第七届全国重味物理与量子色动力学研讨会

Form Factor / Wilson Line Duality and OPE

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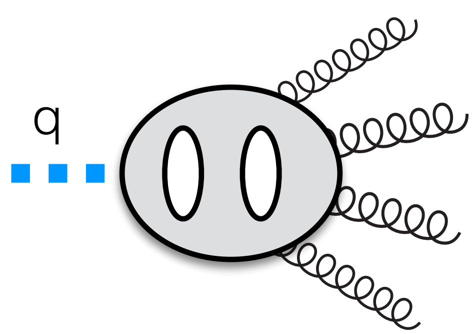
Based on: Y. Guo, L. Wang, GY, Commun.Theor.Phys. 77 (2025) [2209.06816]； Y. Guo, J. Shen, GY to appear

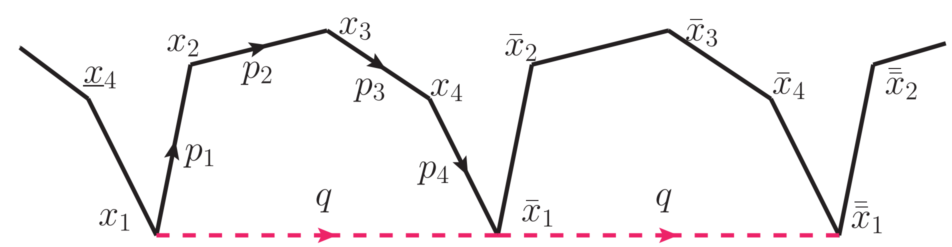
2025.04.18-22，南京

Plan

A perturbative story: bootstrapping a two-loop form factors

A non-perturbative story: FF/WL duality and OPE





We consider N=4 SYM theory (maximally SUSY 4D gauge theory)

We consider the light-like limit: $q^2 \to 0$

Plan

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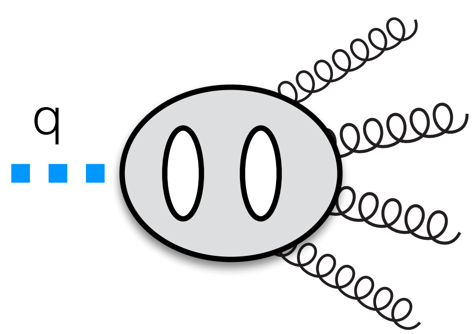

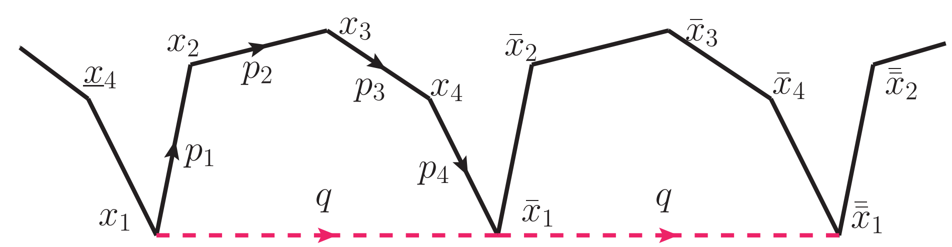

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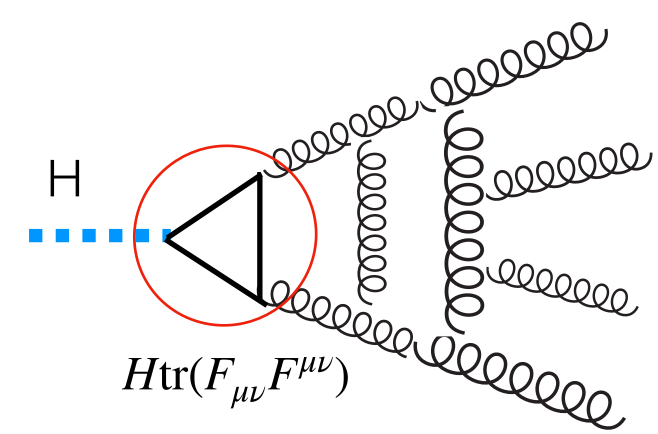
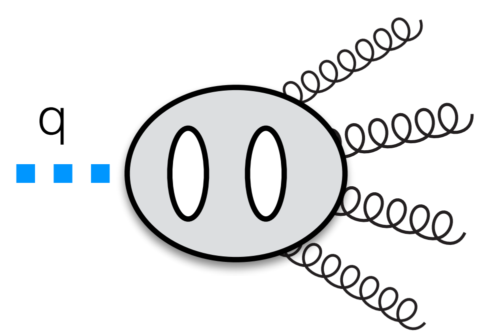
We consider the light-like limit: $q^2 \to 0$

In analogy with Higgs-4 Gluons amplitudes:



A bootstrap computation

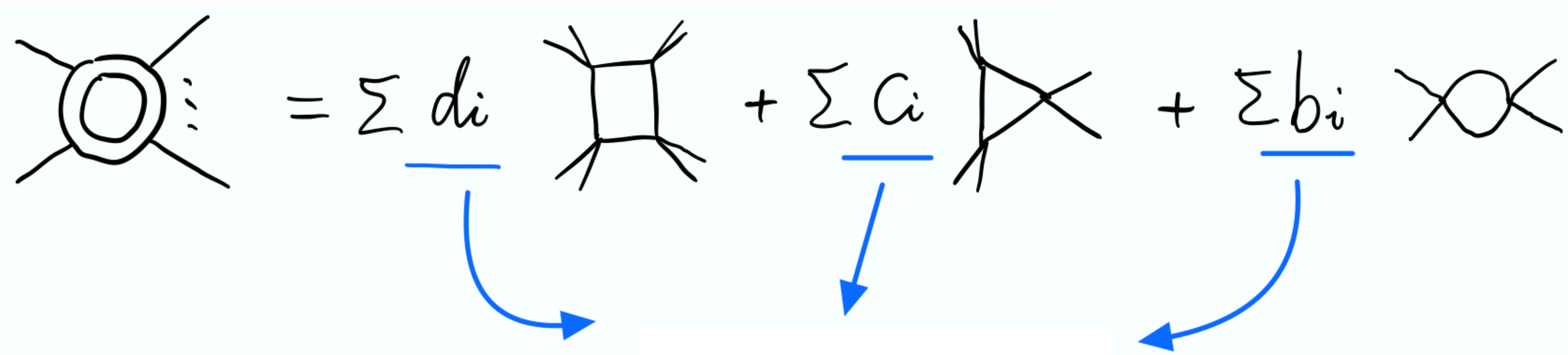


Bootstrap strategy

Based on the fact:

any amplitude or form factor can be expanded in a set of integral basis

Consider one-loop amplitudes:



What we really want

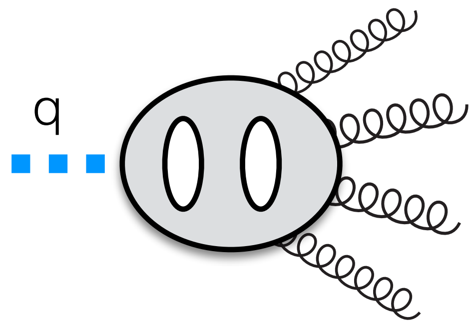
Ansatz in master integrals → **Physical constraints** → Solution of coefficients

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

$$\mathscr{F}^{(l)} = \sum_i C_i I_i^{(l)}$$

Master-integral bootstrap Guo, Wang, GY (2021)

