Multi-Loop and Multi-Leg Feynman integrals

On the analytic computation of 2loop 6point and 3loop 5point Feynman integrals

Heavy Flavor Physics and QCD Conference
2025.04.19

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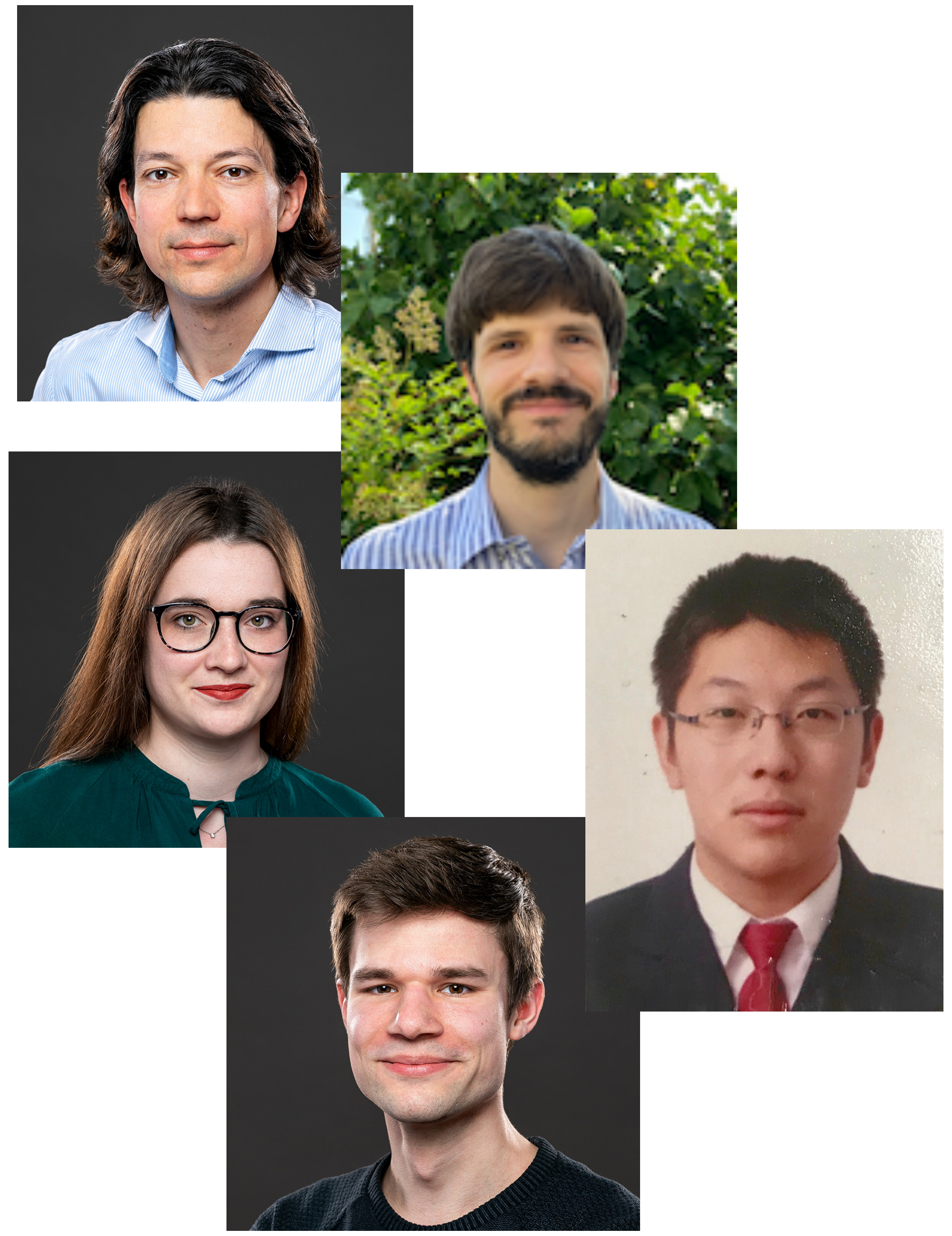
University of Science and Technology of China

Based on

Integral evaluation



2loop 6point

Group

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847

Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697

3loop 5point

Group

Based on package development

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, *arXiv: 2502.20778*

Refer to Zihao Wu's talk

Based on perturbative QCD computations

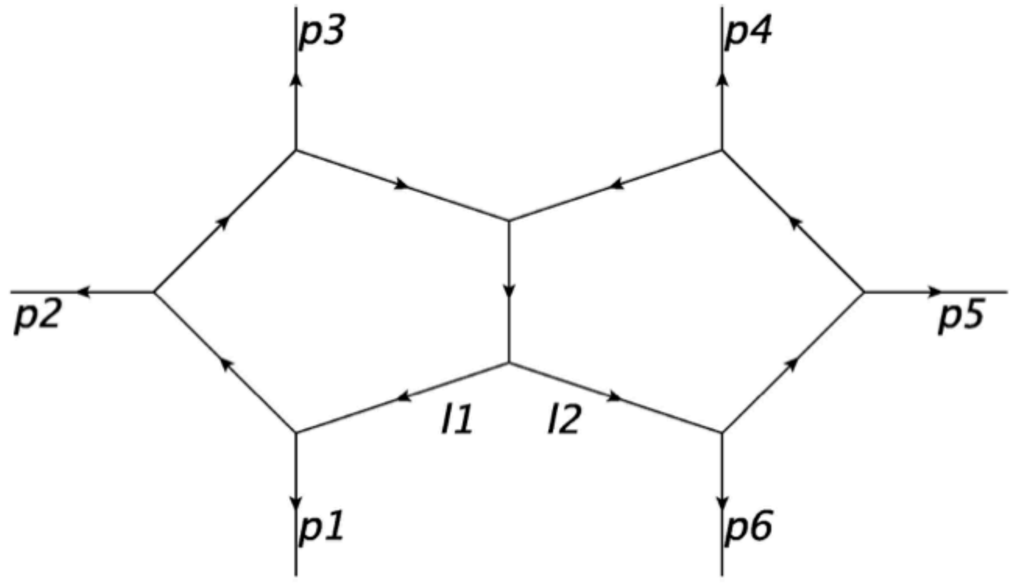
"Two-loop amplitudes for $O(\alpha_s^2)$ corrections to W$\gamma\gamma$ production at the LHC"
Badger, Hartanto, Wu, *YZ*, Zoia, JHEP12(2024)221

