(non-)Analyticity of Single-Inclusive Hadron Production

SIA@N3LO

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with: Chuan-Qi He，Hongxi Xing，Tong-Zhi Yang，2503.20441

See also: 2006.10534 with Hao Chen, Tong-Zhi Yang, Yu Jiao Zhu
2411.11595 with Zhen Xu

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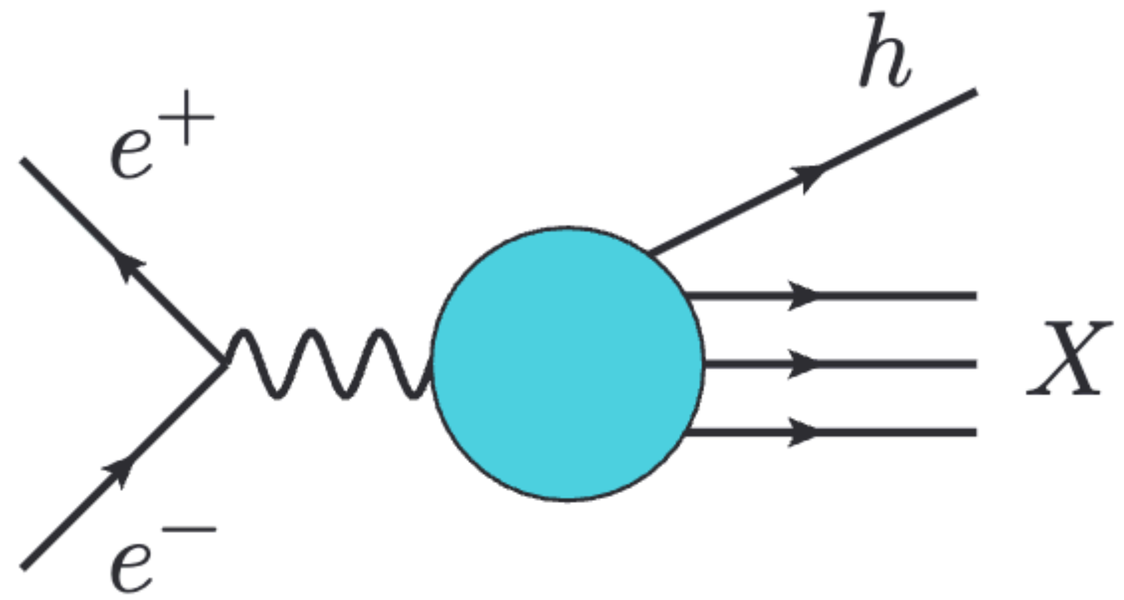
Single-inclusive hadron production

- Fragmentation functions (FFs) are non-perturbative ingredients in factorization theorem connecting partonic and hadronic cross section

- Single-inclusive hadron production:



$$\frac{1}{\sigma_0}\frac{d^2\sigma^h}{dx\,d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)F_T^h(x,q^2) + \frac{3}{4}\sin^2\theta\, F_L^h(x,q^2) + \frac{3}{4}\cos\theta\, F_A^h(x,q^2)$$

$$F^h(x,q^2) = \sum_i \int_x^1 \frac{dz}{z} C_i\left(z, \alpha_s(\mu), \frac{q^2}{\mu^2}\right) D_i^h\left(\frac{x}{z}, \mu^2\right)$$

pert. Wilson coeff. **non-pert. FFs**

- Very active research in recent years: excellent data from BES, LHC, et al.

see e.g. Jun Gao's talk

- Theoretically, closely connects with lightray operators

Existing perturbative results

- The Wilson coefficients can be calculated in pQCD
  - NLO Curci, Furmanski, Petronzio, 1980; Furmanski, Petronzio, 1980; Floratos, Kounnas, Lacaze, 1981
  - NNLO Mitov, Moch, 2006
- The evolution of non-pert. FFs are governed by the timelike DGLAP equation

$$\frac{\partial}{\partial \ln \mu^2} D_i^h\left(x,\mu^2\right) = \sum_j \int_x^1 \frac{dz}{z} P_{ji}\left(z, \alpha_s\left(\mu^2\right)\right) D_j^h\left(\frac{x}{z},\mu^2\right)$$

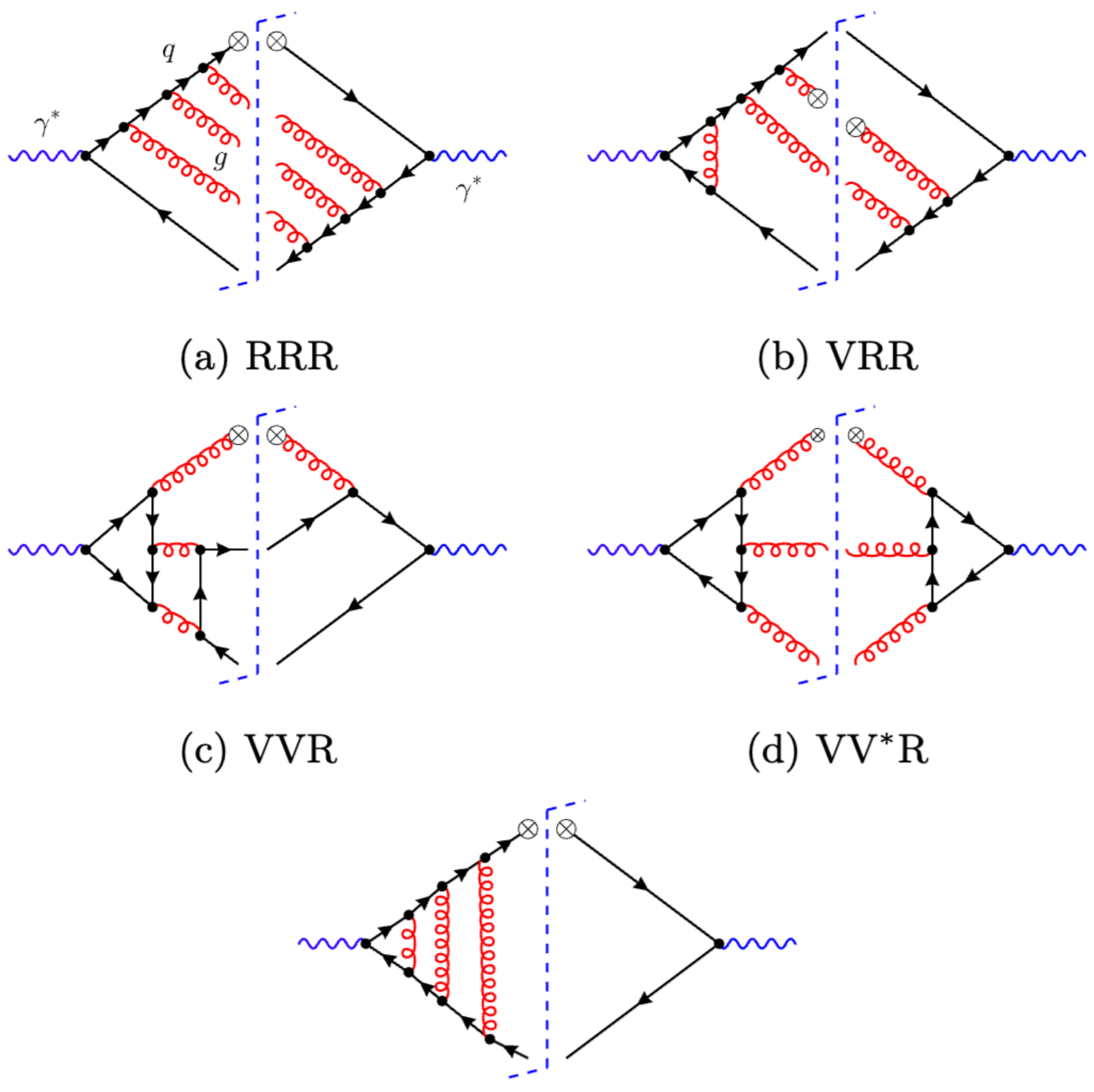
Known to three loops!

Chen et al., 2020; Mitov, Moch, Vogt, 2006; Moch, Vogt, 2008; Almasy, Moch, Vogt, 2011

- Important to push the calculation for Wilson coeff. to N3LO

Diagrams and master integrals

- Standard Feynman diagram methods for calculation



(a) RRR (b) VRR

(c) VVR (d) VV*R

(e) VVV

Many different partonic channels

Complicated and easy to make mistake

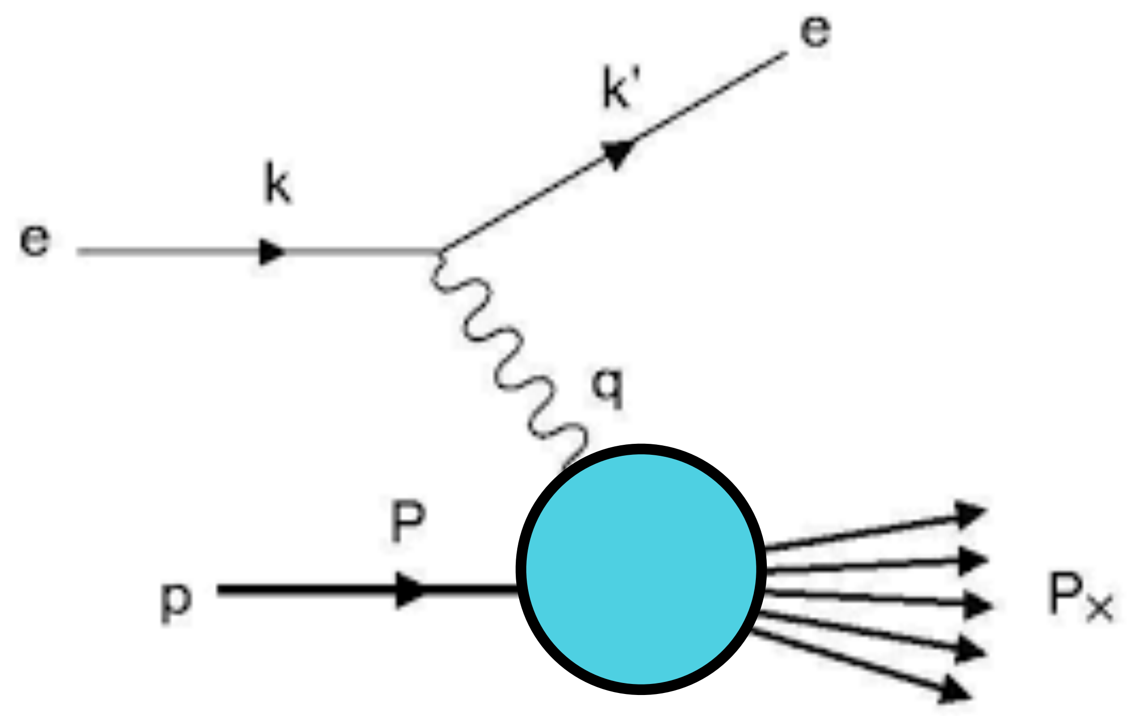
See recent progress in MI calculation: Magerya, Fakeshazy, 2025

In general very difficult. Can we find shortcut to the results?