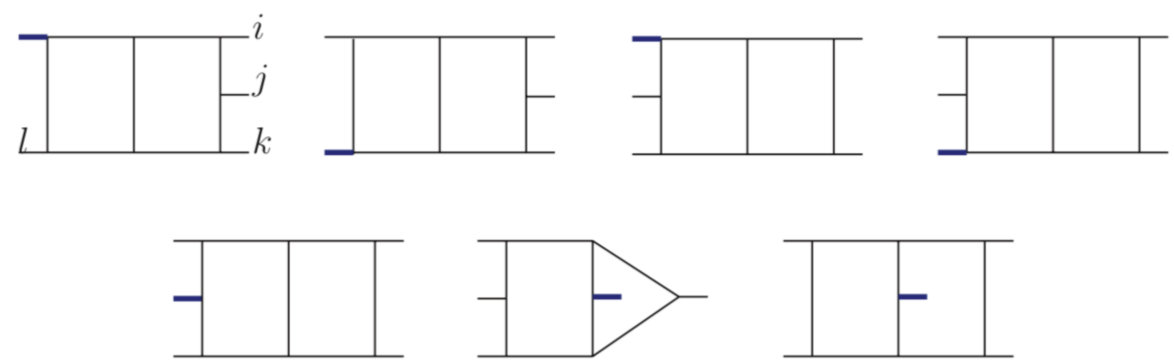
Two-loop ansatz



$$F_4^{(2)} = F_4^{(0)} \sum_{i=1}^{590} C_i \, I_i^{(2)}$$

$I_i^{(2)}$

Known uniformly-transcendental master integrals



Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, Lo Presti, Mitev, Page, Papadopoulos, Tommasini, Sotnikov, Wasser, Wever, Zeng, Zhang, Zoia

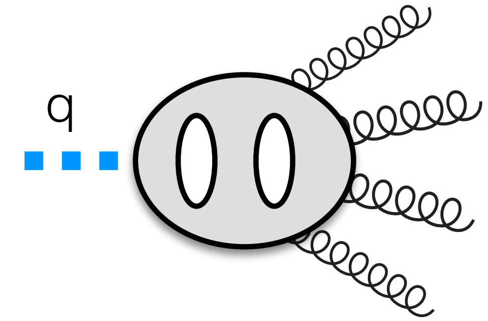
C_i Coefficients to be solved

$$C_i = x_i + y_i B$$

$$B \equiv \frac{s_{12}s_{34} + s_{23}s_{14} - s_{13}s_{24}}{4i\varepsilon(1234)}$$

$x_i \, , y_i$ are integers

Constraints

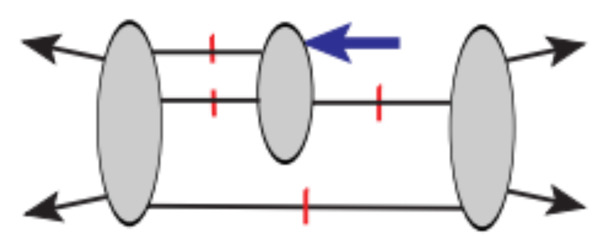


Symmetry $F_4^{(2)}(1,2,3,4)$ cyclicly permuting and flipping external momenta

IR divergences $F_4^{(2)}|_{\text{divergence}} \sim \left(F_4^{(1)}\right)^2 +$ 2-loop cusp/ collinear ADs

Collinear limit $F_4^{(2)}|_{p_i//p_{i+1}} \rightarrow F_3^{(2)}$ (+ splitting functions)

Simple unitarity cuts



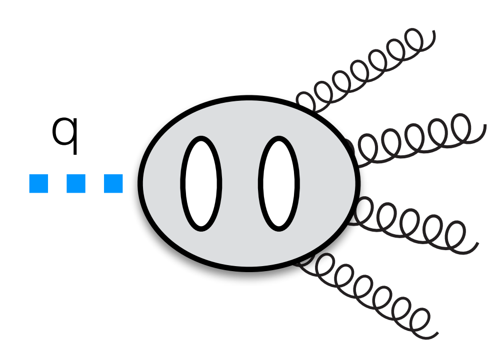
Bootstrapping the two-loop FF

Constraints	Parameters left
Starting ansatz	590×2
Symmetries of external legs	168
IR (Symbol)	109
Collinear limit (Symbol)	43
IR (Function)	39
Collinear limit (Function)	21

Only change $\mathcal{O}(\epsilon)$ terms

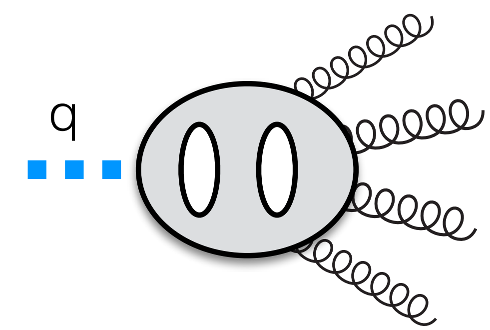
IR divergence
Collinear

} determine the two-loop results (up to finite order) !

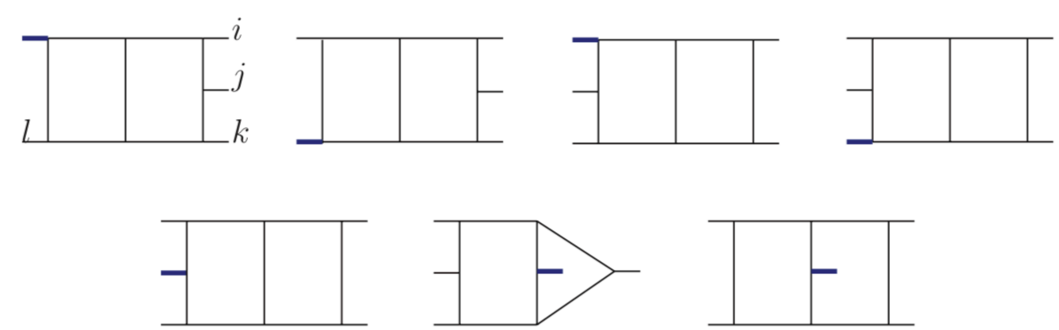


Number of independent variables

Independent Mandelstam variables:



$s_{12}, s_{23}, s_{34}, s_{4q}, s_{1q}$



One would expect 4 dimensionless variables

It turns out that there only 3 independent ones.

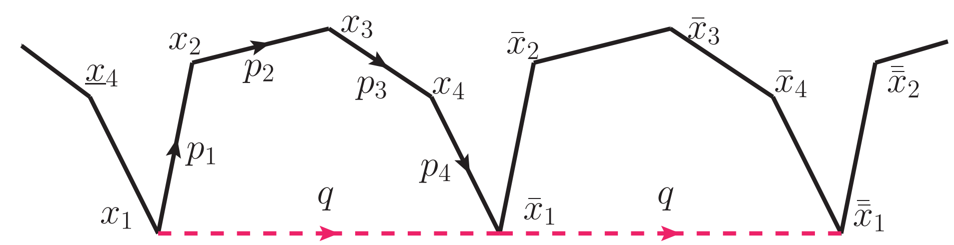
Dual conformal symmetry

The finite remainder depends only on three ratios:

$$u_1 \equiv \frac{s_{12}}{s_{34}} = \frac{x_{13}^2}{x_{3\bar{1}}^2}\,, \quad u_2 \equiv \frac{s_{23}}{s_{14}} = \frac{x_{24}^2}{x_{4\bar{2}}^2}\,, \quad u_3 \equiv \frac{s_{123}s_{134}}{s_{234}s_{124}} = \frac{x_{14}^2 x_{3\bar{2}}^2}{x_{2\bar{1}}^2 x_{4\bar{3}}^2}$$