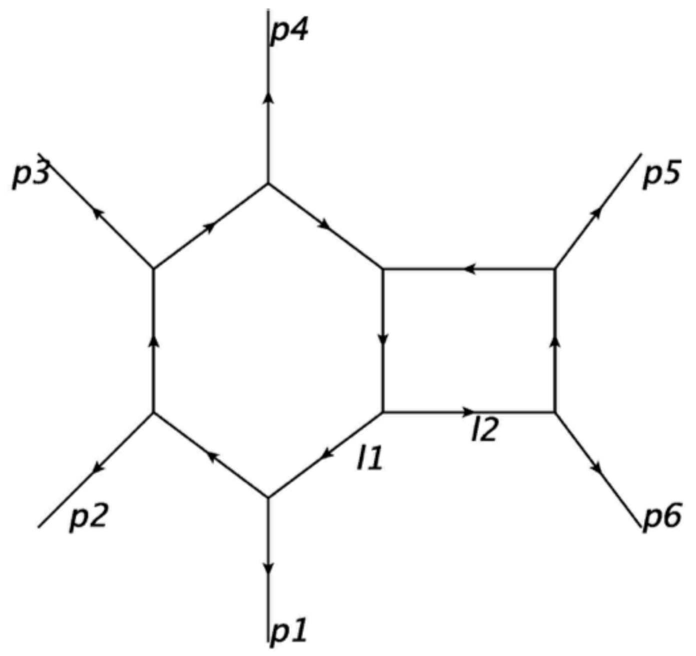
"Full-colour double-virtual amplitudes for associated production
of a Higgs boson with a bottom-quark pair at the LHC"
Badger, Hartanto, Poncelet, Wu, *YZ*, Zoia, *JHEP* 03 (2025) 066

Message

(The first analytic computation of 2loop 8-scale Feynman integrals in DimReg)

