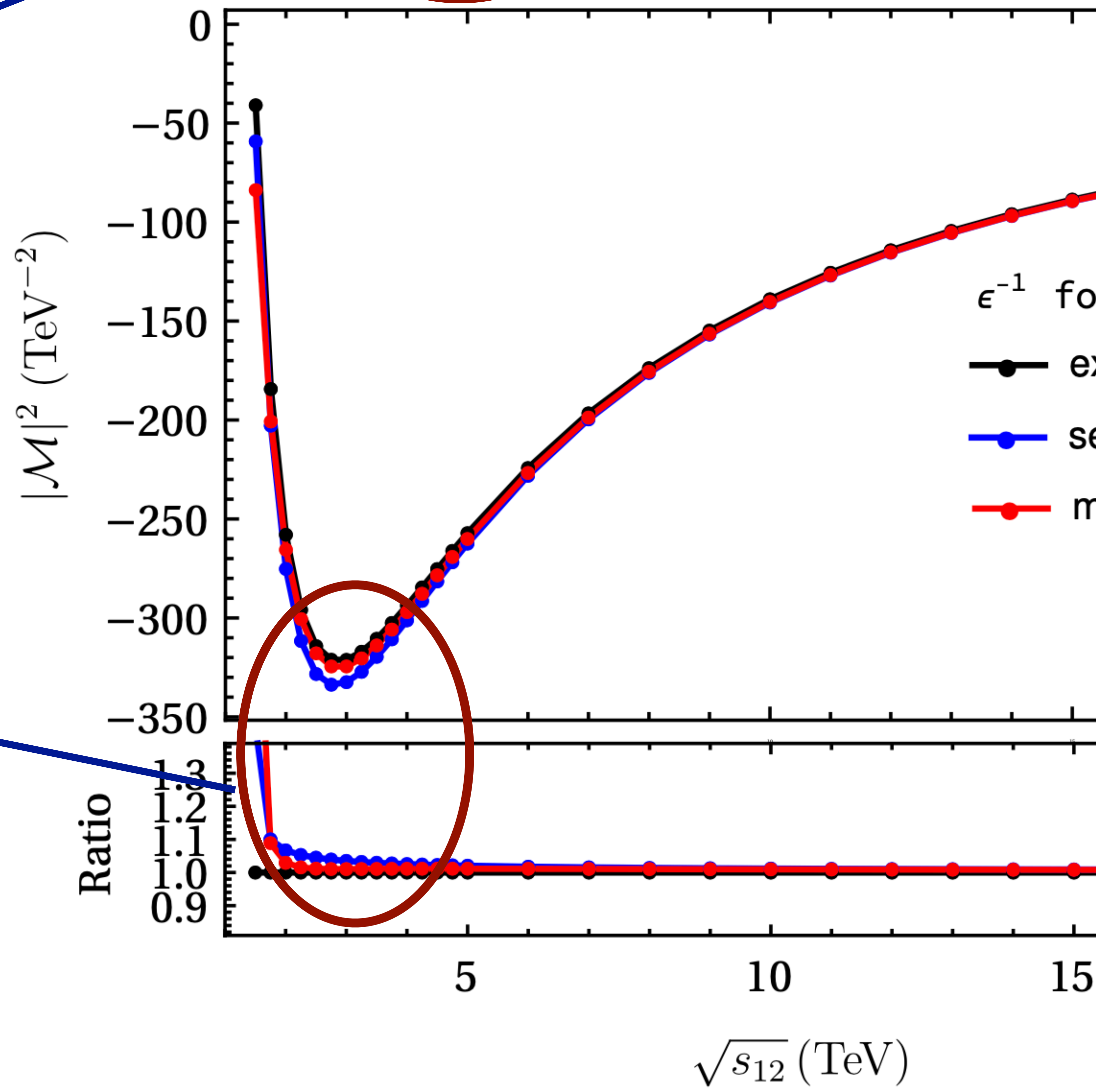
Towards sub-leading factorization

$$\left|\mathcal{M}^{\text{massive}}(\{p\},\{m\})\right\rangle = \prod_i \left(\mathcal{Z}_{[i]}^{(m|0)}(\{m\})\right)^{1/2} \boldsymbol{\mathcal{S}}(\{p\},\{m\}) \left|\mathcal{M}^{\text{massless}}(\{p\})\right\rangle + \mathcal{O}\left(\frac{m^2}{s_{ij}}\right)$$

Power corrections to the factorization formula

Important for intermediate energy range

Important for combining the threshold region and the high-energy region



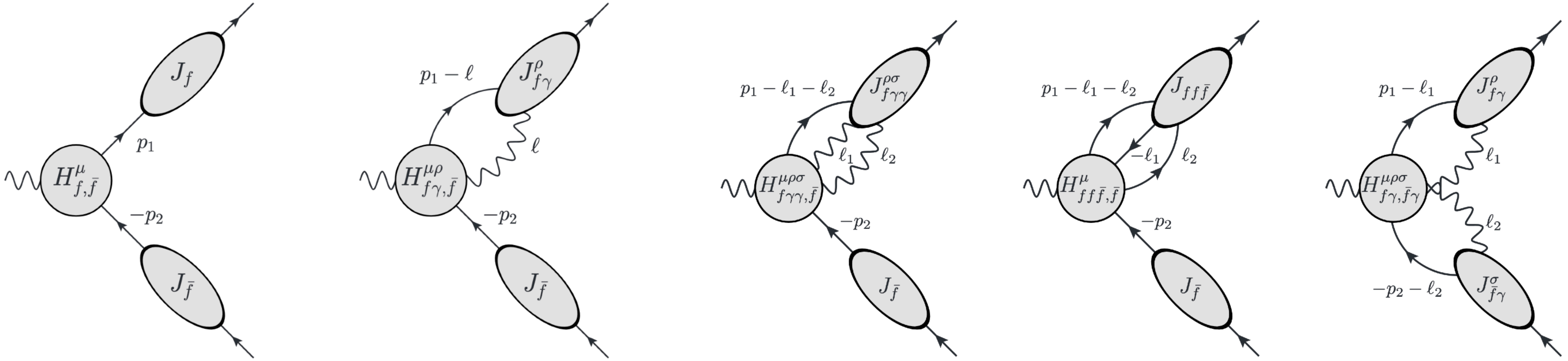
Towards sub-leading factorization

Laenen et al.: 2008.01736
ter Hoeve et al: 2311.16215
van Bijleveld et al.: 2503.10810