which satisfies one dual conformal symmetry

$$\delta_q x_i^\mu \equiv \frac{1}{2} x_i^2 q^\mu - (x_i \cdot q)\, x_i^\mu$$

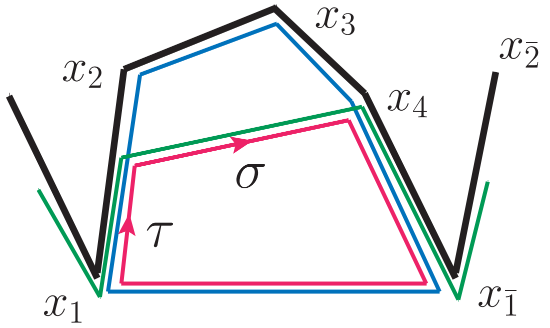


$$q^2 = 0$$

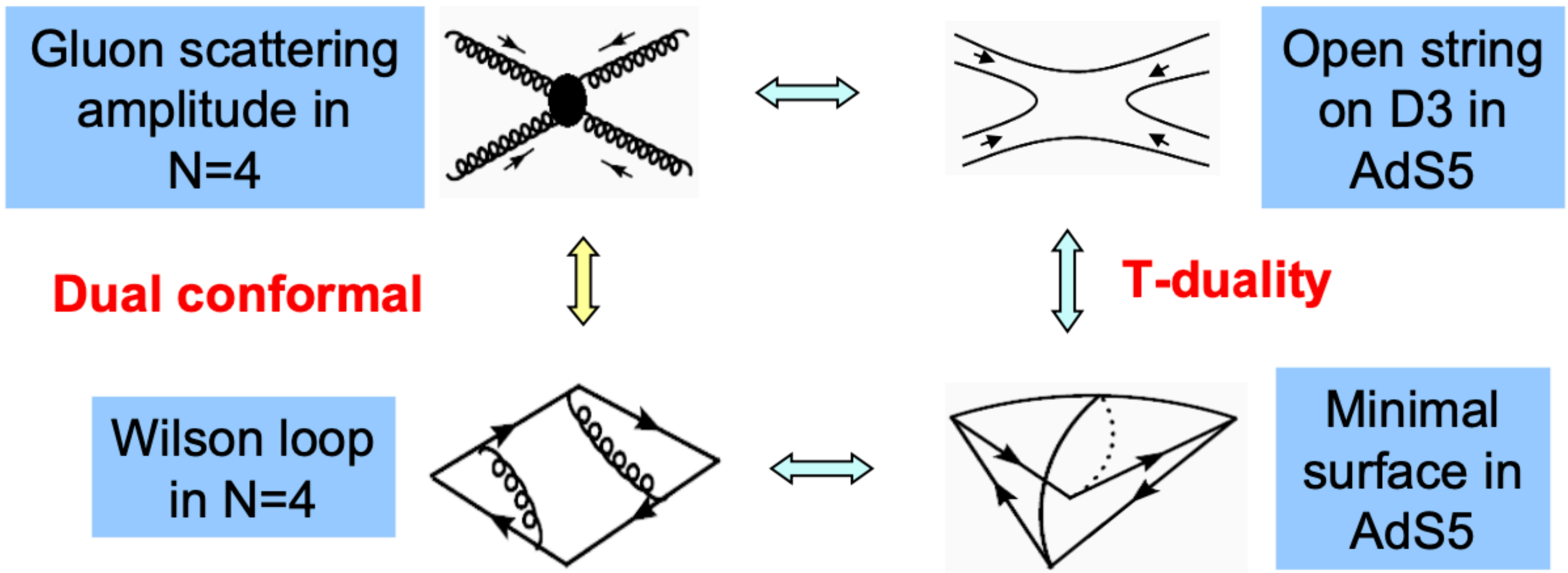
$$\delta_q F_4^{(2)} \simeq 0$$

This is a non-trivial two-loop check to the FF / WL duality.