analytic computation of all 2loop 6point massless planar integrals is done



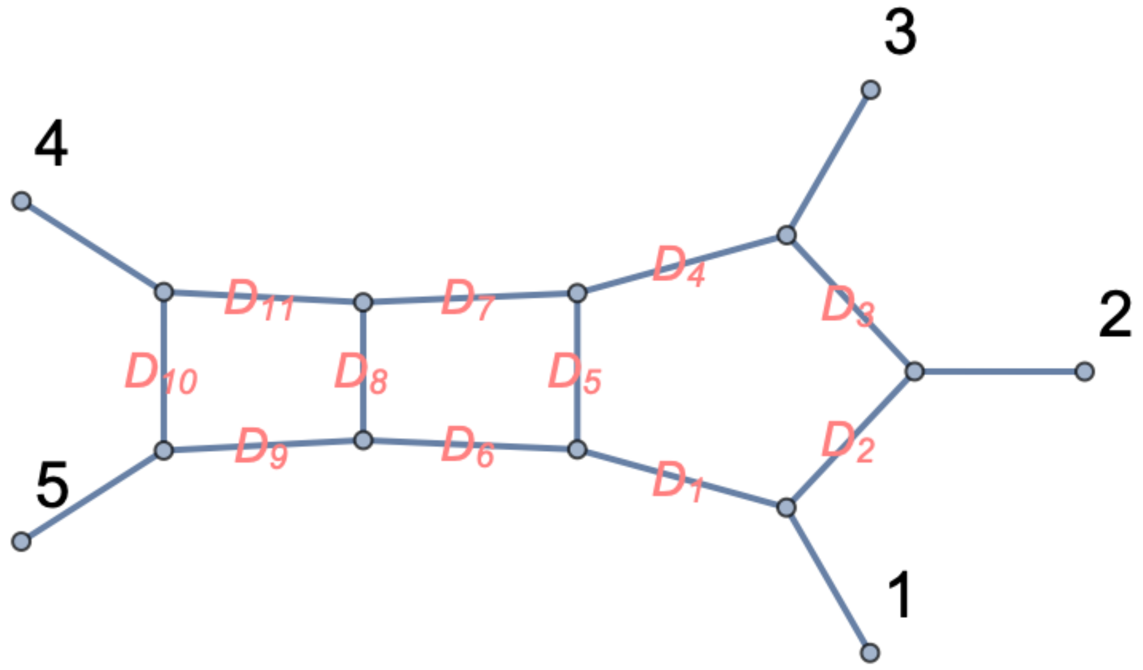


Henn, Matijasic, Miczajka, Peraro, Xu, YZ

JHEP 08(2024) 027

arXiv: 2501.01847

The first analytic computation of 3loop 5-point Feynman integral family



Liu, Matijasic, Miczajka, Xu, Xu, YZ,

arXiv: 2411.18697

Outline

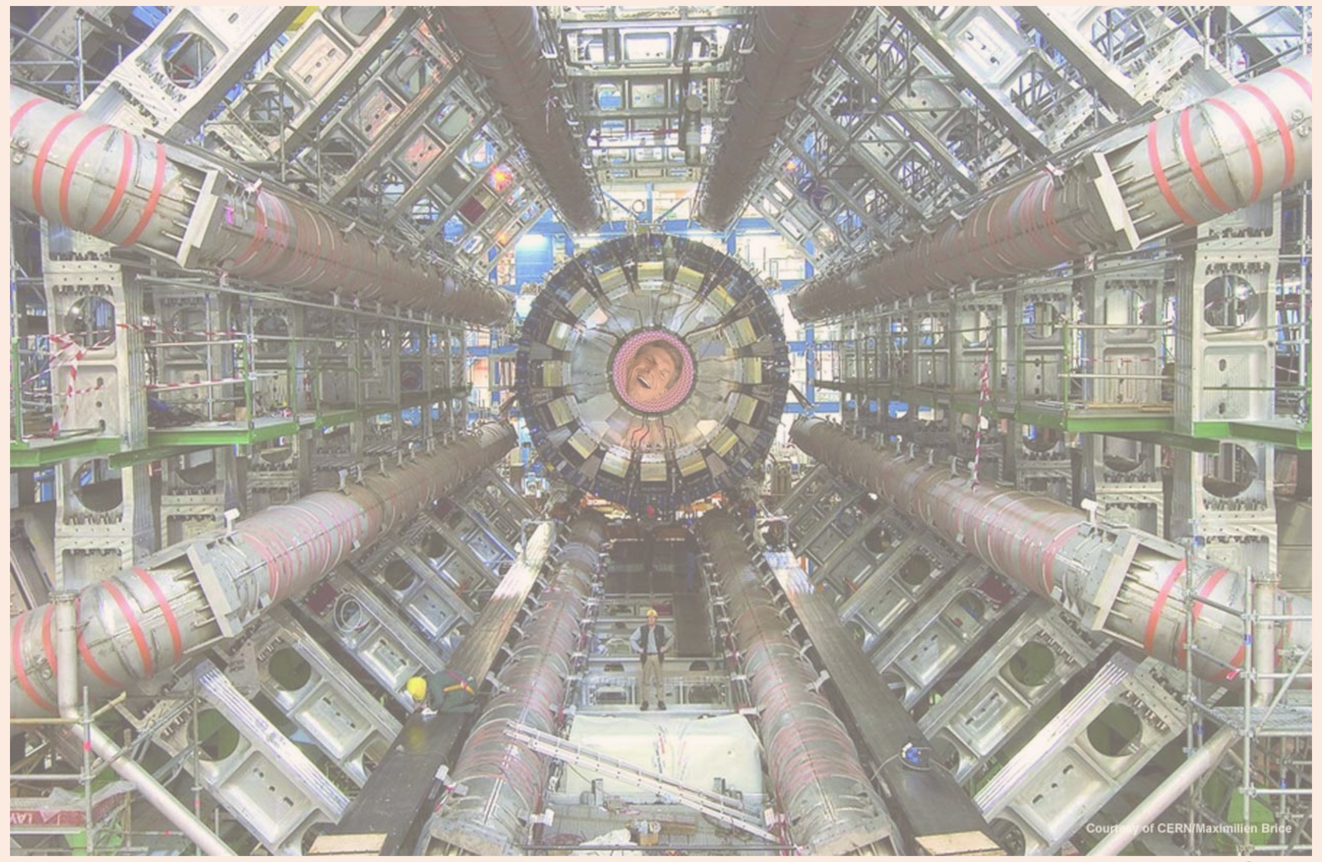
Why Feynman integrals? Why analytic?

Case 1: **2loop 6point Feynman integrals**

Case 2: 3loop 5point Feynman integrals

Summary and Outlook

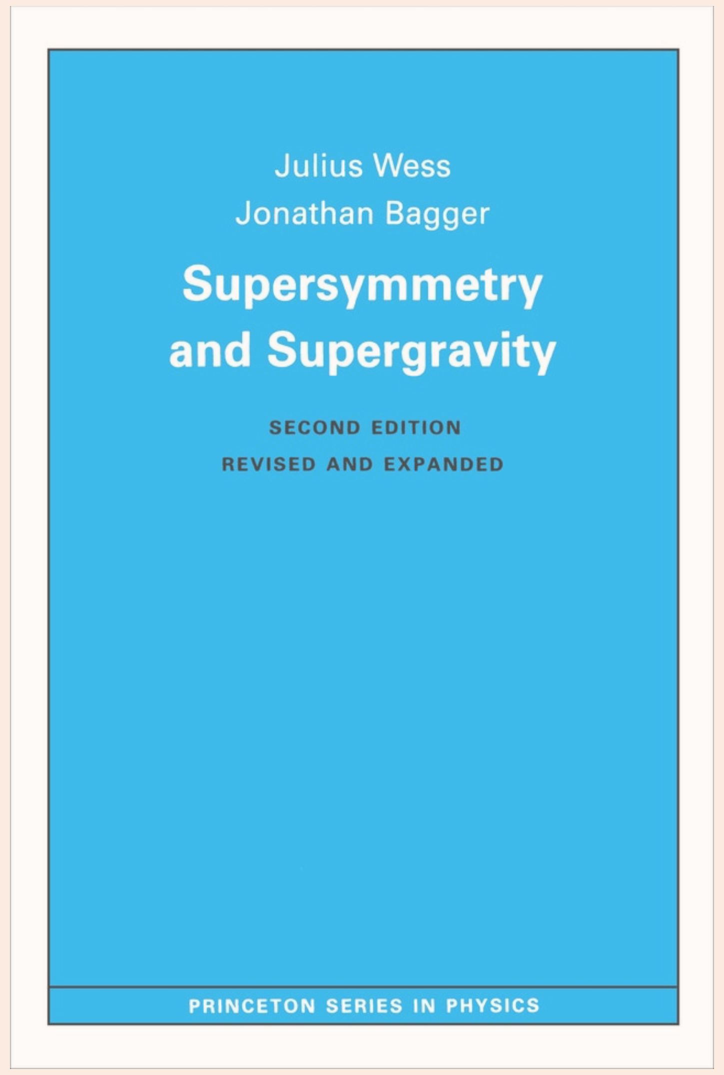
Why Feynman integrals?