Partial results available at the next-to-leading power

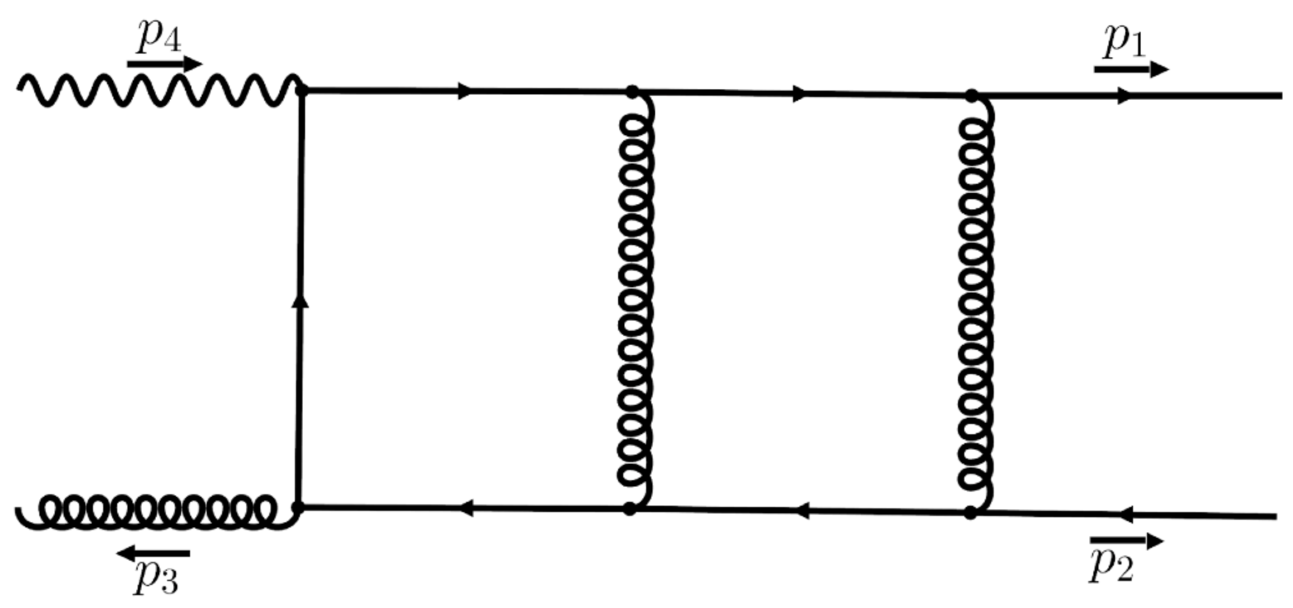
$$\mathcal{M}_{\text{coll.}} = \left(\prod_{i=1}^{n} J_f^i\right) \otimes H_f\, S + \sum_{i=1}^{n}\left(\prod_{j\neq i} J_f^j\right)\left[J_{f\gamma}^i \otimes H_{f\gamma}^i + J_{f\partial\gamma}^i \otimes H_{f\partial\gamma}^i\right] S$$
$$+ \sum_{i=1}^{n}\left(\prod_{j\neq i} J_f^j\right) J_{f\gamma\gamma}^i \otimes H_{f\gamma\gamma}^i\, S + \sum_{i=1}^{n}\left(\prod_{j\neq i} J_f^j\right) J_{fff}^i \otimes H_{fff}^i\, S$$
$$+ \sum_{1\leq i\leq j\leq n}\left(\prod_{k\neq i,j} J_f^k\right) J_{f\gamma}^i J_{f\gamma}^j \otimes H_{f\gamma,f\gamma}^{ij}\, S + \mathcal{O}(\lambda^3)\,,$$