DIS at N3LO

- A closely related process is DIS

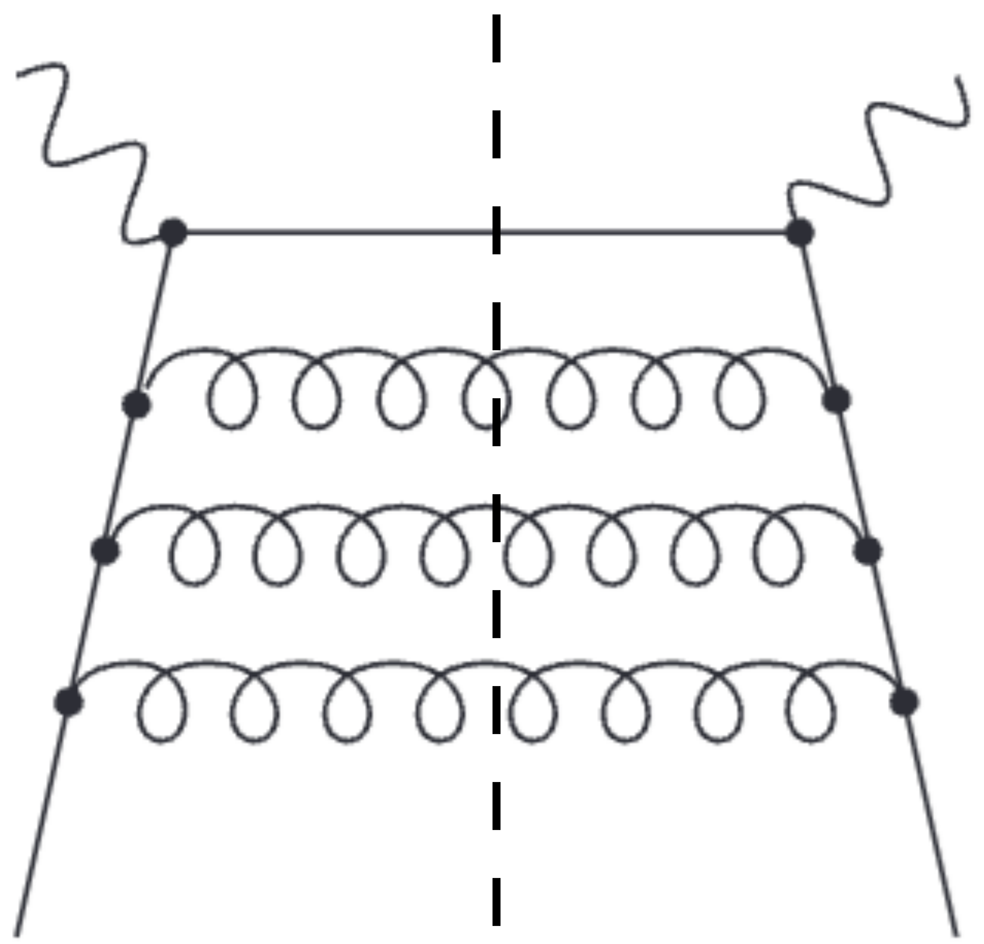
- Structure function factorization into PDFs and perturbative Wilson coeff.



$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\ \mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2\left(x,Q^2\right)}{x} + y^2 F_1\left(x,Q^2\right)\right]$$

$$F_k\left(x,Q^2\right) = \sum_i \int_x^1 \frac{d\xi}{\xi} C_{k,i}\left(\frac{x}{\xi}, \frac{Q^2}{\mu_F^2}, \alpha_s\left(\mu_R^2\right)\right) f_i\left(\xi, \mu_F^2\right)$$

$$\frac{df_i}{d\ln Q^2} = \sum_j \int_x^1 \frac{d\xi}{\xi} P_{ij}\left(\frac{x}{\xi}\right) f_j(\xi)$$

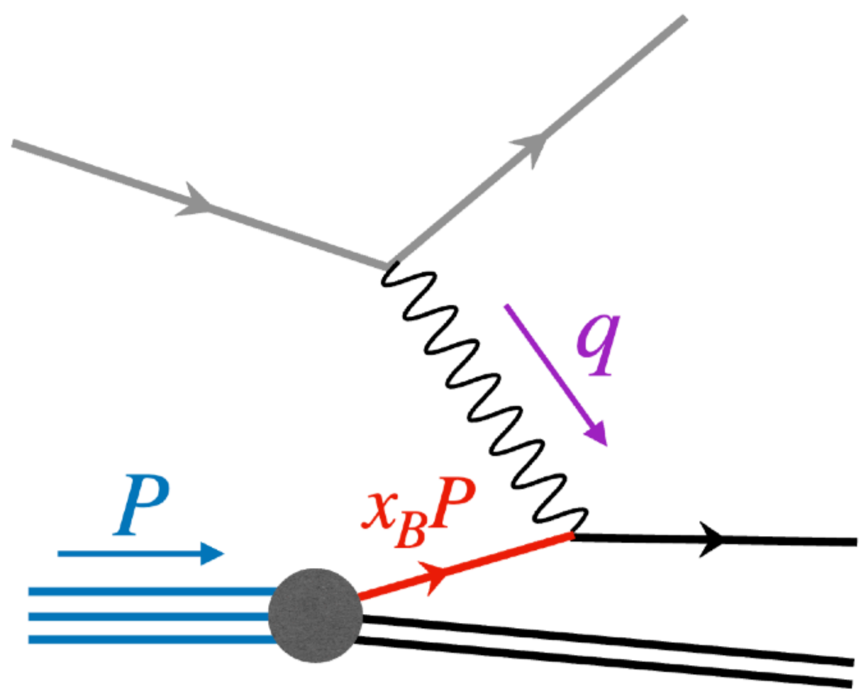


Both Wilson coeff. and space-like DGLAP kernel are known to 3-loops!

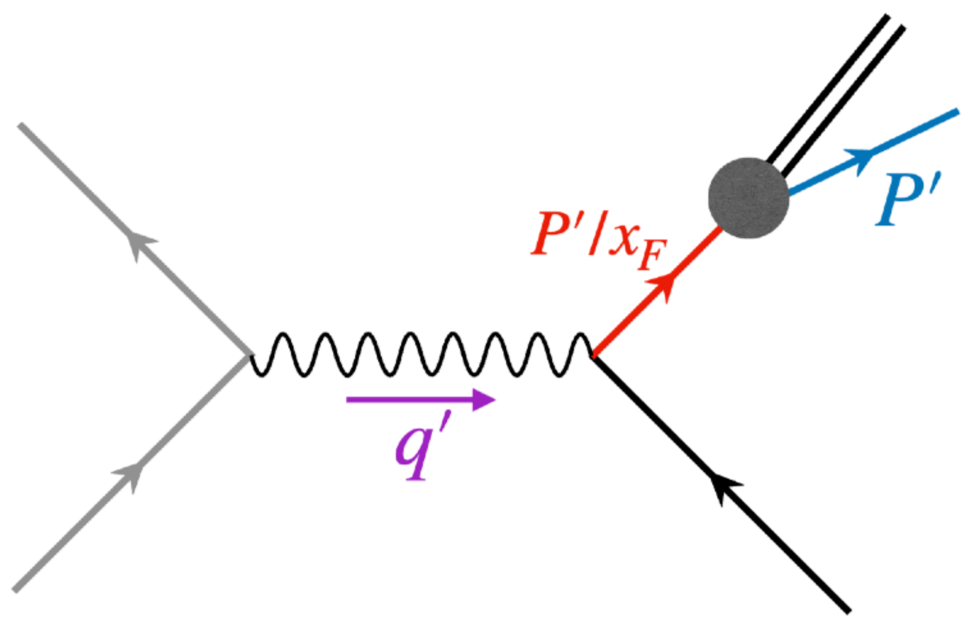
Key simplification: related to imaginary part of forward scattering amplitude