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

Feynman integrals



Formal theory

N=8 supergravity UV finiteness

Gravitational wave template computations

Why *analytic* Feynman integrals?

- Auxiliary Mass Flow or Secdec methods slow or not available yet

for some multi-loop multi-leg Feynman integrals

3loop 5point Feynman integrals

- Theoretical aspects of quantum field theory

for examples: 2loop N=4 SYM theory spacelike splitting amplitude

Henn, Ma, Xu, Yan, *YZ*, Zhu, arXiv 2406.14604

- Quantum field theory computation of gravitational wave

analytic continuation/ Fourier transform is sometimes needed

Current status of Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Liu, Matijasic, Miczajka, Xu, Xu, YZ, arXiv:2411.18697

Henn, Matijasic, Miczajka, Peraro, Xu, YZ JHEP 08(2024) 027, arXiv:2501.01847

Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056

Goal of analyticity

Feynman integral

arguments related to
Letters, algebraic function of kinematics

$$I = \sum_{i=-2L} \epsilon^i \sum_{\alpha} c_\alpha G(W_{\alpha_1}, \ldots, W_{\alpha_{2L+i}}; z)$$

Goncharov polylogarithm function

$$G(\mathbf{0}_k; z) = \frac{1}{k!}(\log z)^k, \quad G(a_1, \ldots, a_k; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots a_k; t)$$

Dimensional regularization parameter

well studied function with Hopf algebra structure

For more complicated cases, iterative integral of elliptic functions, Calabi-Yau functions can appear

Canonical Differential Equation

Uniformly transcendental (UT) basis determination
Canonical differential equation

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$ Henn 2013

Solving differential equation
with boundary value

$$\int d\log(W_{i_1}) \circ \ldots \circ d\log(W_{i_k})$$

$\to$ polylogarithm functions or one-fold integration

analytic result

Canonical Differential Equation, new insights

Better Integration-by-parts (IBP) reduction

NeatIBP, Wu, Boehm, Ma, Xu, *YZ* 2023
Comput.Phys.Commun. 295 (2024) 108999

Blade, Guan, Liu, Ma, Wu 2024
Comput.Phys.Commun. 310 (2025) 109538

Alphabet searching

Effortless, Matijasic, Miczajka to appear
https://github.com/antonela-matijasic/Effortless

BaikovLetter, Jiang, Liu, Xu, Yang, 2401.07632

PLD, Fevola, Mizera, Telen
Comput. Phys. Commun. 303 (2024) 109278

SOFIA Correia, Giroux, Mizera
2503.16601

Solving differential equation

Novel representation of one-fold integration

Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697

2loop 6point Feynman integrals

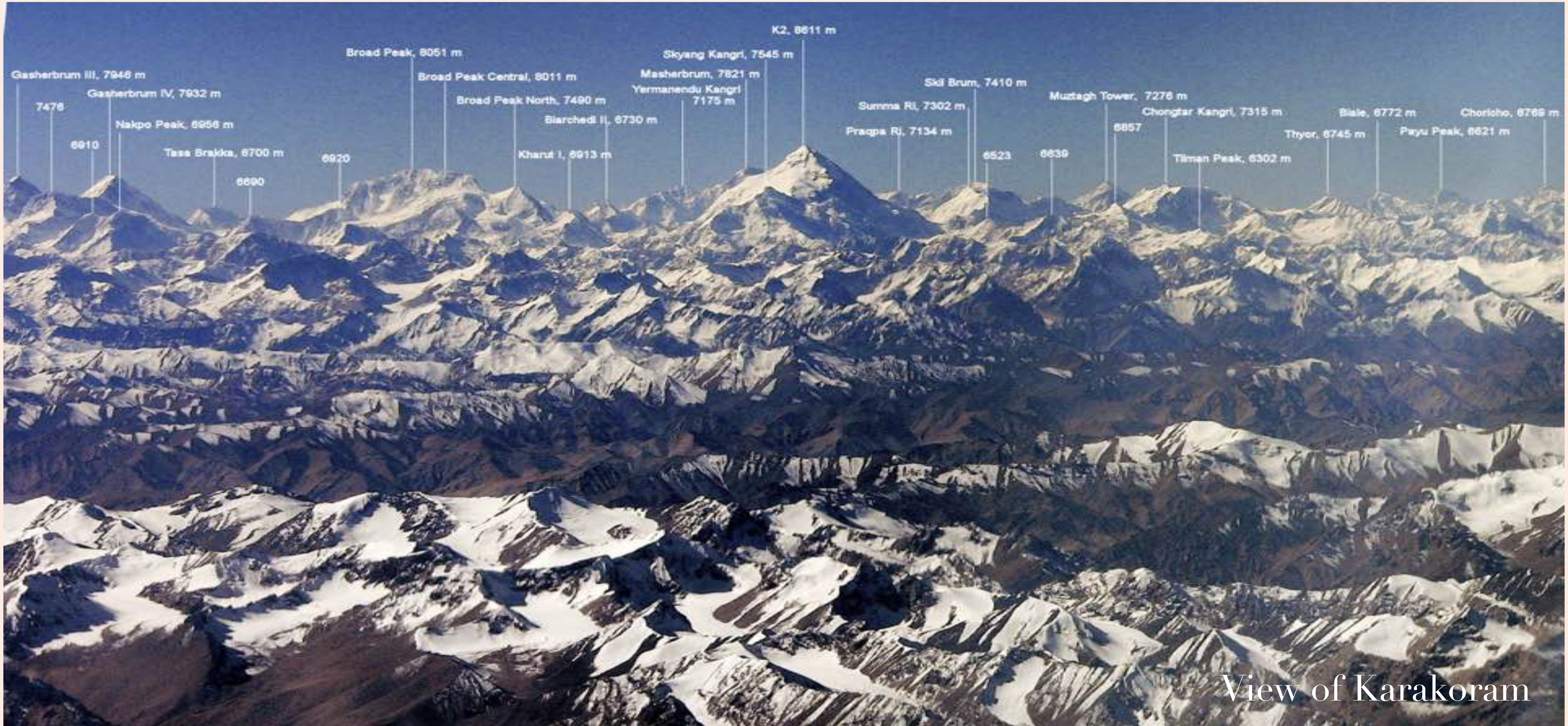
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *arXiv:2501.01847*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