- Analysis in the collinear region
- Validated against $1 \to 2$ form factors



Towards sub-leading factorization

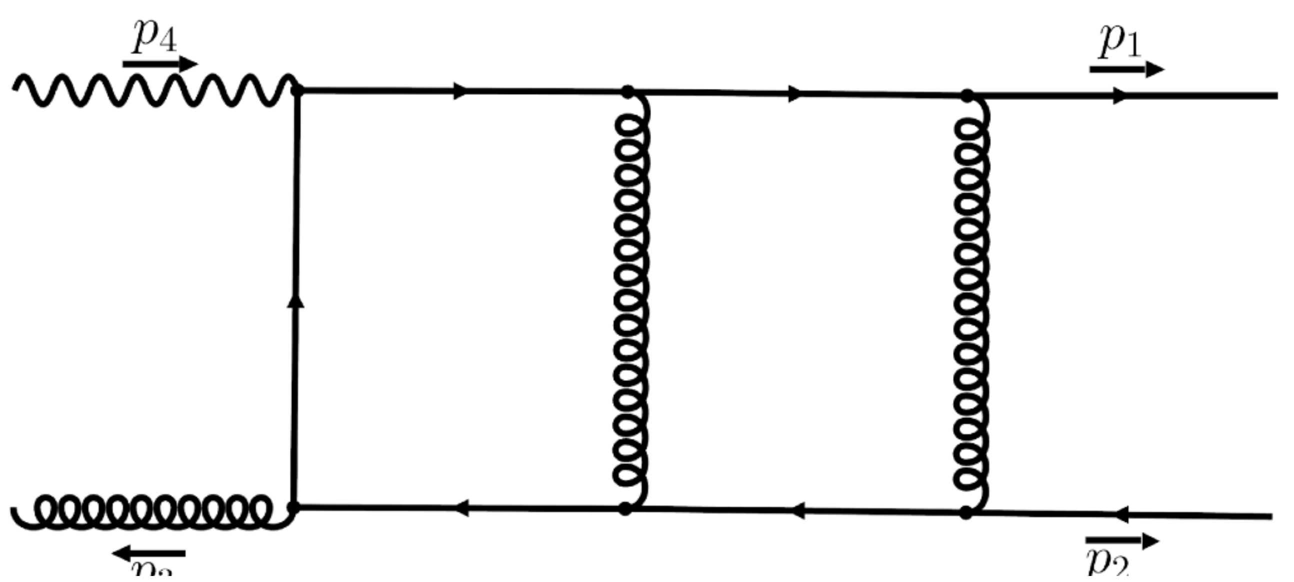
Ongoing: analyzing sub-leading corrections in $1 \to 3$ form factors

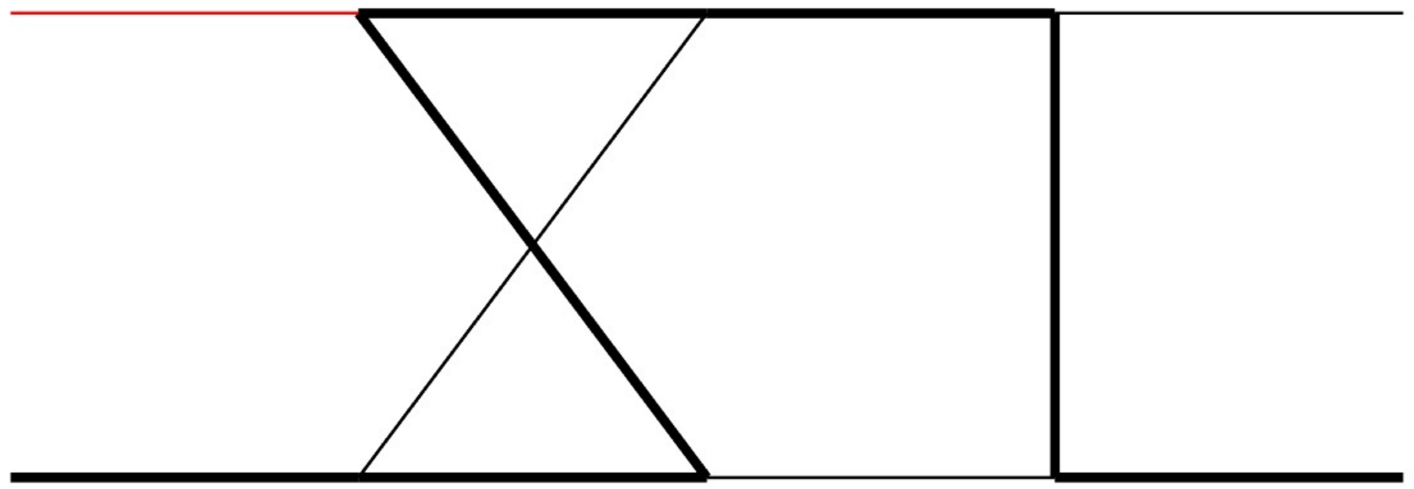


- Small-mass expansion of the full form factor (planar contributions)
- Using differential equations w.r.t. m^2 to set up relations among expansion coefficients
- Solving differential equations w.r.t. other kinematic invariants