Structure of Wilson Coefficients

$$x_B = \frac{-q^2}{2P \cdot q}$$



(a) DIS



(b) e^+e^-

$$x_F = \frac{2P' \cdot q'}{q'^2}$$

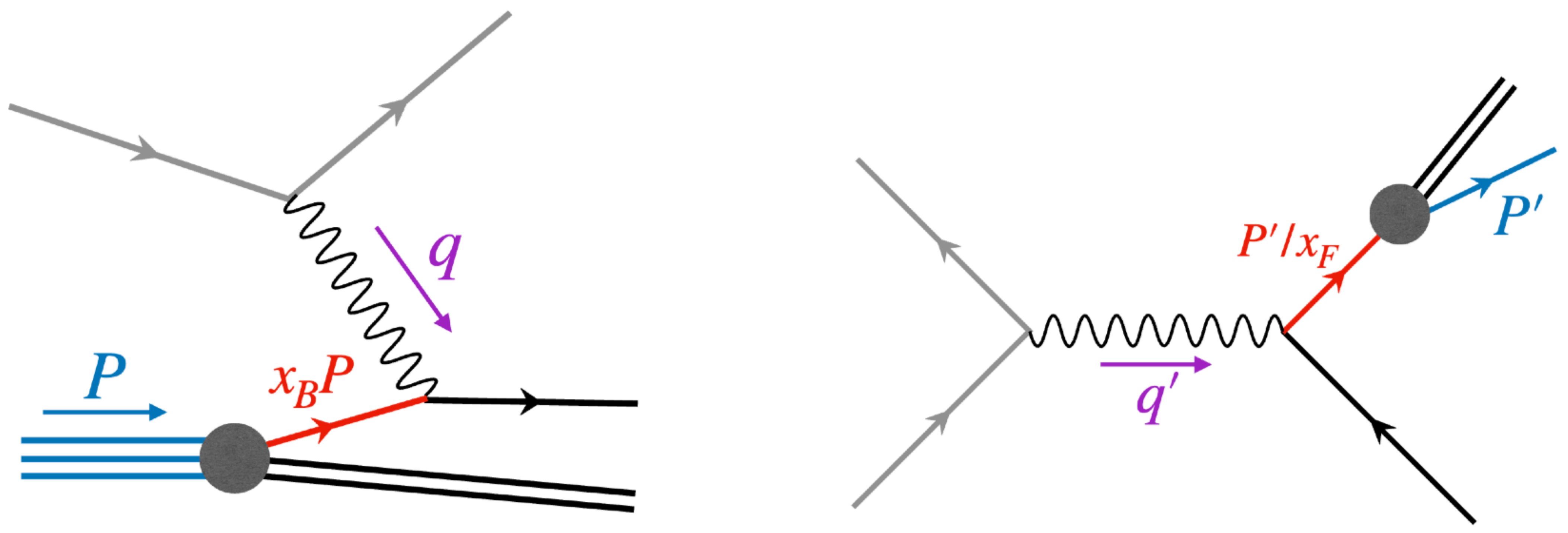
$$C_{\text{DIS}}(x_B, \alpha_s) \qquad C_{\text{SIA}}(x_F, \alpha_s)$$

naive analytic continuation relation: $P \to -P' \,, q \to q' \quad \Rightarrow x_B \to \frac{1}{x_F}$

Can we exploit results for DIS Wilson coeff. to derive results for SIA?

Structure of Wilson Coefficients

$$x_B = \frac{-q^2}{2P \cdot q}$$



(a) DIS   (b) e^+e^-

$$x_F = \frac{2P' \cdot q'}{q'^2}$$

The critical part of the analytic continuation is that of powers of $\ln(1-x_B)$, which is given by

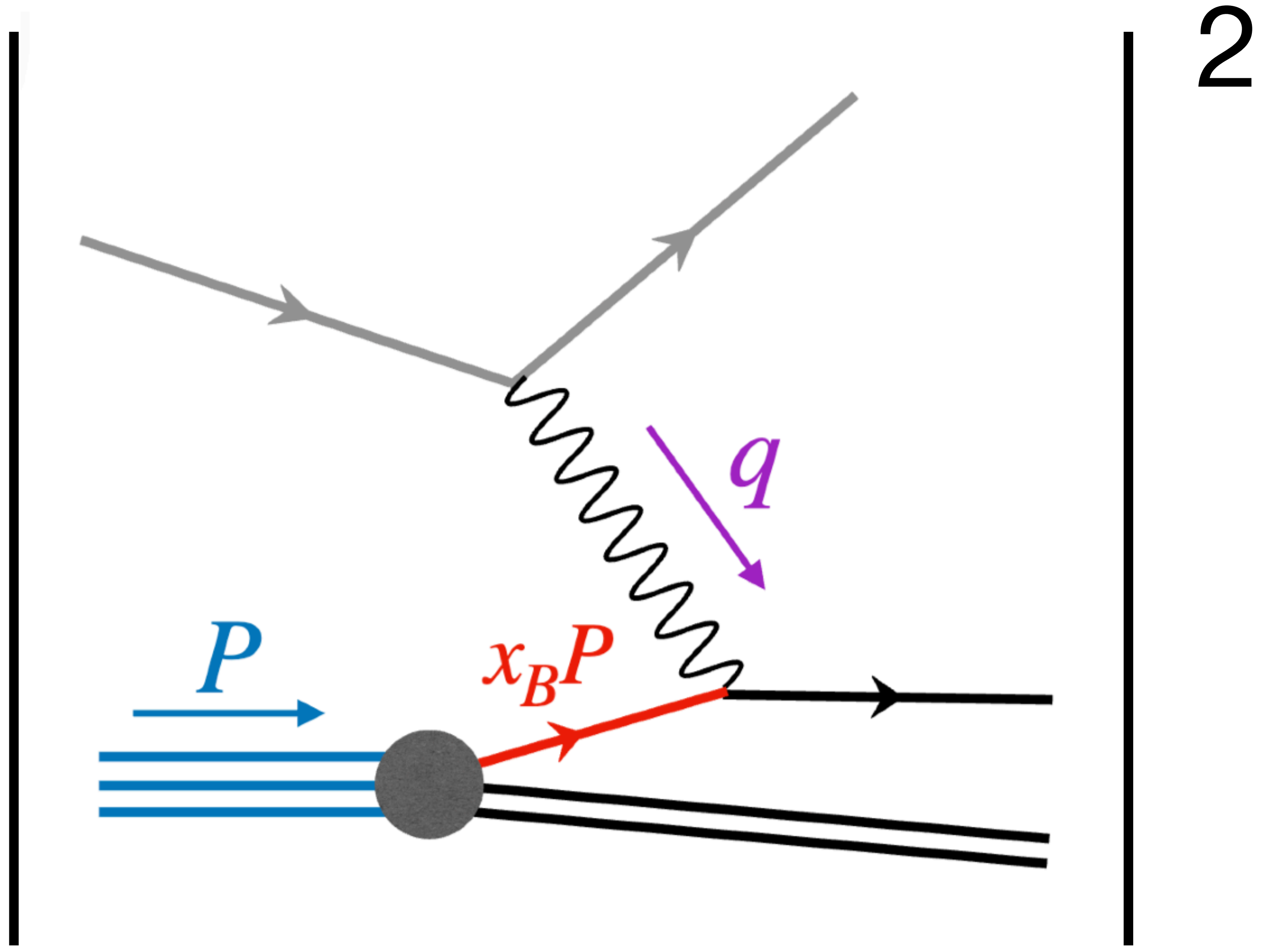
$$x_B \to \frac{1}{x_F}$$

$$\ln(1-x_B) \xrightarrow{AC_\kappa} \ln(1-x_F) - \ln x_F + \kappa\, i\pi \quad \text{with} \quad \kappa = 0 \text{ or } 1\,. \tag{8}$$

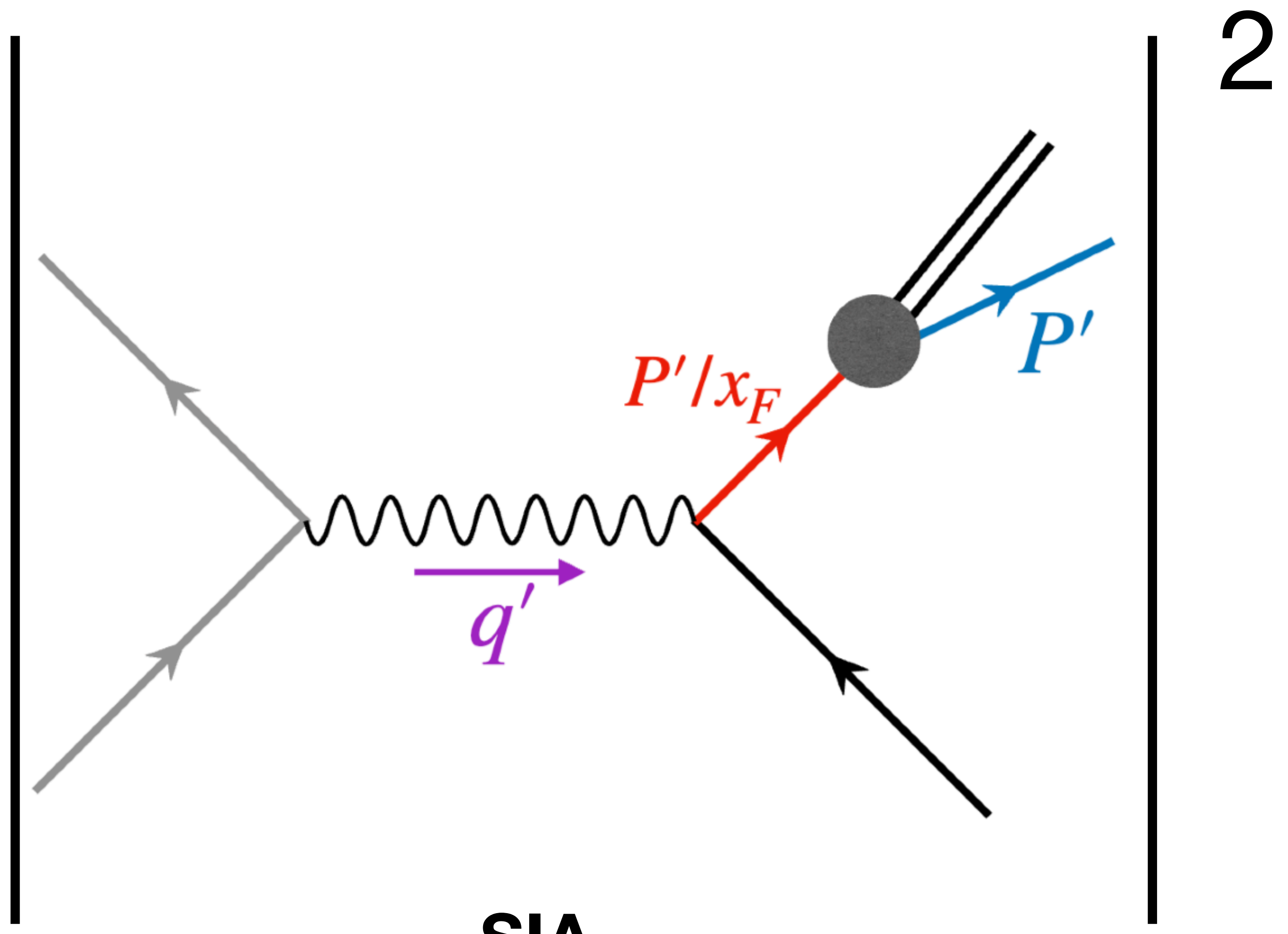
For $\kappa = 1$ the real part is taken in the end. It is not clear at all that beyond NLO Eq. (8) is applicable, in either form, to quantities such as physical evolution kernels instead of to (classes of) Feynman diagrams, see the discussions in Refs. [5,8,16].