FF/WL duality and OPE



Amplitudes/Wilson loop duality



Weak coupling

Strong coupling

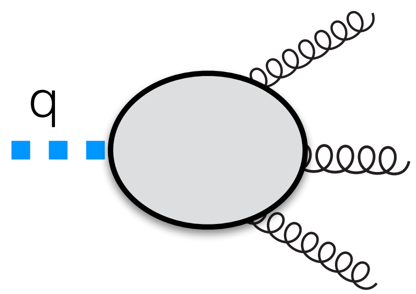
Alday, Maldacena, 2007

Drummond, Korchemsky, Sokatchev, 2007

Brandhuber, Heslop, Travaglini, 2007

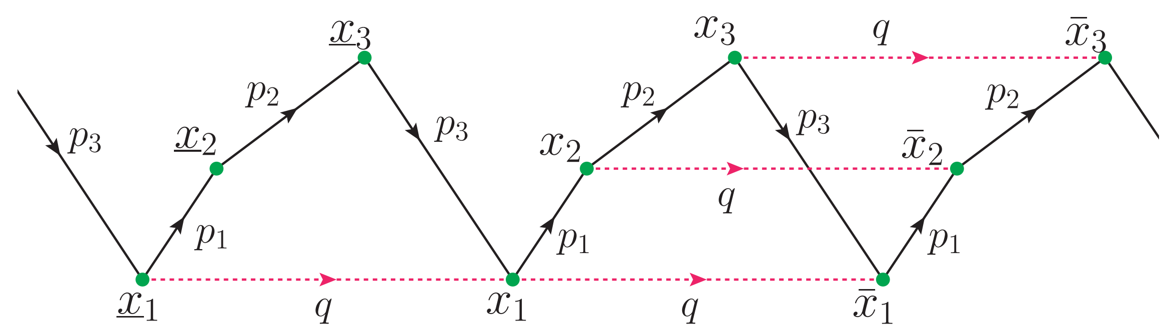
Form factor/Wilson loop duality

Form factor

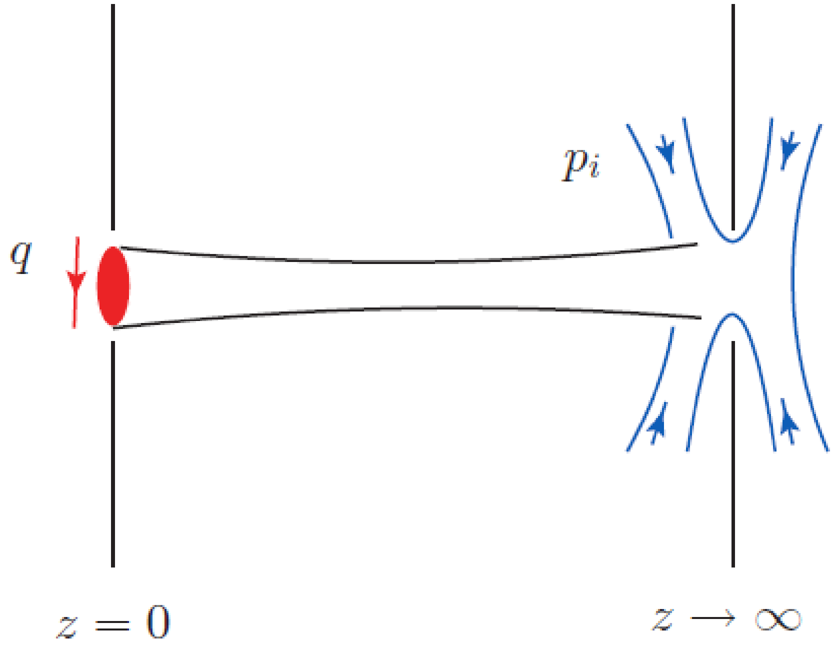


$x_i - x_{i+1} = p_i$

Periodic Wilson line

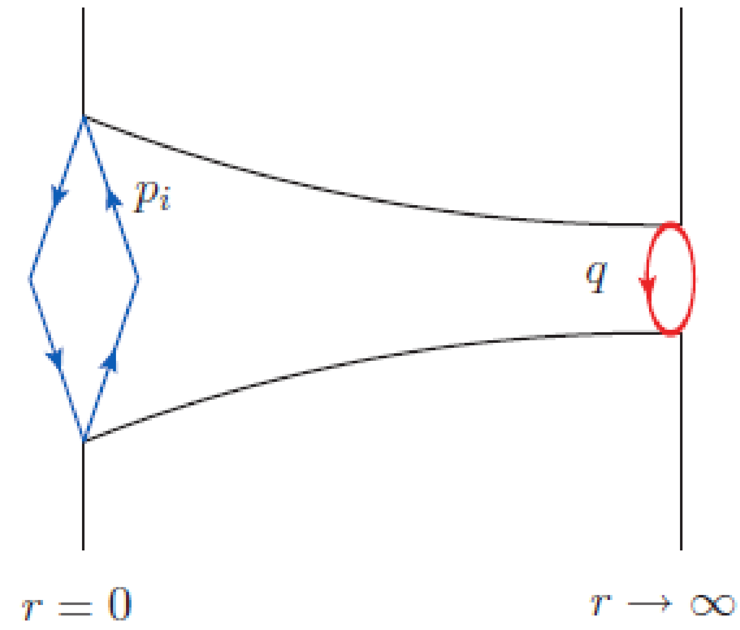


Strong coupling as string minimal surfaces Alday, Maldacena 2007



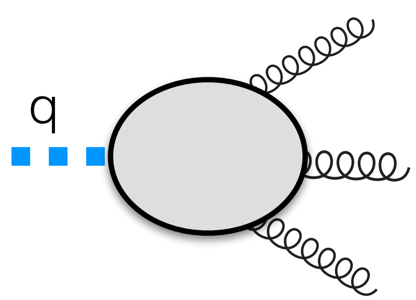
T-duality

$r = \frac{1}{z}$



Form factor/Wilson loop duality

Form factor



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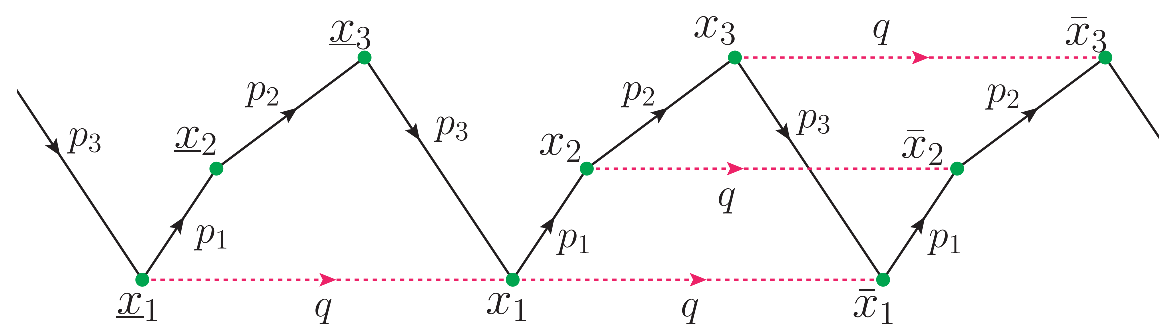
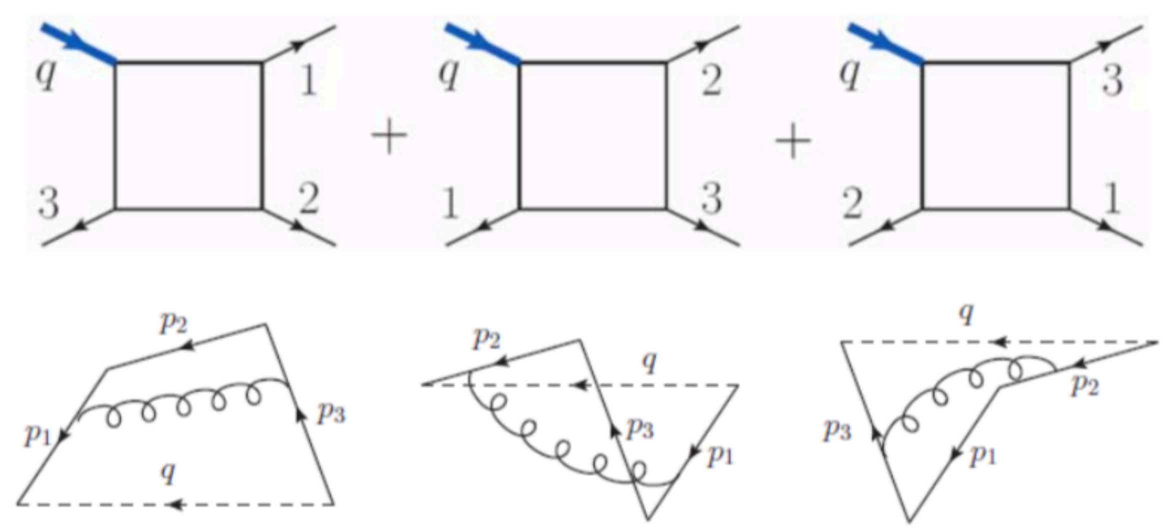


Weak coupling

Full Wilson loop prescription is only known at one loop..



Brandhuber, Spence, Travaglini, GY 2010

OPE of null-polygon Wilson loops

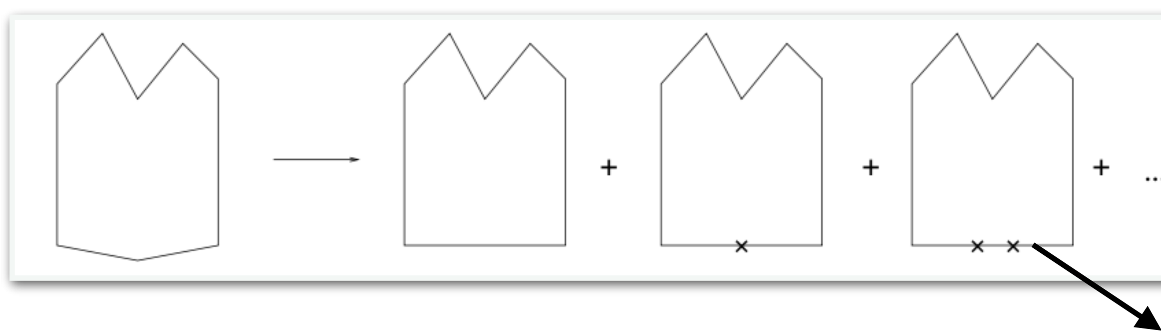
A new non-perturbative framework:

Alday, Gaiotto, Maldacena, Sever, Vieira 2010

"An Operator Product Expansion for Polygonal null Wilson Loops"