The status of art for analytic computations

2loop Feynman integral: Scale frontier



2loop Feynman integral: Scale frontier

2loop 5point massless	*Gehrmann, Henn, Lo Presti 2015* *Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019*	5 scales
2loop 5point one-mass	*Papadopoulos, Tommasini, Wever 2019* *Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020* *Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023*	6 scales
2loop 5point two-mass	*Cordero, Figueiredo, Kraus, Page and Reina 2023* for leading-Color pp→ttH amplitudes with a light-quark loop	7 scales
2loop 6point massless	*Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2024* *for NNLO 4 jets production, 2 jets+ 2photons*	8 scales!

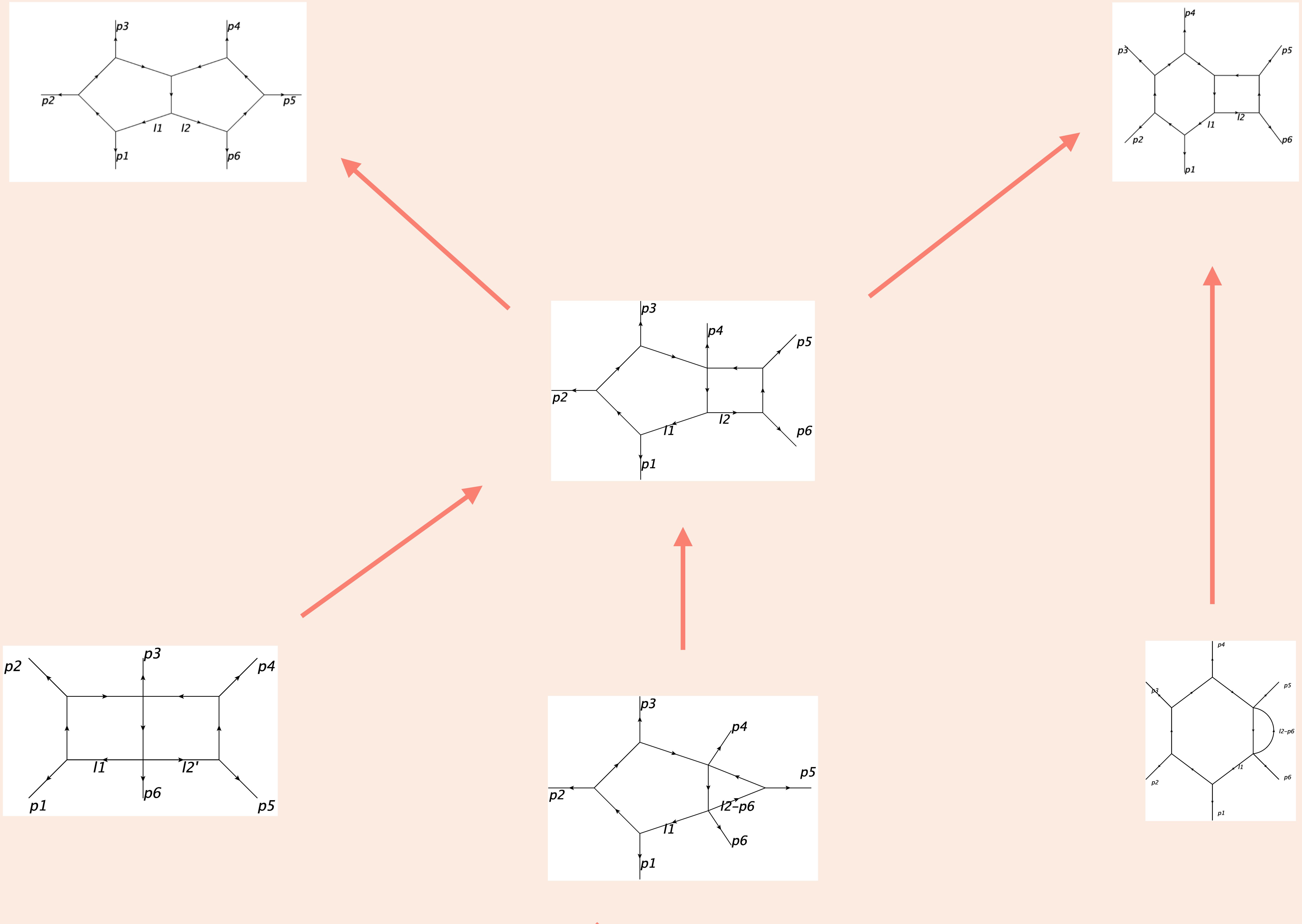
$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