Towards sub-leading factorization

Ongoing: analyzing sub-leading corrections in $1 \to 3$ form factors





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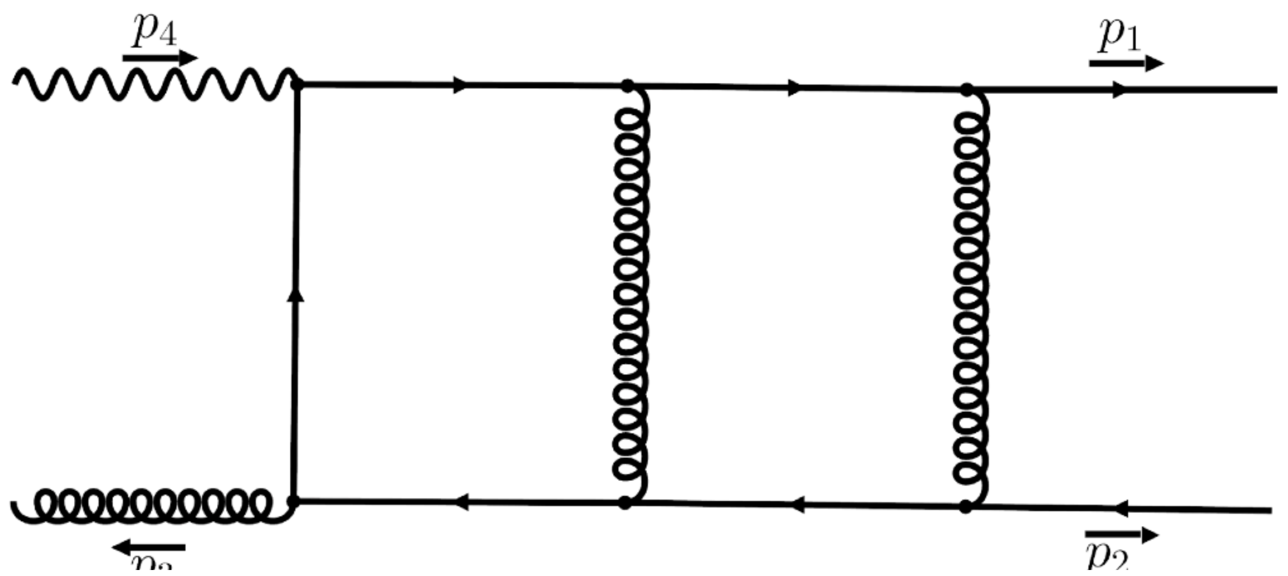
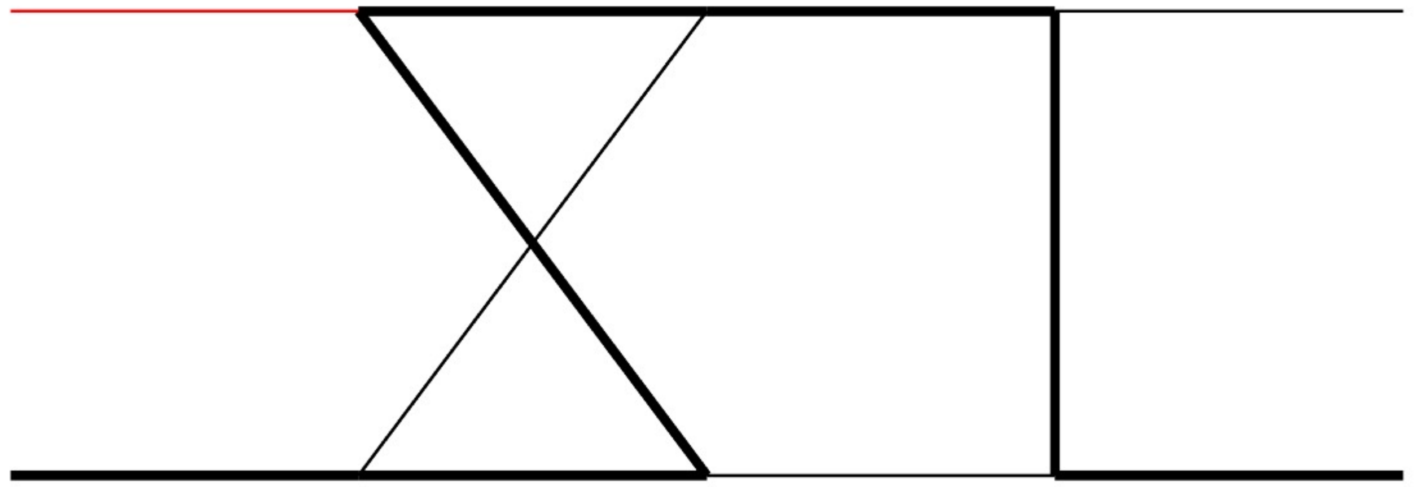
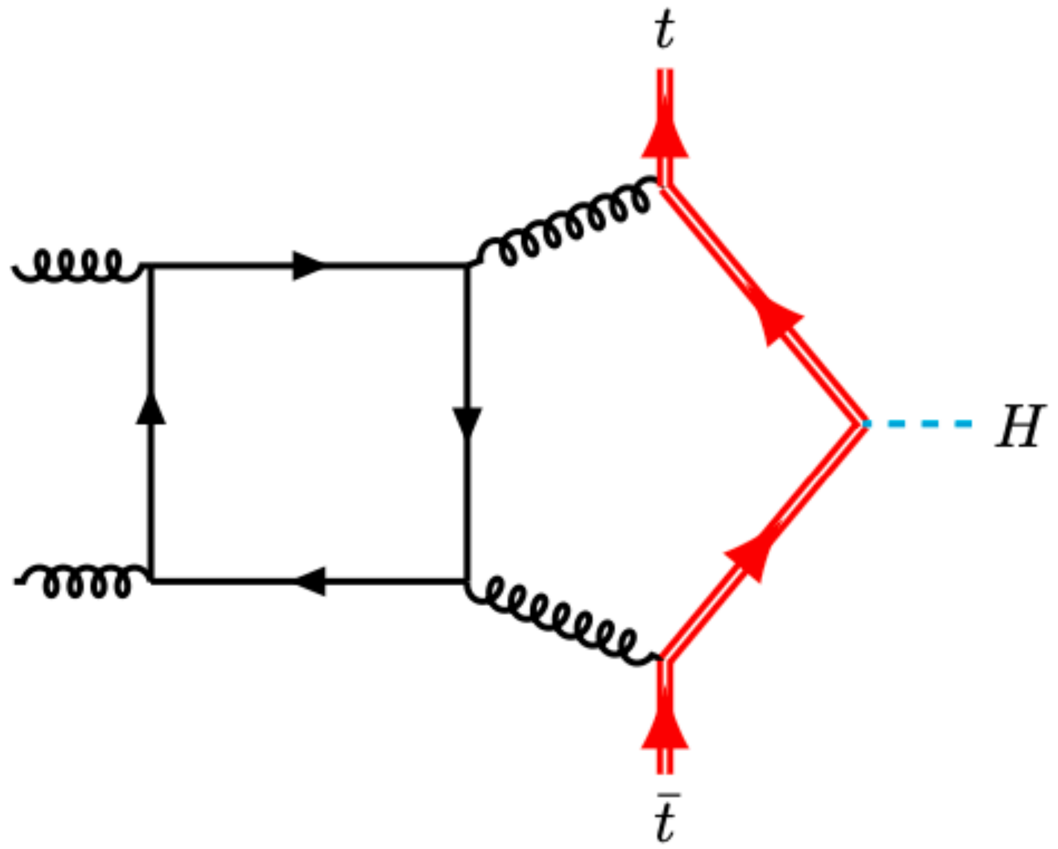
Bottleneck towards non-planar families: integral reduction