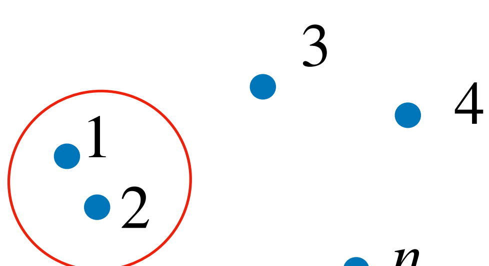
OPE limit ⟶ Collinear limit

but with all sub-leading terms



GKP states Basso 2010

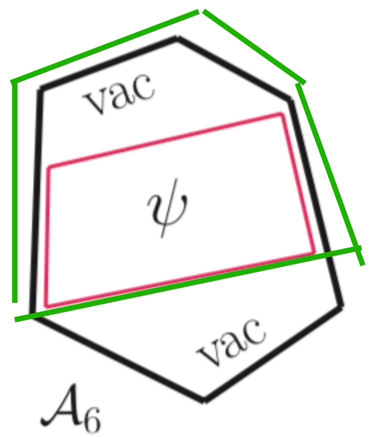
In analogy with OPE for local operators:



$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \ldots \mathcal{O}_n \rangle = \sum_i e^{-tE_i} \, C_{12i} \, C_{i3\ldots n}$$

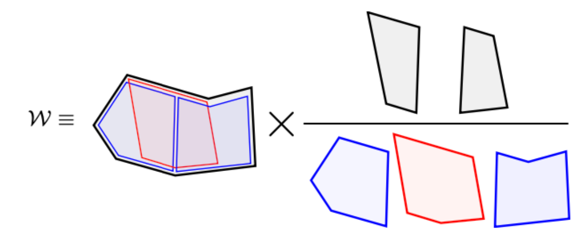
Pentagon decomposition

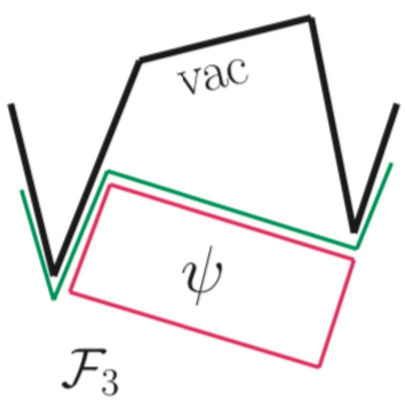
A system framework to perform OPE is "pentagon decomposition":



Basso, Sever, Vieira (2013)

$$\mathcal{W}_{\mathcal{A}_6} = \sum_{\psi} e^{-E_\psi \tau + i p_\psi \sigma + i m_\psi \phi}\, \mathbb{P}(0|\psi)\, \mathbb{P}(\psi|0)$$

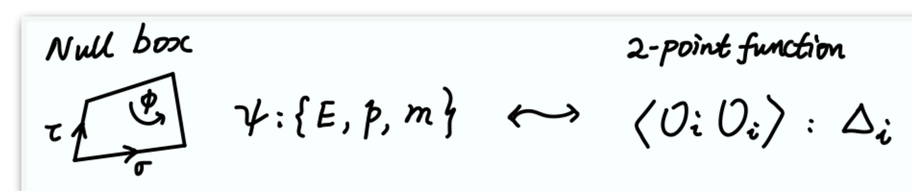


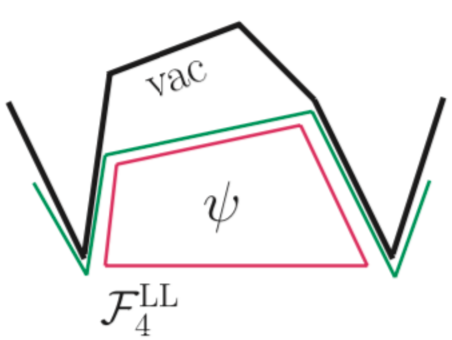


Sever, Tumanov, Willhelm (2021)

Form factor ($q^2 \neq 0$)

$$\mathcal{W}_{\mathcal{F}_3} = \sum_{\psi} e^{-E_\psi \tau + i p_\psi \sigma}\, \mathbb{P}(0|\psi)\, \mathbb{F}(\psi)$$

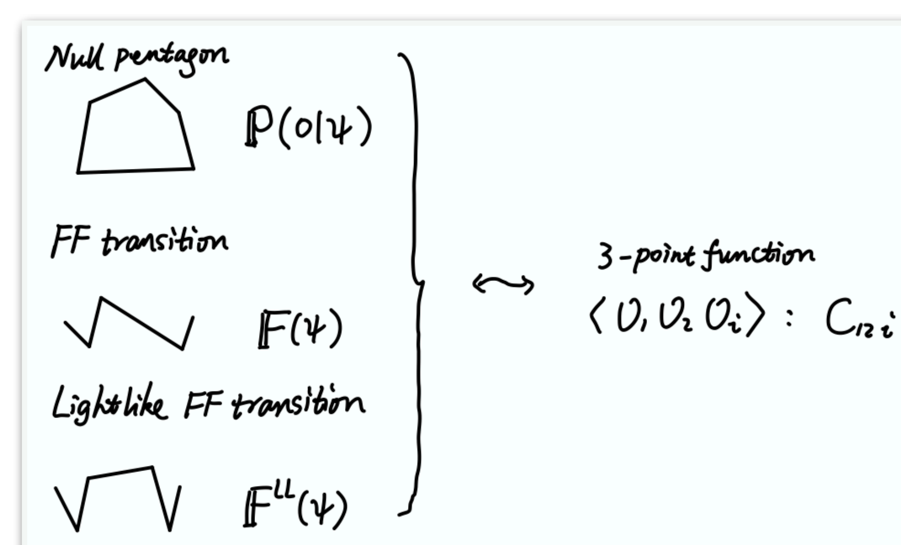




Guo, Wang, GY (2022)

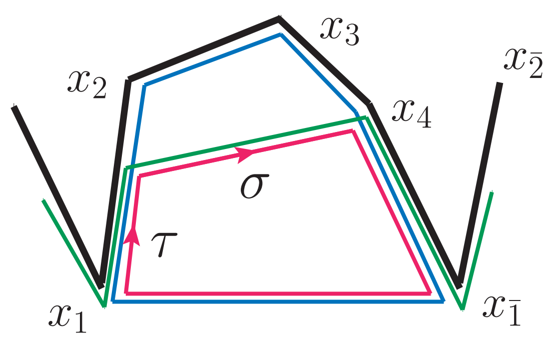
Form factor ($q^2 = 0$)

$$\mathcal{W}_4^{\mathrm{LL}} = \sum_{\psi} e^{-E_\psi \tau + i p_\psi \sigma + i m_\psi \phi}\, \mathbb{P}(0|\psi)\, \mathbb{F}^{\mathrm{LL}}(\psi)$$



Lightlike Form factor OPE

Momentum cross ratios (Form factors) ⟶ Geometric parameters (Wilson loops)



$$u_1 = \frac{x_{13}^2}{x_{3\bar{1}}^2} = e^{-2\sigma}, \quad u_2 = \frac{x_{24}^2}{x_{4\bar{2}}^2} = e^{-2\tau}$$

$$u_3 = \frac{x_{14}^2 x_{3\bar{2}}^2}{x_{2\bar{1}}^2 x_{4\bar{3}}^2} = \frac{\cosh(\sigma-\tau)+\cos(\phi)}{\cosh(\sigma+\tau)+\cos(\phi)}$$