All planar 2loop 6point integrals

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, 2501.01847

267
UT integrals

202
UT integrals



Feynman integrals, Scheme dependence

external momenta d=4

$$G\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix} = 0$$

9-*1* =*8* Mandelstam variables

external momenta d

9 Mandelstam variables

$s_{12}, s_{13}, s_{14}, s_{15}, s_{23}, s_{24}, s_{25}, s_{34}, s_{35}$

number of master integrals also depend on the scheme

Momentum Twistor

external momenta $d=4$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1,\dots 6 \qquad \tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle\langle i-1,\, i\rangle},$$

A particular parameterization

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_2x_1}+\frac{1}{x_1} & \frac{1}{x_2x_1}+\frac{1}{x_2x_3x_1}+\frac{1}{x_1} & \frac{1}{x_2x_1}+\frac{1}{x_2x_3x_1}+\frac{1}{x_2x_3x_4x_1}+\frac{1}{x_1} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_5}{x_2} & x_6 & 1 \\ 0 & 0 & 1 & 1 & x_7 & 1-\frac{x_8}{x_5} \end{pmatrix}$$

$$x_1 = s_{12}$$

$$x_2 = -\frac{\mathrm{Tr}_+(1234)}{2s_{12}s_{34}}$$

$$x_3 = -\frac{\mathrm{Tr}_+(1345)}{2s_{45}s_{13}}$$

$$x_4 = -\frac{\mathrm{Tr}_+(1456)}{2s_{56}s_{14}}$$

$$x_5 = \frac{s_{23}}{s_{12}}$$

$$x_6 = -\frac{\mathrm{Tr}_+(1532)+\mathrm{Tr}_+(1542)}{2s_{15}s_{12}}$$

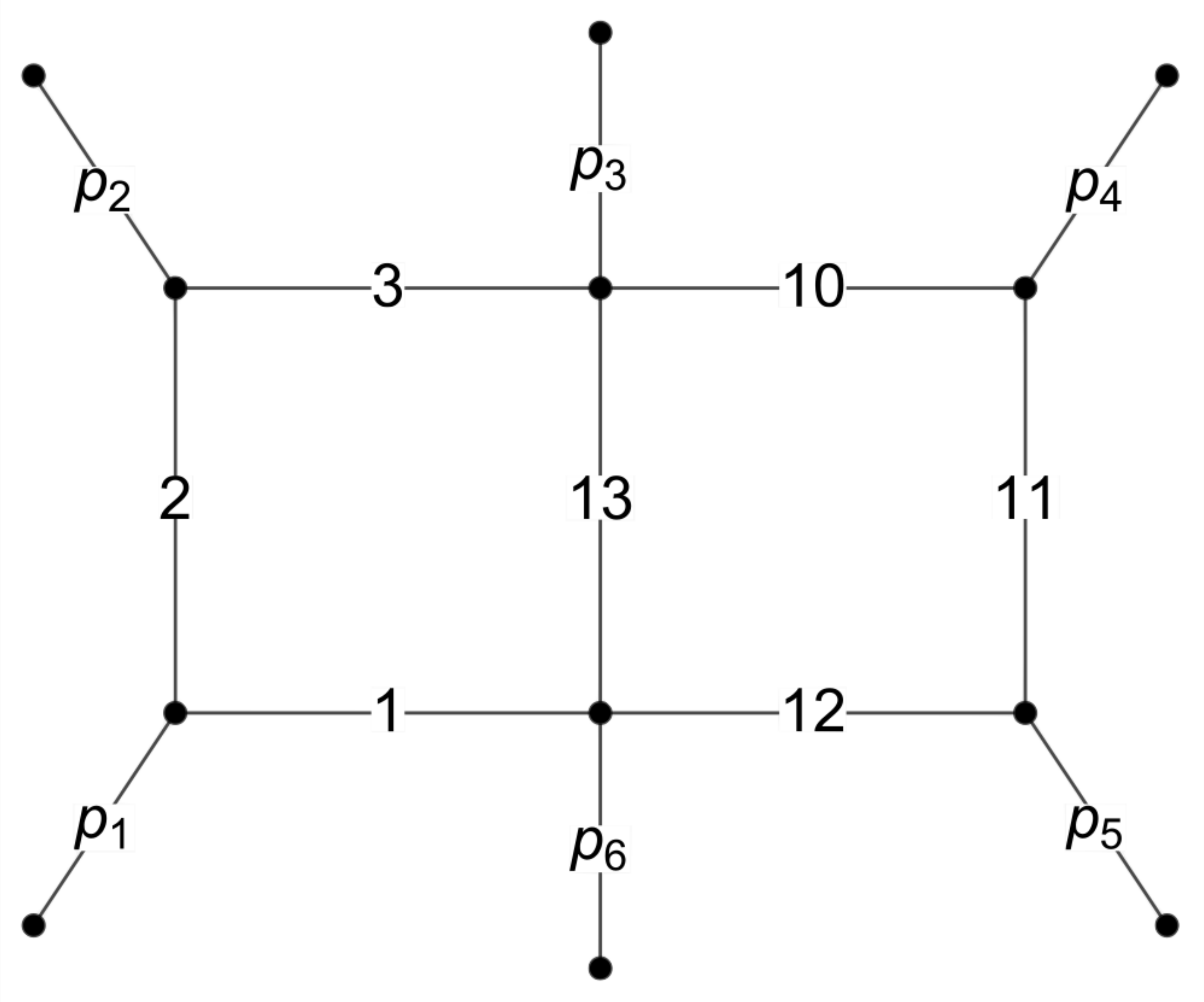
$$x_7 = 1+\frac{\mathrm{Tr}_+(1542)+\mathrm{Tr}_+(1543)}{2s_{15}s_{23}}$$