Towards sub-leading factorization

Ongoing: analyzing sub-leading corrections in $1 \rightarrow 3$ form factors







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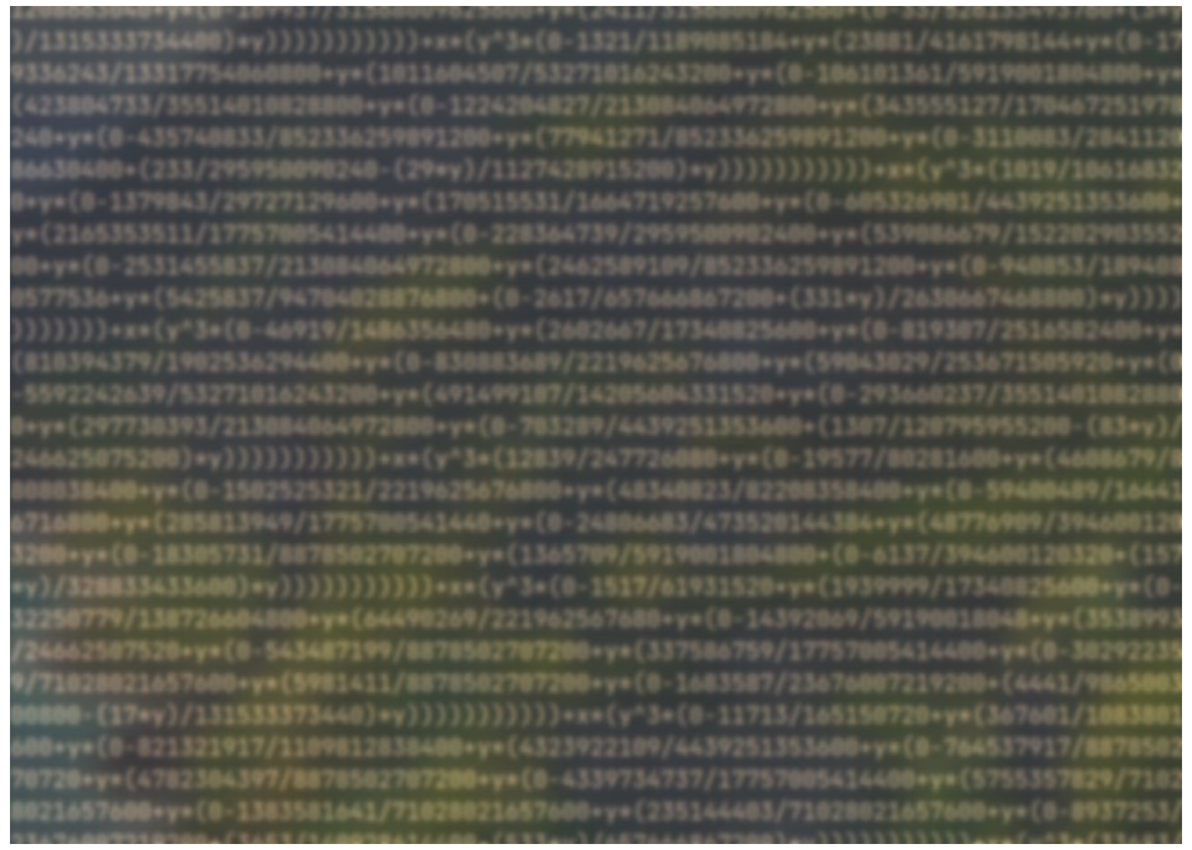
Bottleneck towards non-planar families: integral reduction

Improved reduction techniques are also necessary to tackle more complicated processes

Reduction coefficients as a power expansion

A reduction coefficients is a (very large) rational expression of m^2 and other variables $\vec{x}$