Almasy, Moch, Vogt, 2011

Failure of naive analytic continuation

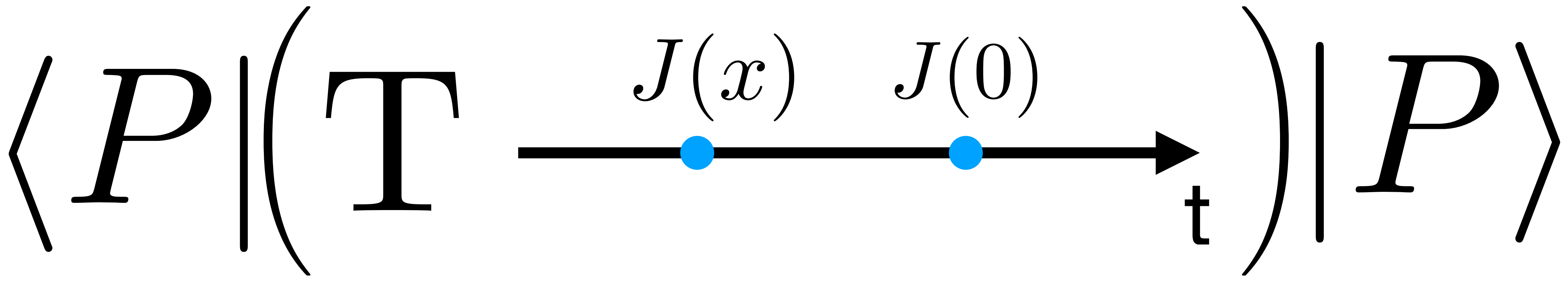


DIS

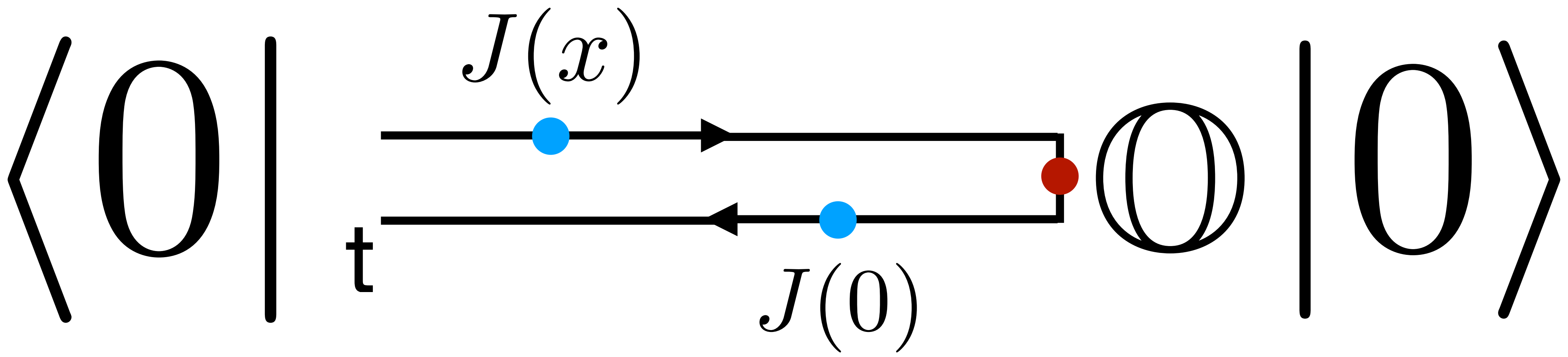


SIA

Wilson coefficient functions are cross section level observable, not analytic!



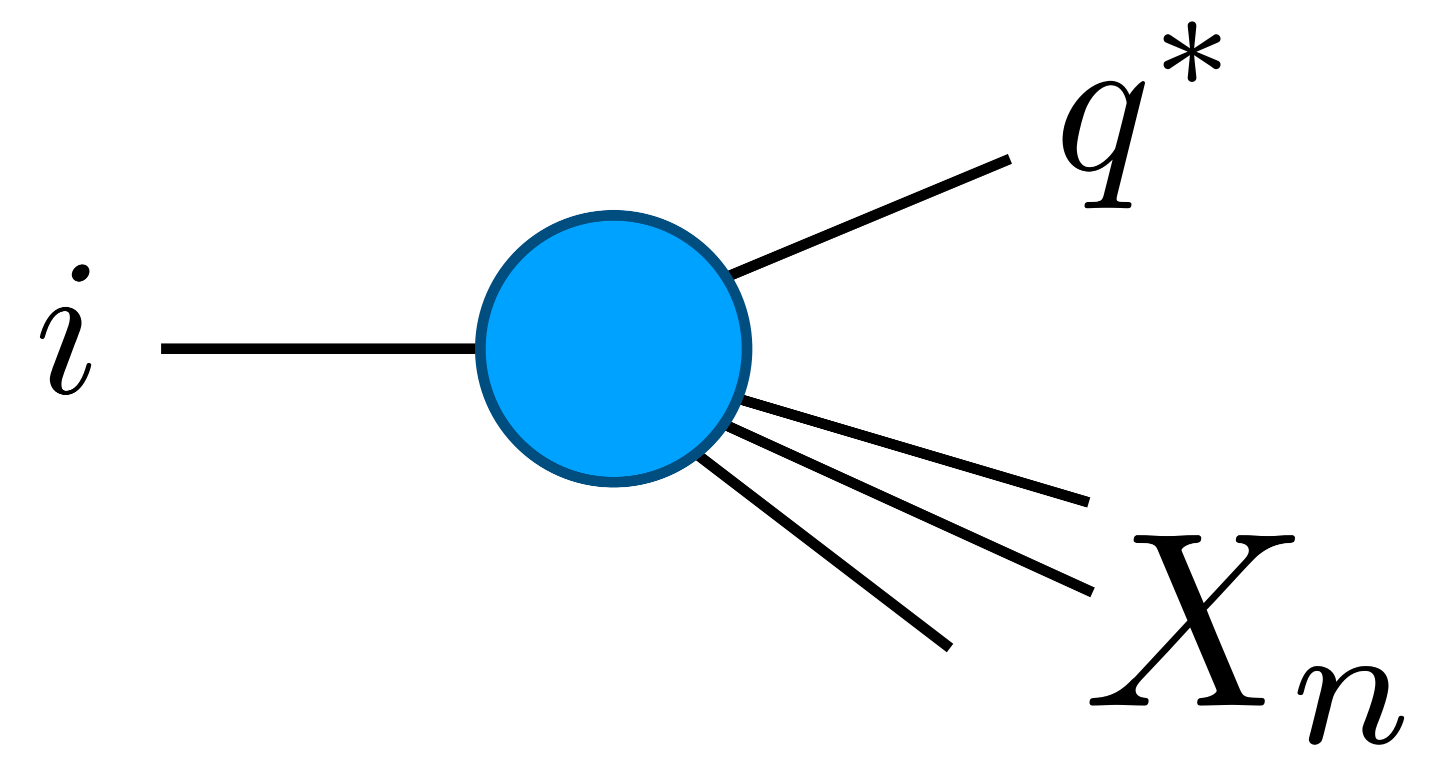
Time-ordered two-point correlator in nucleon state



Wightman 3pt-correlator with Schwinger-Keldysh contour

Solution: Analytic continuation in amplitudes

- Amplitudes are analytic function of external momentum
- Non-analytic dependence on x_B/x_F from singular collinear region
- How to analytic continuation from space-like to time-like?



generalized splitting amplitude

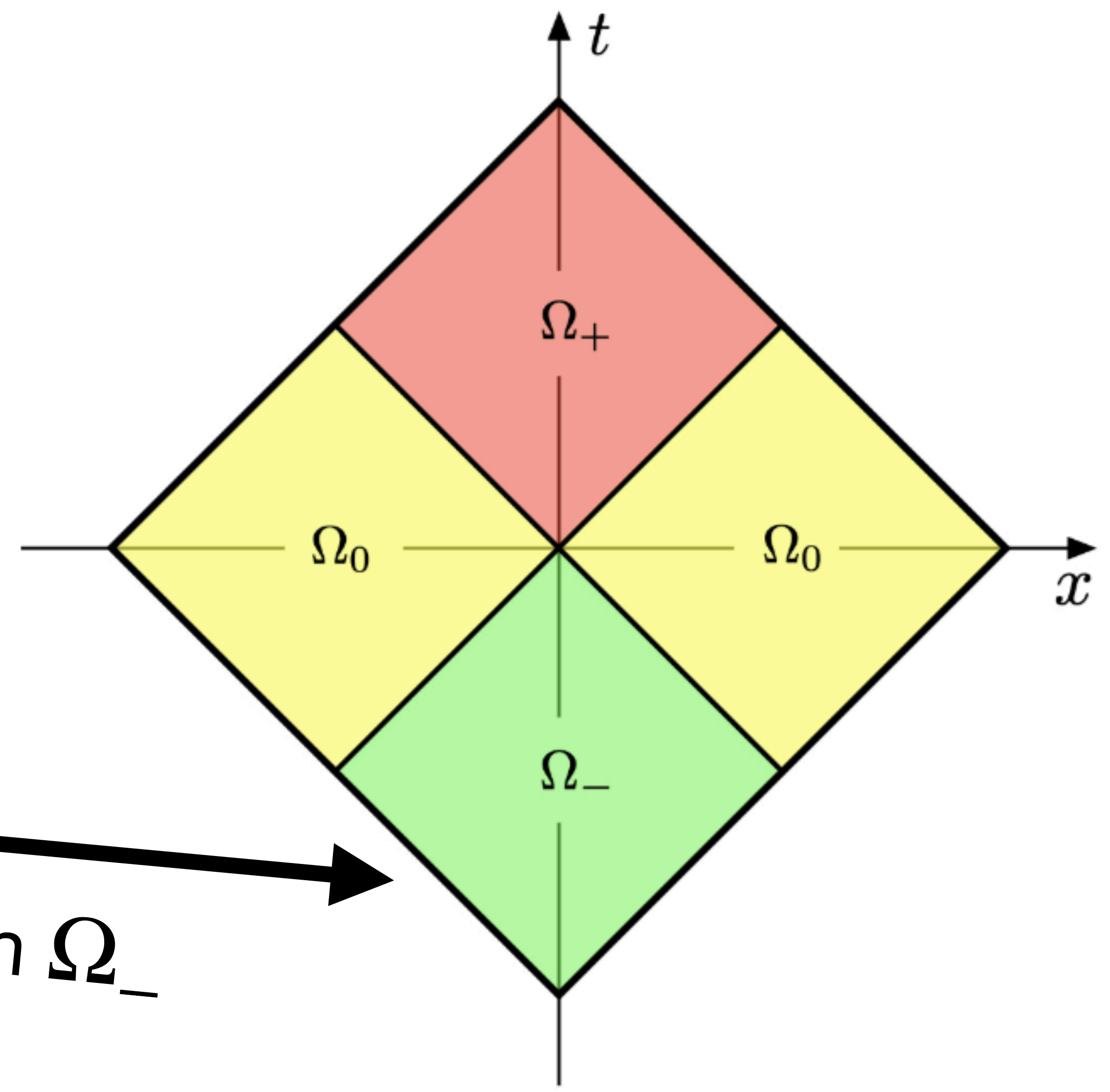
$$\mathbf{Sp}^{\mathrm{S}}_{X_n q^* \leftarrow i} = \int d^d x\, e^{-iP_r \cdot x} \langle X_n | \mathrm{T}\{\chi_n(0) J^i_{P_l}(x)\} | 0 \rangle$$