Collinear limit ⟶ $\tau \to \infty$

One-loop and two-loop large-τ expansion:

$$\mathcal{W}_4^{\mathrm{LL},(1)} = -e^{-2\tau}[1+\cos(2\phi)] + \mathcal{O}(e^{-3\tau})$$
$$\mathcal{W}_4^{\mathrm{LL},(2)} = 2e^{-\tau}\cos(\phi)h_0^{(2)}(\sigma) + \mathcal{O}(e^{-2\tau})$$

$$h_0^{(2)}(\sigma) = 4e^{\sigma}\big[-\mathrm{Li}_3(e^{-2\sigma}) + \mathrm{Li}_2(1-e^{-2\sigma}) - \sigma\mathrm{Li}_2(e^{-2\sigma}) - \zeta_2\big] + (\sigma \to -\sigma)\,.$$

OPE expansion

Compare with OPE prediction:

$$\mathcal{W}_4^{\mathrm{LL},(\ell)}|_{\tau^{\ell-2}e^{-\tau}} = 2\cos(\phi)\int_{\mathbb{R}} \frac{du}{2\pi} e^{2iu\sigma}\, \frac{\big(-E_{3/2}^{(1)}(u)\big)^{\ell-2}}{(\ell-2)!}\left[g^{-4}\mu_F(u)\mathbb{F}_{3,F}^{\mathrm{LL}}(u)\right]_{g=0}$$

We extract the FF transition from the perturbative results:

Leading single-gluon excitation:

$$\mathbb{F}_{F/\bar{F}}^{\mathrm{LL}}(u)\Big|_{g^2} = \frac{2\pi}{(u^2+\frac{1}{4})\cosh(\pi u)}$$

Now we can make predictions to all loop order!

$$\mathcal{W}_4^{\mathrm{LL},(\ell)}|_{\tau^{\ell-1}e^{-\tau}} = 0\,,$$

$$\mathcal{W}_4^{\mathrm{LL},(\ell)}|_{\tau^{\ell-2}e^{-\tau}} = 2\cos(\phi)\int_{\mathbb{R}} \frac{du}{2\pi} e^{2iu\sigma}\, \frac{\big(-E_{3/2}^{(1)}(u)\big)^{\ell-2}}{(\ell-2)!}\left[\frac{-2\pi^2}{(u^2+\frac{1}{4})^2\cosh^2(\pi u)}\right]$$

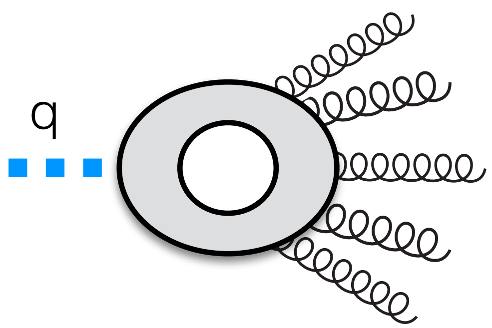
1-loop GKP energy: $E_s^{(1)}(u) = 2\big(\psi(s+iu)+\psi(s-iu)-2\psi(1)\big)$

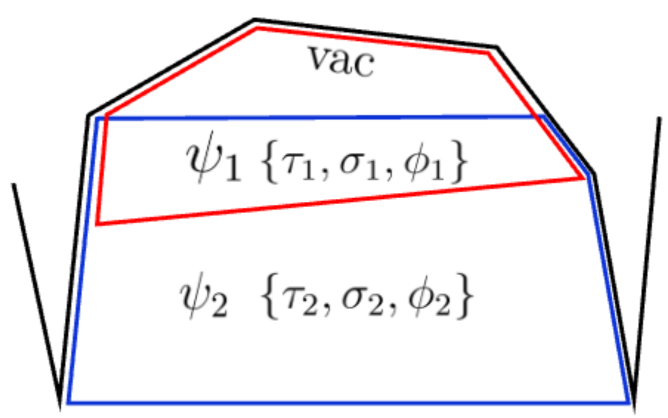
Super non-MHV cases

Generalize to super non-MHV form factors in N=4 SYM

Y. Guo, J. Shen, GY to appear

Five-point form factors





6 independent ratio variables

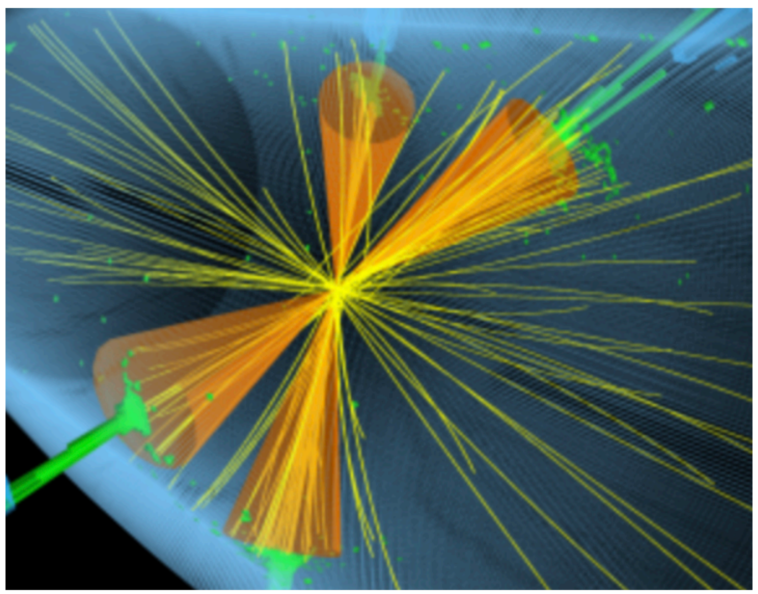
Higher-twist excitations $\sim e^{-2\tau}$

$$\mathcal{W}_4^{(l)\,\mathrm{MHV}}\Big|_{\tau^l e^{-2\tau}} = 0, \qquad \mathcal{W}_4^{(l)\,\mathrm{NMHV}}\Big|_{\tau^{l+1} e^{-2\tau}} = 0,$$

$$\mathcal{W}_4^{(l)\,\mathrm{MHV}}\Big|_{\tau^{l-1} e^{-2\tau}} = -2\cos(\phi)\frac{(-4)^{l-1}}{(l-1)!} \qquad \mathcal{W}_4^{(l)\,\mathrm{NMHV}}\Big|_{\tau^{l} e^{-2\tau}} = -e^{2i\phi}\frac{(-4)^{l}}{l!}.$$

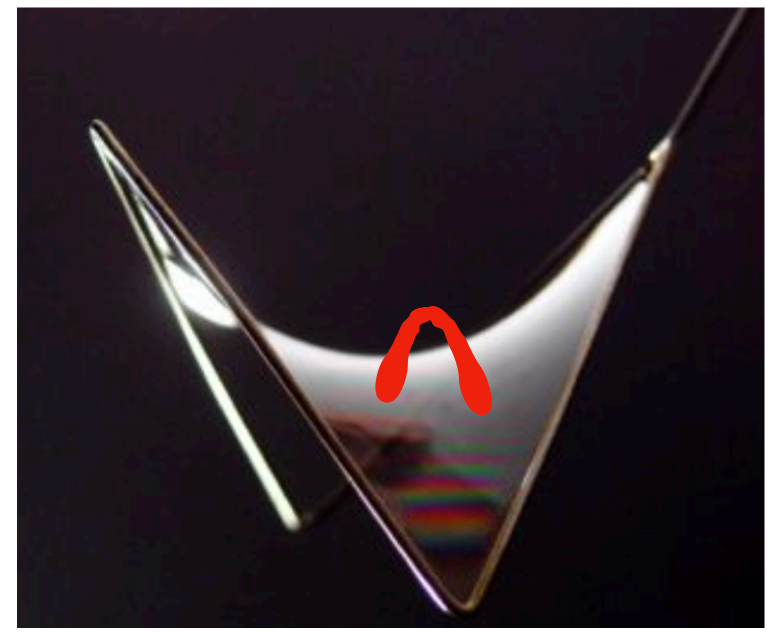
Summary and outlook

A non-perturbative framework for amplitudes/FFs in N=4 SYM



4D scattering

→



2D flux tube

Hard

Possibility to solve non-perturbatively !