$$C(\vec{x}, m^2) =$$



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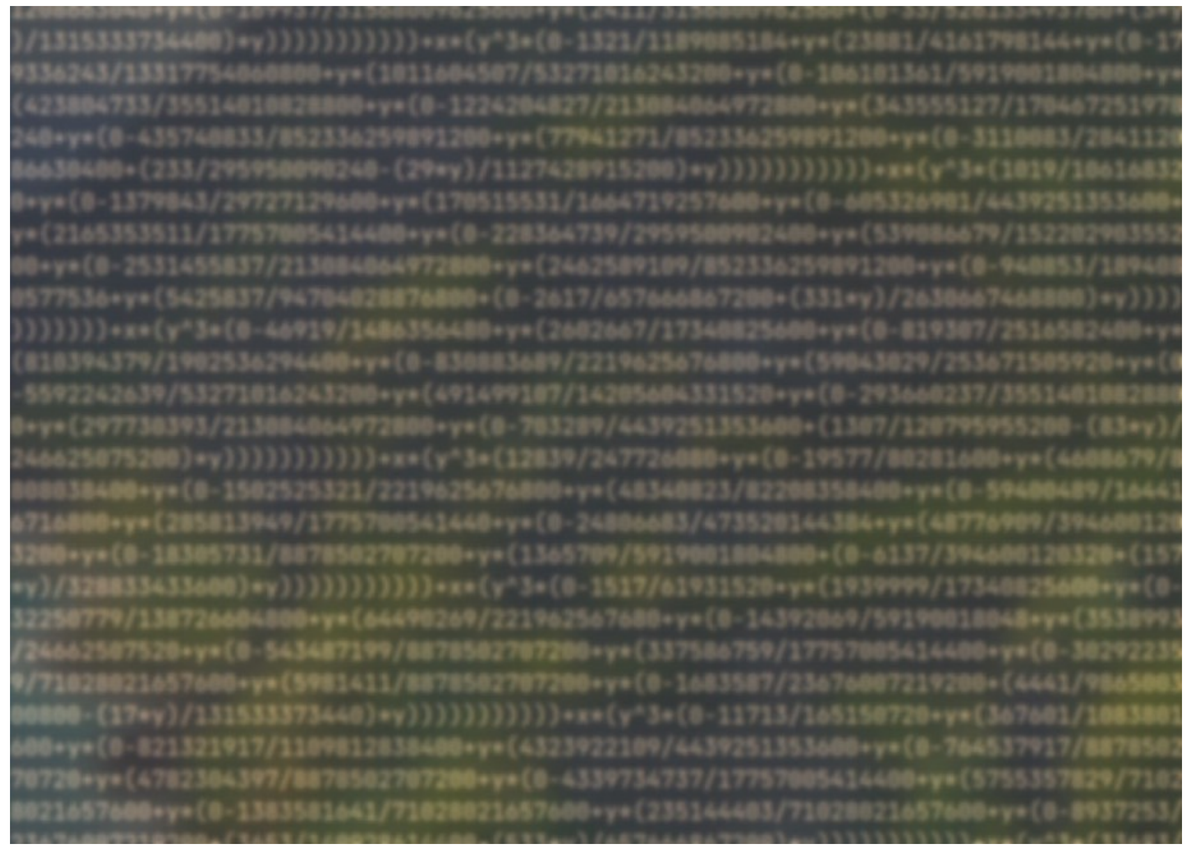
$C(\vec{x}, m^2) =$ 

In practice, we are interested in the first few terms in the expansion

$$C(\vec{x}, m^2) = C_0(\vec{x}) + C_1(\vec{x})\, m^2 + C_2(\vec{x})\, m^4 + \cdots$$

Much simpler expressions

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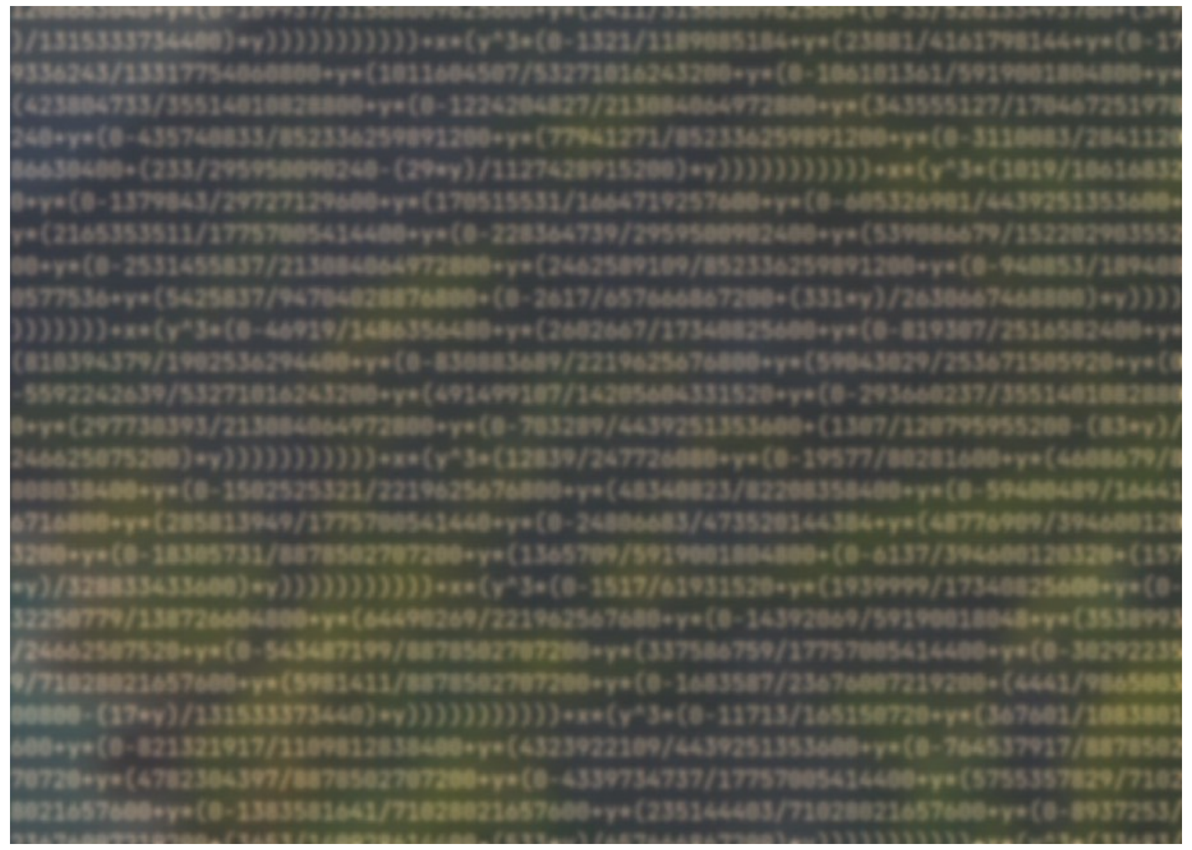
$C(\vec{x}, m^2) =$ 

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Much simpler expressions

Can we obtain $C_i(\vec{x})$ without knowing $C(\vec{x}, m^2)$?