$$\mathrm{T}\{\chi_n(0) J^i_{P_l}(x)\} = \theta(-x^0) [\chi_n(0), J^i_{P_l}(x)]_{\mp} \pm J^i_{P_l}(x) \chi_n(0)$$

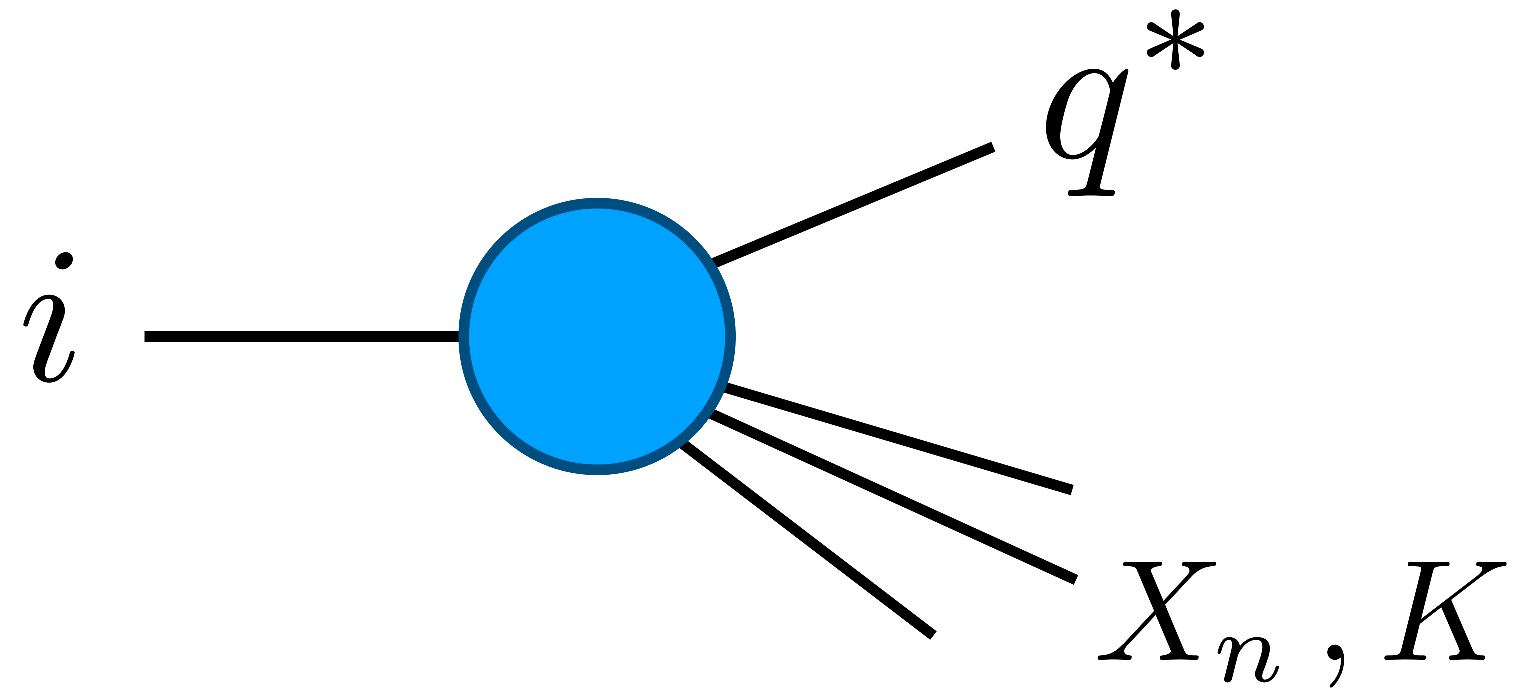
$$\mathbf{Sp}^{\mathrm{S}}_{X_n q \leftarrow i} = \int_{x \in \Omega_-} d^d x\, e^{-iP_r \cdot x} \langle X_n | [\chi(0), J^i_{P_l}(x)]_{\mp} | 0 \rangle$$