Summary and outlook

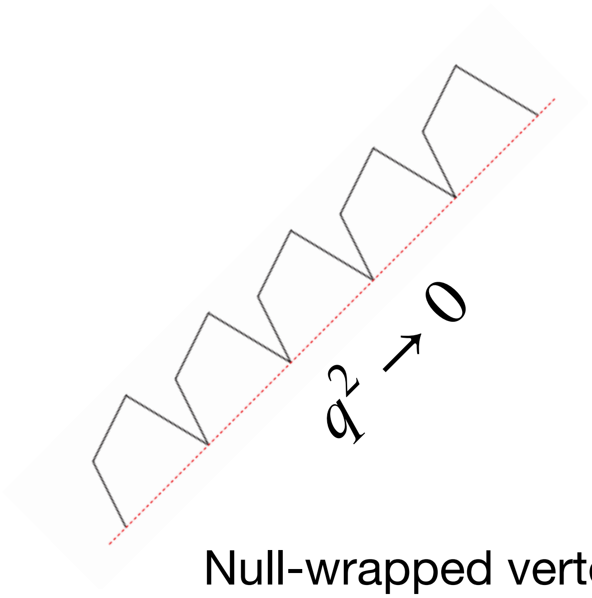
Bootstrapping higher loop results?

Exact solution of FF transition?

FFs of more general operators?

OPE of null-polygon Wilson loops for other theories?

Other meaning of a light-like periodic Wilson loop?



Null-wrapped vertex

Thank you for your attention!

Back up slides

Some details of OPE

$$\mathcal{W}_4^{\mathrm{LL}} = \sum_{\mathbf{a}} \int d\mathbf{u}\, \mathrm{e}^{-E_{\mathbf{a}}(\mathbf{u})\tau + \mathrm{i}p_{\mathbf{a}}(\mathbf{u})\sigma + \mathrm{i}m_{\mathbf{a}}\phi}\, \mathbb{P}_{\mathbf{a}}(0|\mathbf{u})\, \mathbb{F}_{\bar{\mathbf{a}}}^{\mathrm{LL}}(\bar{\mathbf{u}}), \qquad d\mathbf{u} = \prod_{i=1}^{N} \frac{du_i}{2\pi}\mu_{a_i}(u_i).$$

First excitation:

$$\mathcal{W}_4^{\mathrm{LL}} = 1 + 2\cos(\phi) h_0(\tau,\sigma) + \ldots,$$

$$h_0(\tau,\sigma) = \int \frac{du}{2\pi}\, \mu_F(u)\, \mathrm{e}^{-E(u)\tau + \mathrm{i}p(u)\sigma}\, \mathbb{P}(0|u)\, \mathbb{F}_F^{\mathrm{LL}}(u).$$

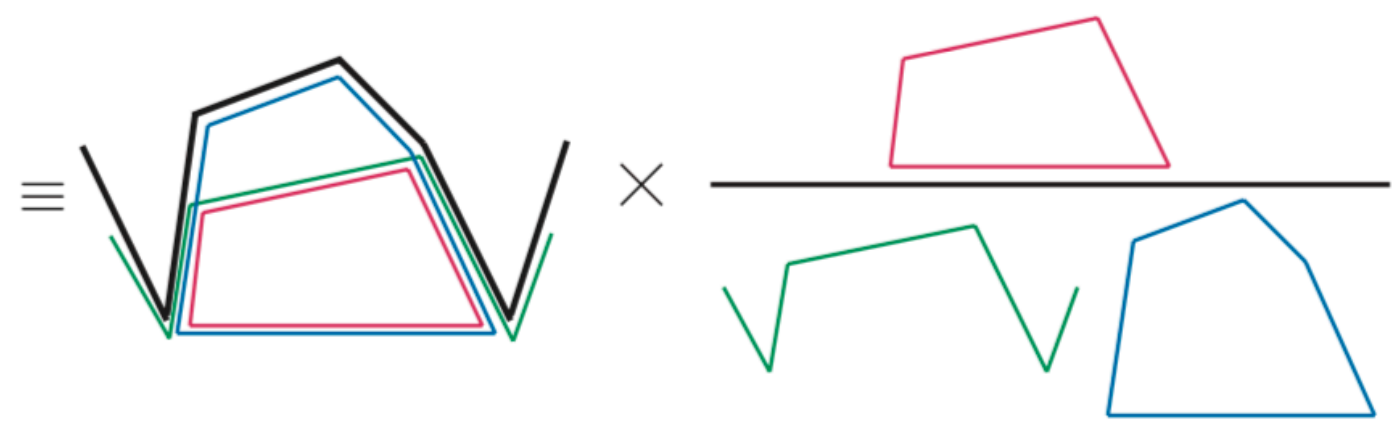
One-loop energy contribution:

$$\mathcal{W}_4^{\mathrm{LL},(\ell)}|_{\tau^{\ell-2}e^{-\tau}} = 2\cos(\phi) \int_{\mathbb{R}} \frac{du}{2\pi} e^{2iu\sigma}\, \frac{\big(-E_{3/2}^{(1)}(u)\big)^{\ell-2}}{(\ell-2)!} \big[g^{-4}\mu_F(u)\mathbb{F}_{3,F}^{\mathrm{LL}}(u)\big]_{g=0}$$

$$\mu_F(u) = -\frac{\pi g^2}{(u^2+\frac{1}{4})\cosh(\pi u)}(1+\mathcal{O}(g^2)). \qquad E_s^{(1)}(u) = 2\big(\psi(s+iu) + \psi(s-iu) - 2\psi(1)\big)$$

OPE regularization

$$\mathcal{W}_4^{\mathrm{LL}} = \frac{\hat{\mathcal{F}}_4^{\mathrm{LL}} \times \mathcal{W}_{\mathrm{sq}}}{\hat{\mathcal{F}}_3^{\mathrm{LL}} \times \mathcal{W}_{\mathrm{pen}}}$$

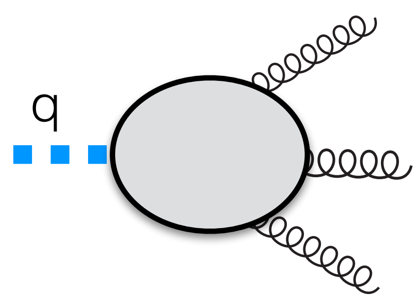


$$\mathcal{W}_4^{\mathrm{LL},(1)} = -\frac{1}{2}\log\left(\frac{(1-u_1)(1-u_2)}{1-u_3}\right)\log\left(\frac{(1-u_1)(1-u_2)u_3}{(1-u_3)u_1}\right)$$

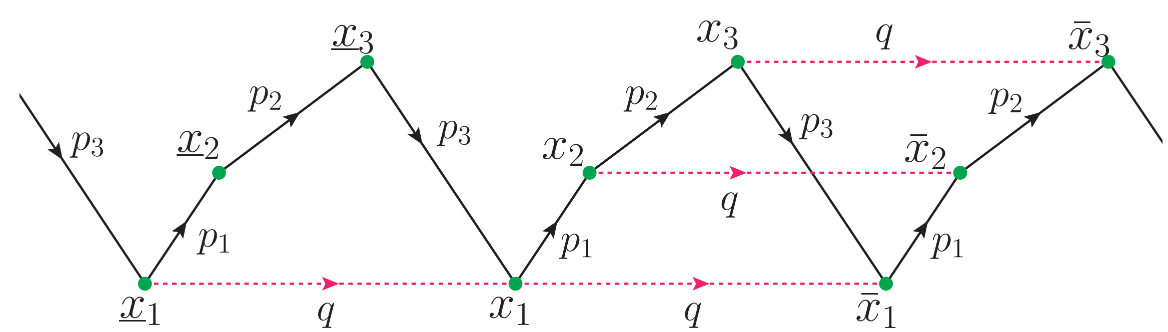
The dual conformal symmetry is manifest at one loop.