$$x_8 = \frac{s_{123}}{s_{12}}$$

Momentum parametrization rationalizes all pseudo scalars

$$\epsilon_{ijkl} \equiv 4i\epsilon_{\mu\nu\rho\sigma}p_i^\mu p_j^\nu p_k^\rho p_l^\sigma\,, \quad \epsilon_{ijkl}^2 = G_{ijkl}$$

Uniformly transcendental (UT) basis determination key step



$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}, \quad i = 1,\ldots,7$$

$$N_1 = -s_{12}s_{45}s_{156},$$

$$N_2 = -s_{12}s_{45}(l_1+p_5+p_6)^2,$$

$$N_3 = \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5+p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix},$$

$$N_4 = \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2-p_6 & p_5 & p_4 & p_1+p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix},$$

$$N_5 = -\frac{1}{4}\frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix},$$

$$N_6 = \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11}(s_{123}+s_{126})}{8},$$

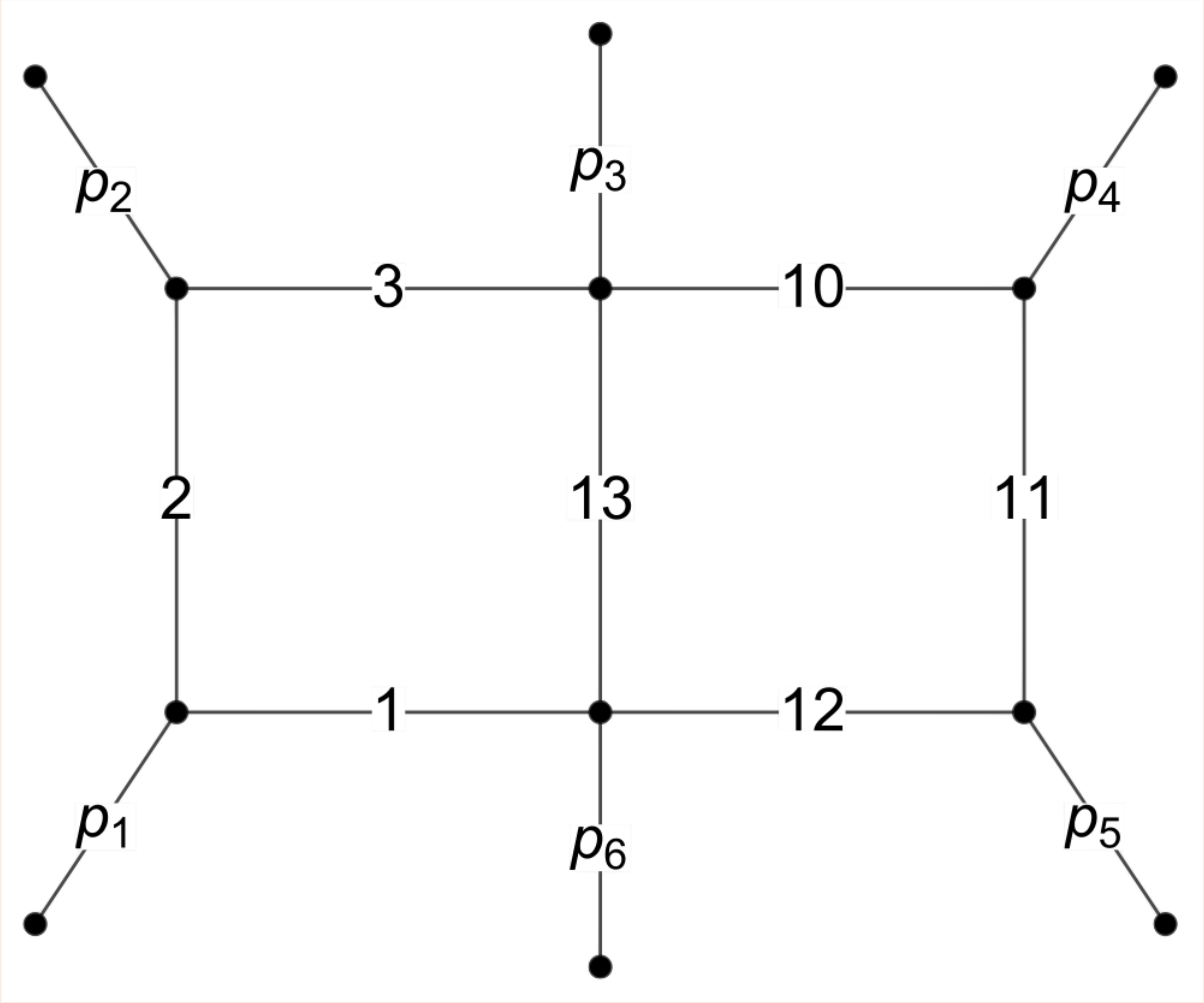
$$N_7 = -\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.$$