vanishing outside lightcone

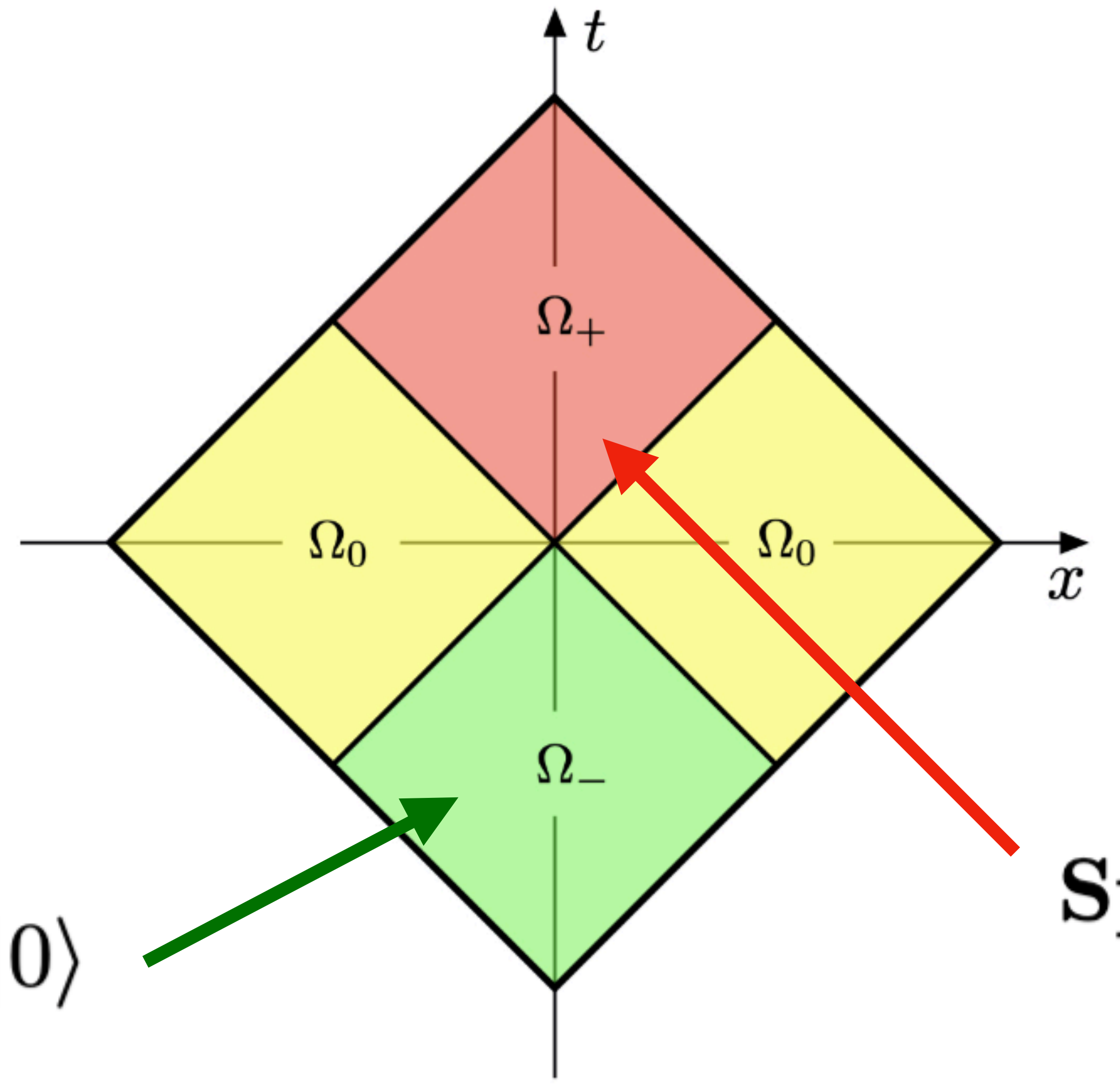
only support in Ω_-

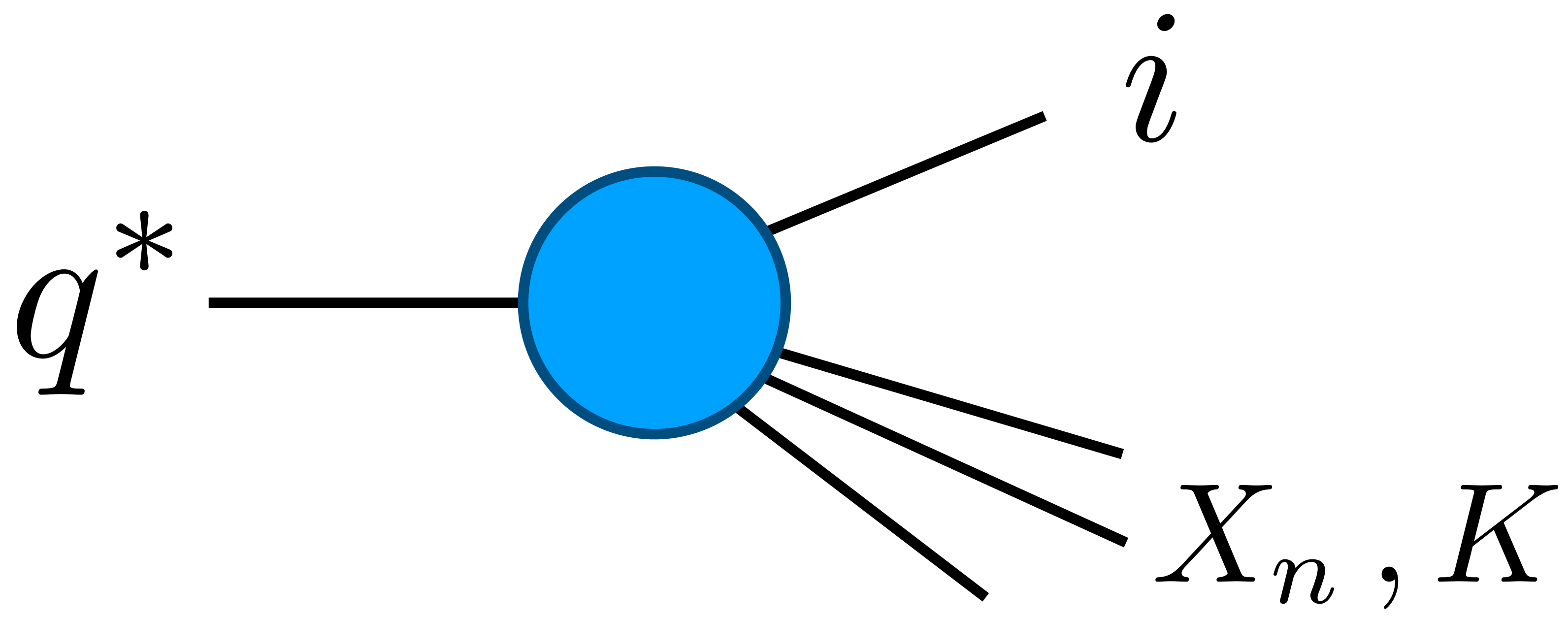


Analytic continuation from space-like infinity



space-like splitting





$$\mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i} = \int_{x\in\Omega_-} d^dx\, e^{-iP_r\cdot x}\langle X_n|[\chi(0), J^i_{P_l}(x)]_\mp|0\rangle$$

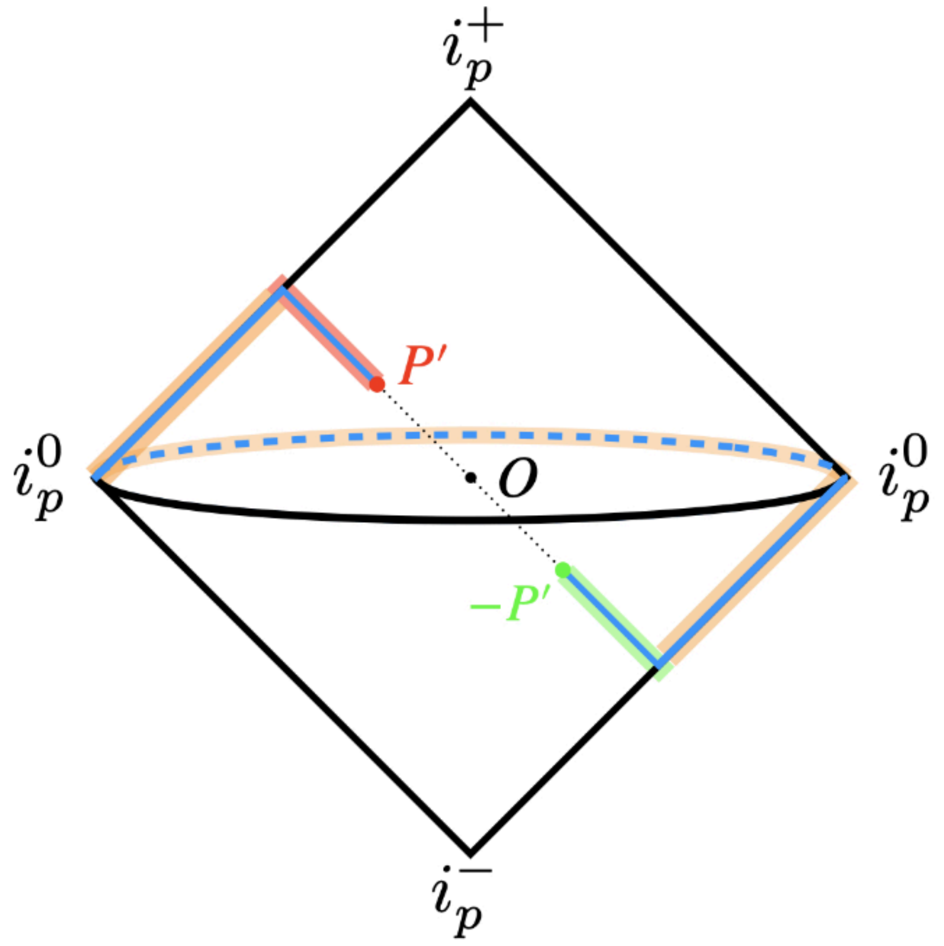
$$\mathbf{Sp}^{\mathrm{T}}_{X_n \bar{\imath}\leftarrow \bar{q}} = \int_{x\in\Omega_+} d^dx\, e^{iP'_r\cdot x}\langle X_n|[J^{\bar{\imath}}_{P'_l}(x), \chi(0)]_\mp|0\rangle$$