Reduction coefficients as a power expansion

Finite field reconstruction has been widely used to reconstruct reduction coefficients (Kira, FIRE, FiniteFlow, …)

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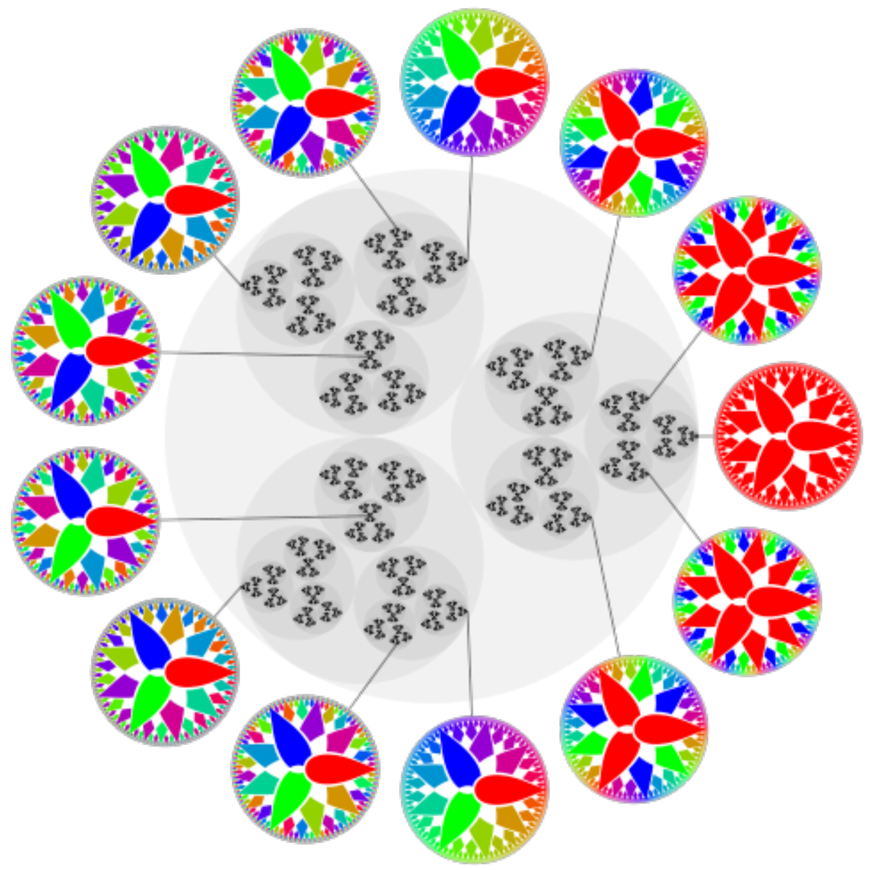
An extension: p-adic numbers



$$\frac{1}{5} = 0.01210121\ldots \ \ (\text{base } 3) = 0\cdot 3^0 + 0\cdot 3^{-1} + 1\cdot 3^{-2} + 2\cdot 3^{-3} + \cdots$$

$$\frac{1}{5} = \ldots 121012102 \ \ (\text{3-adic}) \ = \cdots + 2\cdot 3^3 + 1\cdot 3^2 + 0\cdot 3^1 + 2\cdot 3^0 .$$

Reduction coefficients as a power expansion

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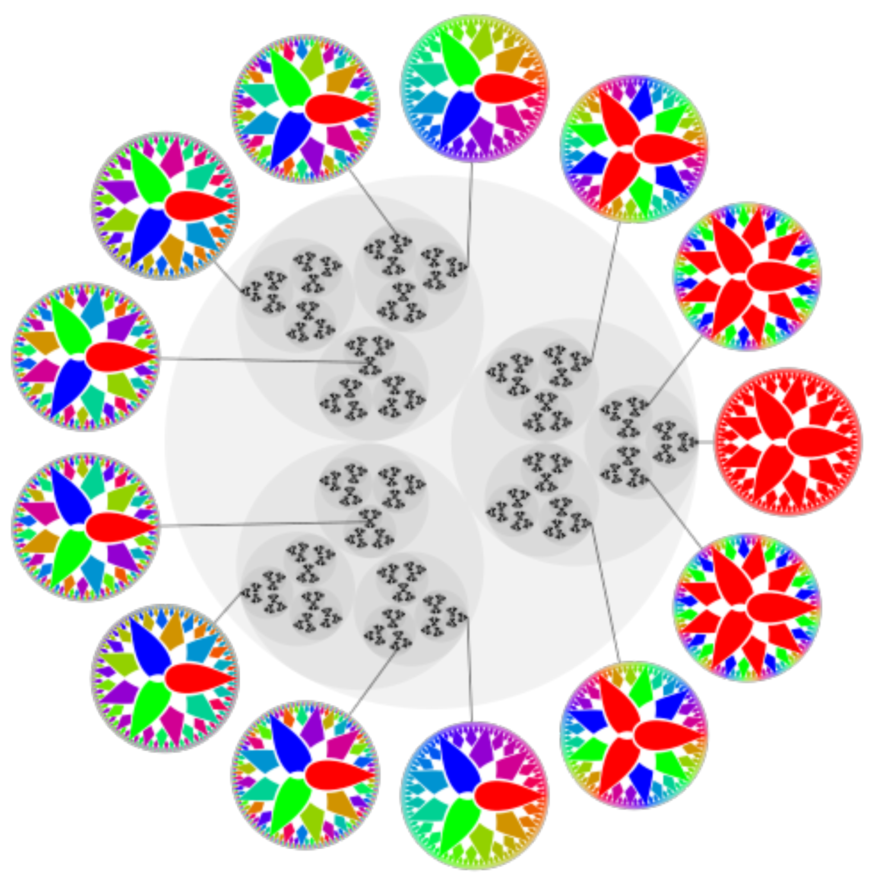