High loops: $$\mathcal{W}_n^{\mathrm{LL}} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{W}_n^{\mathrm{LL},(1)} + \mathcal{R}_n^{\mathrm{LL}}\right]$$

Form factor / Wilson loop duality

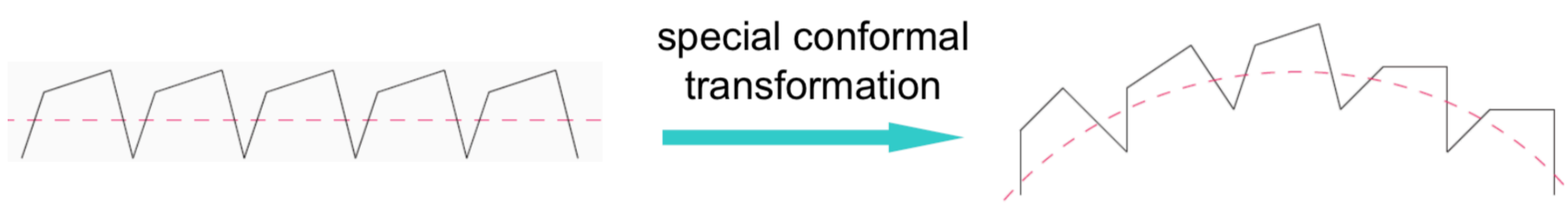


$$x_i - x_{i+1} = p_i$$

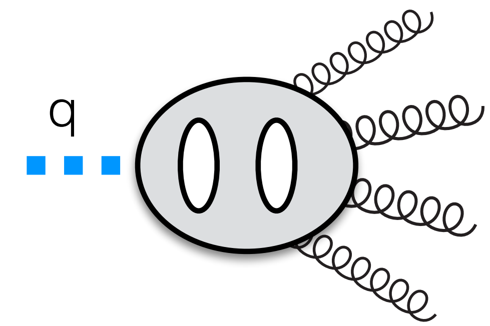


Dual periodic WL picture

No exact dual conformal symmetry for general q.



Bootstrapping the two-loop FF

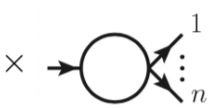


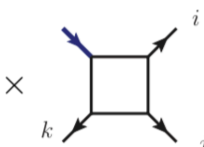
Based on the one-loop results:

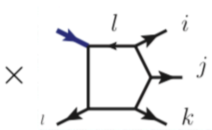
$$\mathcal{F}_4^{\mathrm{LL},(1)} = \mathcal{F}_4^{\mathrm{LL},(0)}\left(\mathcal{G}_1^{(1)} + B\,\mathcal{G}_2^{(1)}\right)$$

Pure functions

$$B \equiv \frac{s_{12}s_{34} + s_{23}s_{14} - s_{13}s_{24}}{4i\varepsilon(1234)}$$

$$I^{(1)}_{\mathrm{Bub}}(1,\ldots,n) = \frac{1-2\epsilon}{\epsilon} \times$$  ,

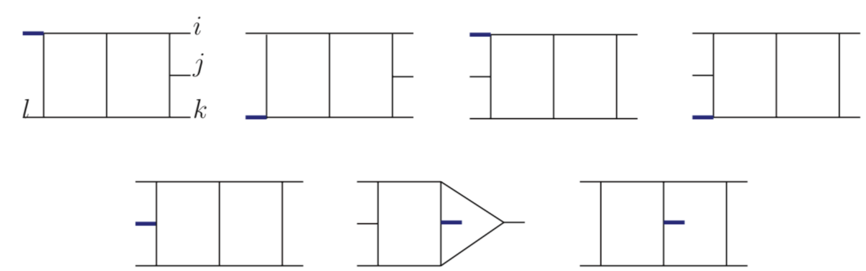
$$I^{(1)}_{\mathrm{Box}}(i,j,k) = (s_{ij}s_{jk} - p_j^2 q^2) \times$$  ,

$$I^{(1)}_{\mathrm{Pen}}(i,j,k,l) = 4i\varepsilon(1234) \times \mu \times$$  .

We propose the ansatz at two loops:

$$\mathcal{F}_4^{\mathrm{LL},(2)} = \mathcal{F}_4^{\mathrm{LL},(0)}\left(\mathcal{G}_1^{(2)} + B\,\mathcal{G}_2^{(2)}\right)$$

$$\mathcal{G}_a^{(2)} = \sum_{i=1}^{590} c_{a,i} I_i^{(2)}$$



UT master integrals: Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, Lo Presti, Mitev, Page, Papadopoulos, Tommasini, Sotnikov, Wasser, Wever, Zeng, Zhang, Zoia

Other 4-point form factors

Ansatz → **Physical constraints** → **Solution**

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR, collinear limits,
unitarity-cut, etc

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

The same strategy has been used to compute four-point form factors of length-3 operators:

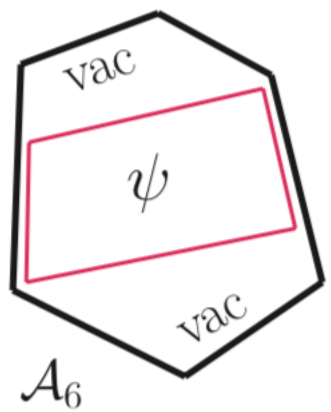
$$F^{(2)}_{\mathrm{tr}(\phi^3)}(1^{\phi},2^{\phi},3^{\phi},4^{g})$$

Guo, Wang, GY 2021

$$F^{(2)}_{\mathrm{tr}(F^3)}(1^{g},2^{g},3^{g},4^{g})$$

Guo, Jin, Wang, GY 2022

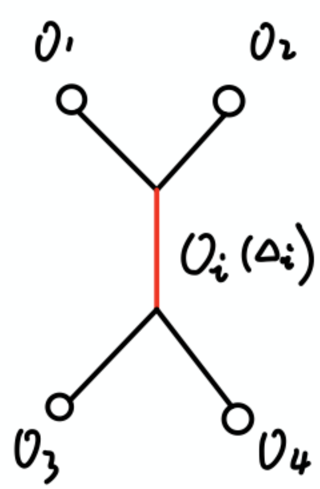
OPE of null-polygon Wilson loops



Basso, Sever, Vieira (2013)   Alday, Gaiotto, Maldacena, Sever, Vieira 2010

$$\mathcal{W}_{\mathcal{A}_6} = \sum_{\psi} e^{-E_\psi \tau + i p_\psi \sigma + i m_\psi \phi}\, \mathbb{P}(0|\psi)\, \mathbb{P}(\psi|0)$$

In analogy with four-point function:



$$G_4 \sim \sum_{O_i} F(\Delta_i)\, C_{12i}\, C_{34i}$$