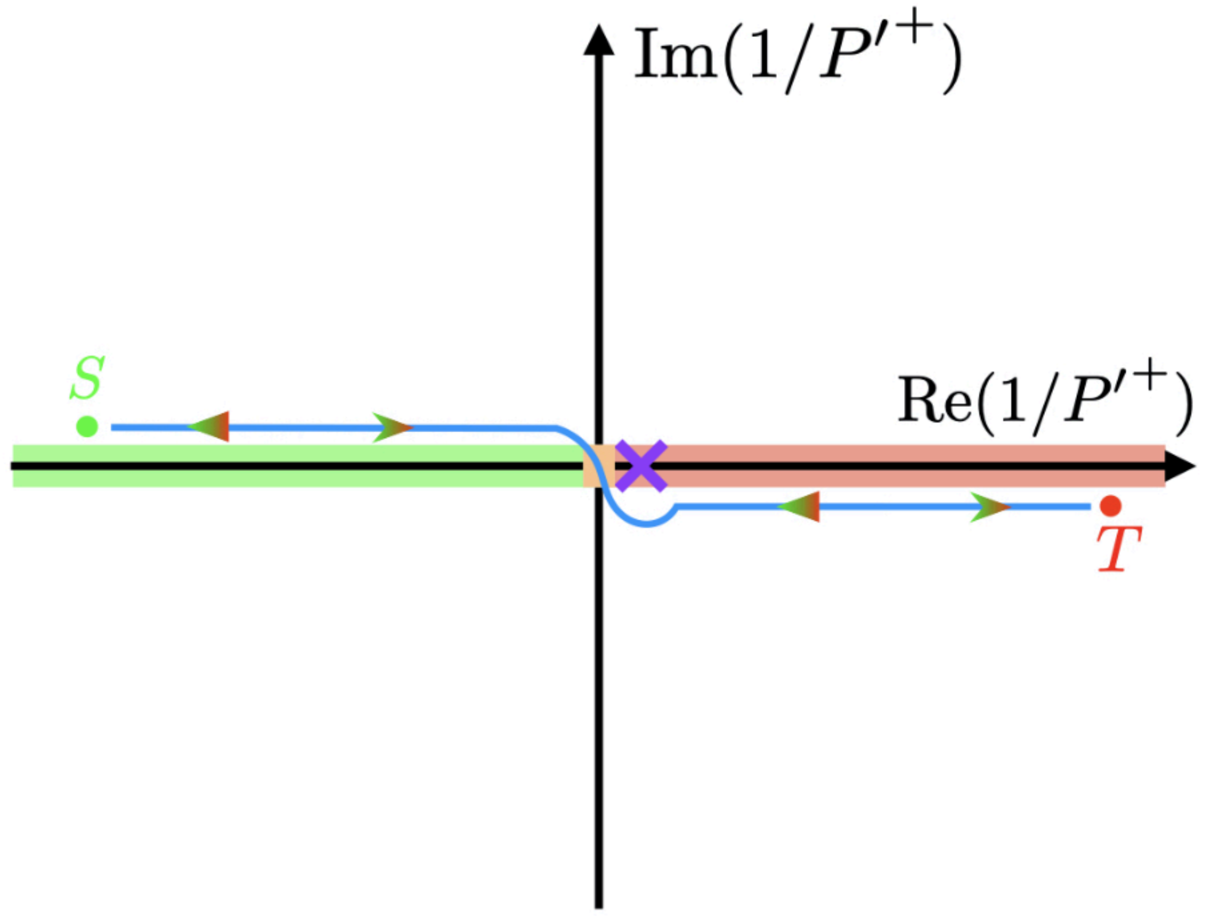
naive continuation

$$P \to -P'\,, q \to q'$$

$$\frac{K^+}{P^+} = 1 - x_B$$

$$\frac{K^+}{P'^+} = \frac{1}{x_F} - 1$$



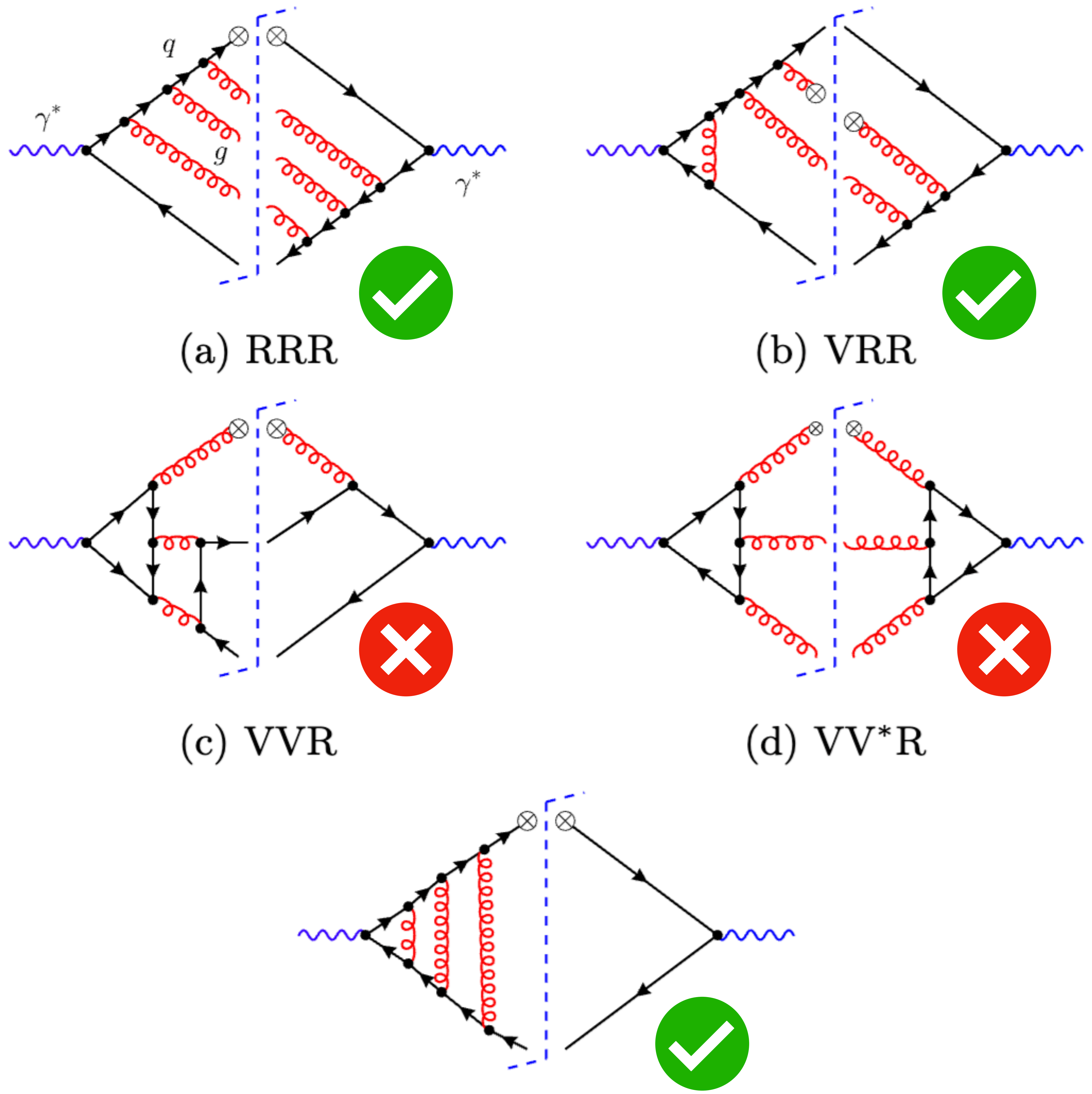


correct continuation

$$\frac{1}{P^+} \to \frac{1}{P^+}e^{i\pi} = \frac{1}{P'^+}$$

$$(1 - x_B) \to (1 - x_B)e^{i\pi} = \left(\frac{1}{x_F} - 1\right)$$

Decomposition of parton contribution



(a) RRR (b) VRR (c) VVR (d) VV*R (e) VVV

$$(1 - x_B) \to (1 - x_B)e^{i\pi} = \left(\frac{1}{x_F} - 1\right)$$

Correct continuation prescription, only work on amplitude

RRR: analytic function of x_F

VRR: non-analytic part pure imaginary

VVV: non-analytic part independent of x_F

VVR and VV*R: non-trivial non-analytic part, must be treated separately

A perturbative prescription

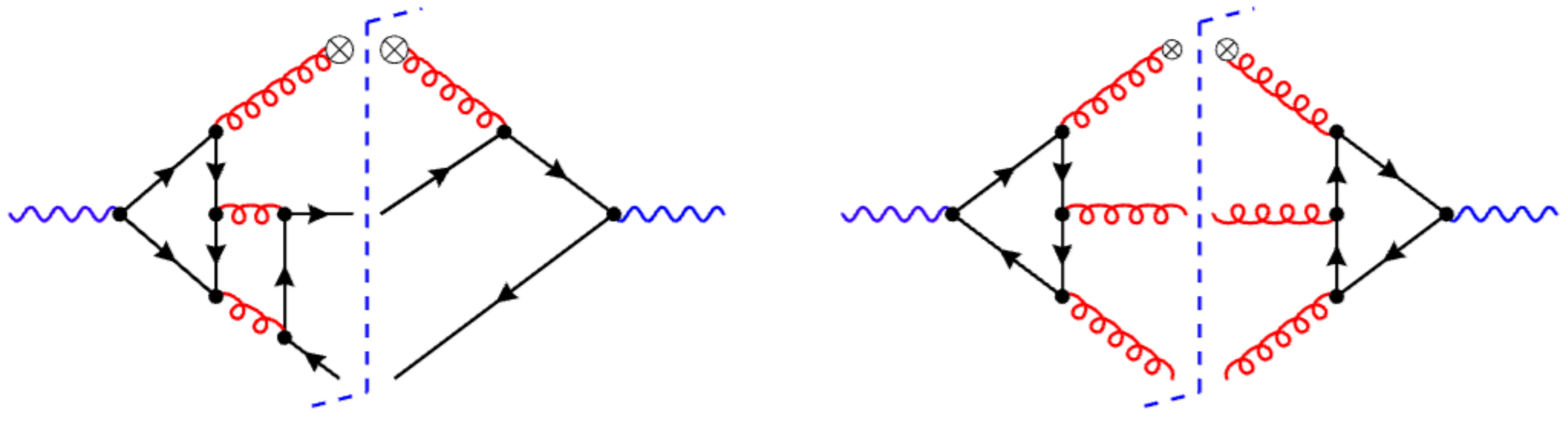
$$d\sigma^{(3),\,\mathrm{SIA}} = \mathrm{Re}\Bigg[\mathcal{AC}\Big(d\sigma^{(3),\,\mathrm{DIS}}\Big)$$

$$- \mathcal{AC}\Big(d\sigma^{(3),\,\mathrm{DIS}}_{\mathrm{VVR}} + \mathrm{c.c.}\Big) - \mathcal{AC}\Big(d\sigma^{(3),\,\mathrm{DIS}}_{\mathrm{VV^*R}}\big|_{x_B^* \to x_B}\Big)$$

$$+ \Big\{\mathcal{AC}\Big(d\sigma^{(3),\,\mathrm{DIS}}_{\mathrm{VVR}}\Big) + \mathrm{c.c.}\Big\} + \mathcal{AC}\Big(d\sigma^{(3),\,\mathrm{DIS}}_{\mathrm{VV^*R}}\Big)\Bigg]$$

Subtract incorrect continuation contribution

Add back to correct results



(c) VVR (d) VV*R

$$d\sigma^{\mathrm{SIA}}(x_F) = A(x_F, \alpha_s) + B(\alpha_s)\delta(1 - x_F)$$

normal function and plus distribution in x_F

cannot be determined from analytic continuation

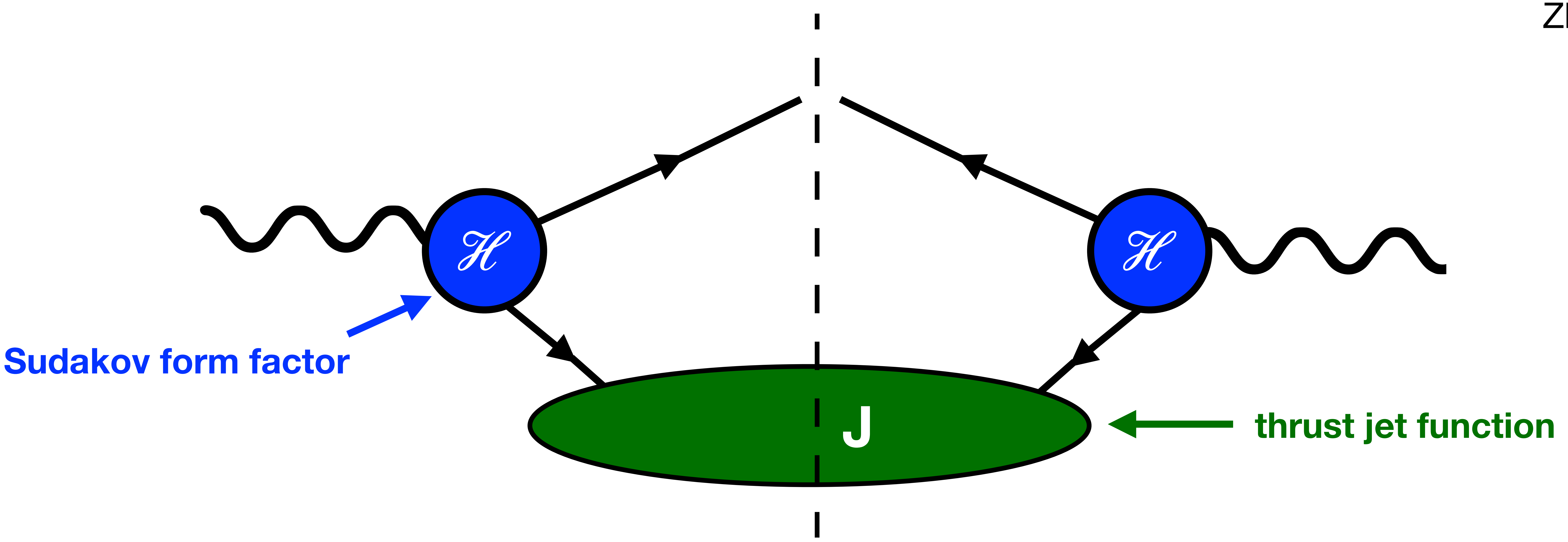
Sum rule:

$$\int_0^1 dx\, \sigma^{\mathrm{SIA}}(x) = \sigma_{\mathrm{tot}}$$

B from threshold resummation

- In the threshold limit, x_F->1, can derive a factorization theorem for SIA

Zhen Xu, HXZ, 2024



known to three loops:

Bruser, Ze Long Liu, Stahlhofen, 2018

$$\frac{1}{\sigma_0}\frac{d\sigma_T(x,Q^2)}{dx} = \sum_q \int_x^1 \frac{d\eta}{\eta}\, C_q^{(\mathrm{SIA})}\left(\frac{x}{\eta}, Q^2, \mu^2, \mu_F^2\right) D_q(\eta, \mu_F^2)$$

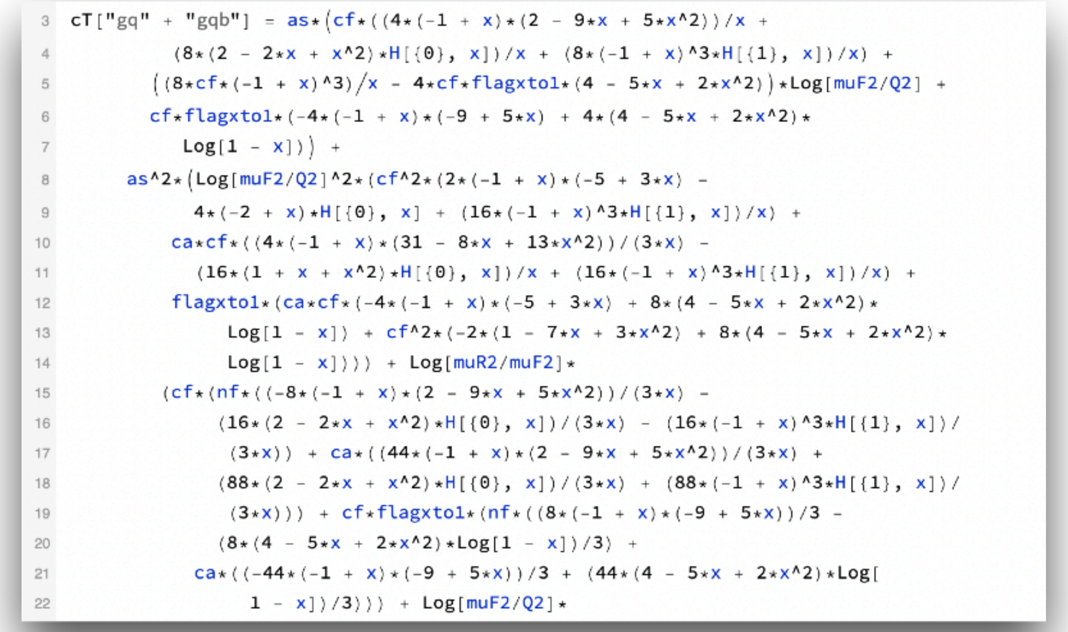
$B\delta(1-x_F)$ determine from factorization formula agrees completely with the one from sum rule

A very strong check

The explicit N3LO results

- Analytic results completely written in harmonic polylogarithms

Ancillary files (details):
- SIA_LN3LO.m
- SIA_TN3LO.m
- readme.txt

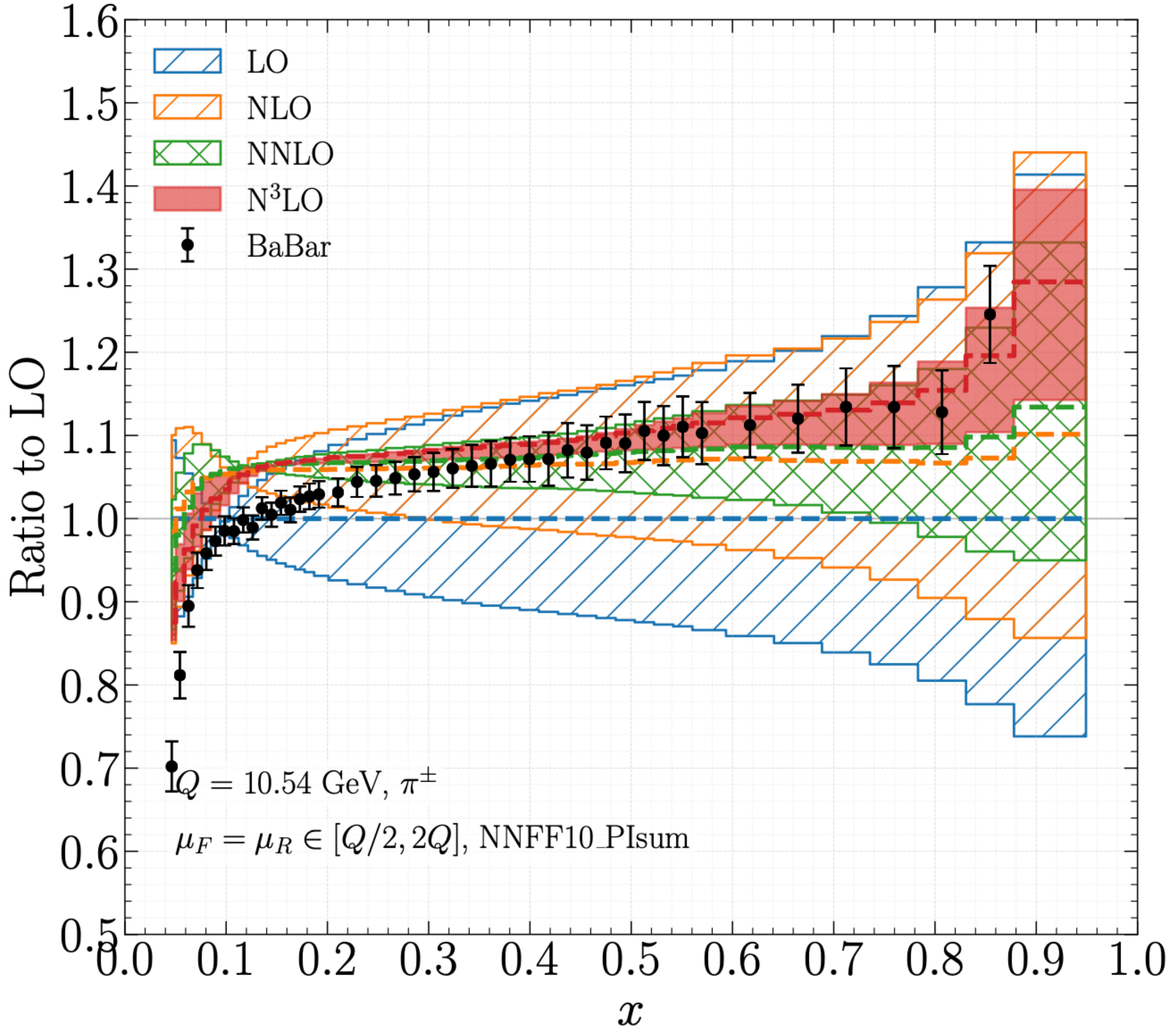


```
cT["gq" + "gqb"] = as*(cf*((4*(-1 + x)*(2 - 9*x + 5*x^2))/x +
        (8*(2 - 2*x + x^2)*H[{0}, x])/x + (8*(-1 + x)^3*H[{1}, x])/x) +
      ((8*cf*(-1 + x)^3)/x - 4*cf*flagxto1*(4 - 5*x + 2*x^2))*Log[muF2/Q2] +
      cf*flagxto1*(-4*(-1 + x)*(-9 + 5*x) + 4*(4 - 5*x + 2*x^2)*
        Log[1 - x])) +
    as^2*(Log[muF2/Q2]^2*(cf^2*(2*(-1 + x)*(-5 + 3*x) -
          4*(-2 + x)*H[{0}, x] + (16*(-1 + x)^3*H[{1}, x])/x) +
        ca*cf*((4*(-1 + x)*(31 - 8*x + 13*x^2))/(3*x) -
          (16*(1 + x + x^2)*H[{0}, x])/x + (16*(-1 + x)^3*H[{1}, x])/x) +
        flagxto1*(ca*cf*(-4*(-1 + x)*(-5 + 3*x) + 8*(4 - 5*x + 2*x^2)*
            Log[1 - x]) + cf^2*(-2*(1 - 7*x + 3*x^2) + 8*(4 - 5*x + 2*x^2)*
            Log[1 - x]))) + Log[muR2/muF2]*
       (cf*(nf*((-8*(-1 + x)*(2 - 9*x + 5*x^2))/(3*x) -
            (16*(2 - 2*x + x^2)*H[{0}, x])/(3*x) - (16*(-1 + x)^3*H[{1}, x])/
             (3*x)) + ca*((44*(-1 + x)*(2 - 9*x + 5*x^2))/(3*x) +
            (88*(2 - 2*x + x^2)*H[{0}, x])/(3*x) + (88*(-1 + x)^3*H[{1}, x])/
             (3*x))) + cf*flagxto1*(nf*((8*(-1 + x)*(-9 + 5*x))/3 -
            (8*(4 - 5*x + 2*x^2)*Log[1 - x])/3) +
          ca*((-44*(-1 + x)*(-9 + 5*x))/3 + (44*(4 - 5*x + 2*x^2)*Log[
              1 - x])/3))) + Log[muF2/Q2]*
```

- N3LO corrections are substantial

- Necessary to further reduce scale uncertainties

- Indicate better agreement with data?



Conclusion

- We have obtained N3LO prediction for single-inclusive hadron production
- Method: amplitude level analytic continuation
- Useful for precision determination fragmentation functions
- Can be used to compute one-point energy correlator at N3LO
- Future: understand analytic continuation at cross section level

Thank you very much!