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→ **Similar!**

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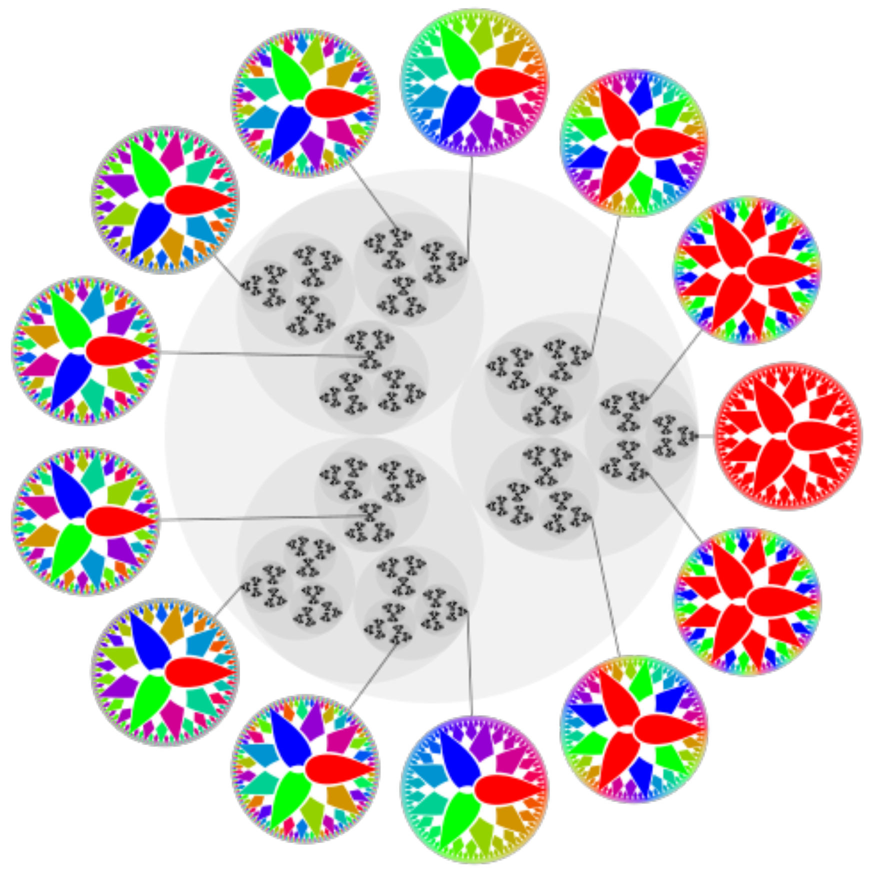

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→ **Similar!**

Initial studies show that p-adic reconstruction of the first 2~3 terms can be orders-of-magnitude faster than finite field reconstruction of the full expression

Can be applied to expansion in ϵ as well…

Reduction with intersection theory

Intersection theory provides a promising approach for integral reduction

Frellesvig et al.: 1901.11510, 1907.02000, 2008.04823

$$\langle\varphi| = \sum_{i=1}^{\nu} c_i \langle e_i| \qquad \langle e_i|d_j\rangle = \delta_{ij} \quad \longrightarrow \quad c_i = \langle\varphi|d_i\rangle$$

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A lot of development for the computation of intersection numbers: successful application to 2-loop 5-point problems (11-layer intersection numbers) Brunello et al.: 2401.01897, 2408.16668

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Reduction with intersection theory is usually formulated in the Baikov representation, we recently reformulated it in the **Feynman parametrization** Lu, Wang, LLY: 2411.05226

- **Fewer variables** → fewer layers in multivariable intersection numbers
- **Simpler polynomials** → easier manipulation

Reduction with intersection theory and branch representation

Lu, Wang, LLY: 2411.05226

It turns out that our reformulation can be combined with a **new representation** modifying the Feynman parametrization in a clever way

Huang, Huang, Ma: 2412.21053

See the talk by Y.-Q. Ma for more details about this new representation

$$I = \int_0^\infty J(\boldsymbol{X})\, \mathrm{d}^n \boldsymbol{X}$$

→ $n = 2L + 1$ for L-loop integrals (independent of the number of external legs)

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2-loop 6-point: **3-layer** intersection numbers instead of **9** (Feynman parameters)

3-loop 5-point: **5-layer** intersection numbers instead of **11** (Feynman parameters)

6 layers "for free"!

Reduction with intersection theory and branch representation

Lu, Wang, LLY: 2411.05226

It turns out that our reformulation can be combined with a **new representation** modifying the Feynman parametrization in a clever way

Huang, Huang, Ma: 2412.21053

See the talk by Y.-Q. Ma for more details about this new representation

$$I = \int_0^\infty J(\boldsymbol{X})\, \mathrm{d}^n \boldsymbol{X}$$

→ $n = 2L + 1$ for L-loop integrals (independent of the number of external legs)

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To appear → **stay tuned!**

Summary

- Factorization is extremely important in heavy flavor physics, including top quark physics
- Bound-state effects near $t\bar{t}$ threshold: important for top quark mass measurement, and confirmed by experiments
- High-energy factorization: resummation of large logarithms and construction of approximate multi-loop amplitudes
- Partial results for high-energy factorization beyond leading power
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  - Reduction coefficients as a power expansion using p-adic numbers
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Thank you!