Chiral numerator *(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)* / Gram determinant correspondence

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12].$$

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *JHEP08(2024)027*

Chiral numerator to UT integral numerators



linear combination

$$\mathcal{N}_A = s_{45}\big(\langle 15\rangle[52] + \langle 16\rangle[62]\big) l_1 \cdot (\lambda_2\tilde{\lambda}_1)\,,$$
$$\mathcal{N}_B = s_{45}\big([15]\langle 52\rangle + [16]\langle 62\rangle\big) l_1 \cdot (\lambda_1\tilde{\lambda}_2)\,.$$

parity even

$$\mathcal{N}_A + \mathcal{N}_B = -\frac{1}{2} s_{12}s_{45}(l_1 + p_5 + p_6)^2 + \frac{1}{2} s_{12}s_{45}s_{156} + \ldots$$

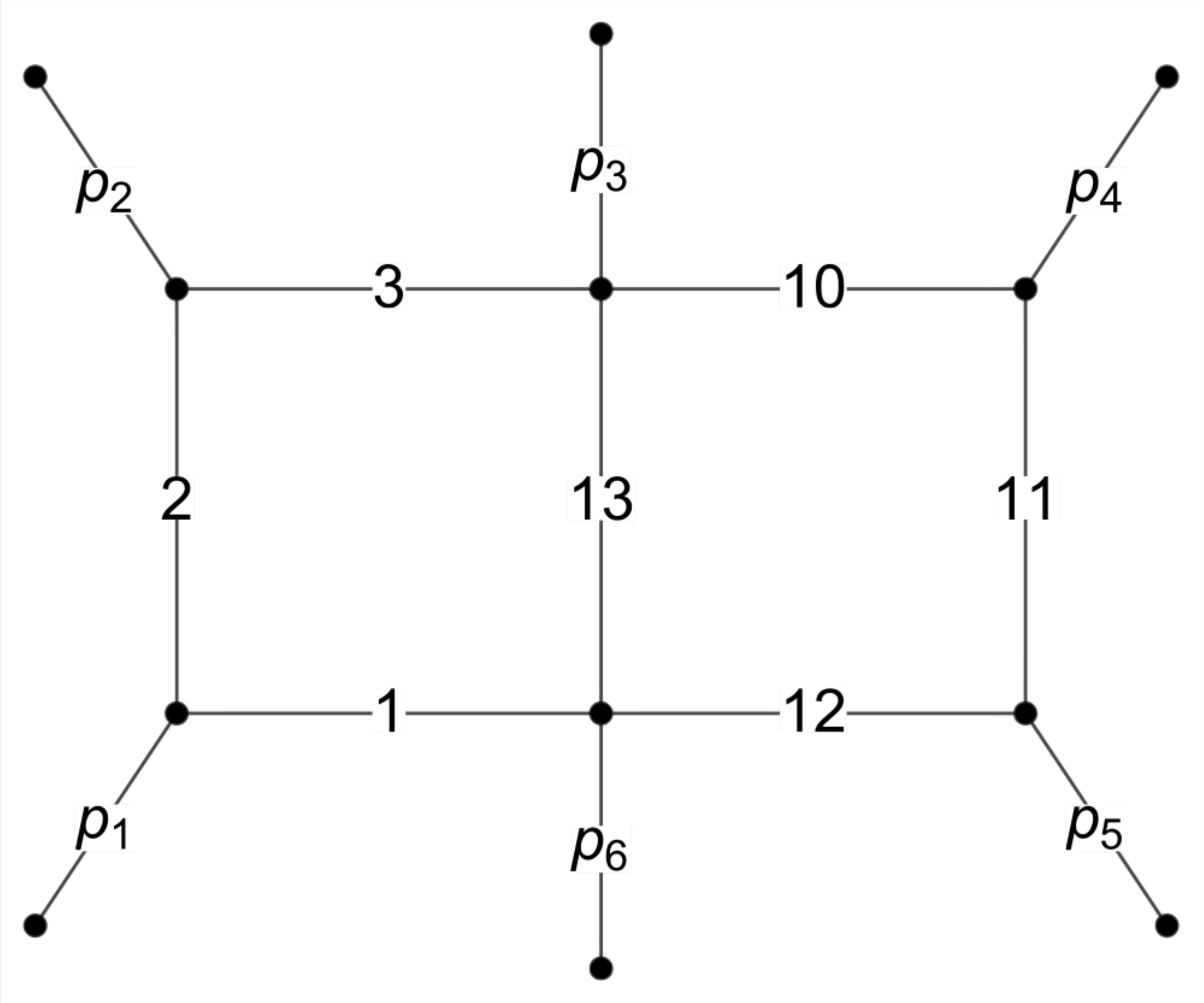
parity odd

$$\mathcal{N}_A - \mathcal{N}_B = \frac{-8 s_{45} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5+p_6 \\ p_5 & p_1 & p_2 & p_6 \end{pmatrix}}{\epsilon_{5126}},$$

Chiral numerator to UT integral numerators

quadratic combination

$$s_{24}\frac{\langle 15\rangle}{\langle 42\rangle}(l_1\cdot\lambda_2\tilde{\lambda}_1)(l_2'\cdot\lambda_4\tilde{\lambda}_5)$$



One-loop
hexagon leading singularity

parity even

$$\frac{1}{8}G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5 \end{pmatrix}+\frac{D_2D_{11}(s_{123}+s_{126})}{8}$$

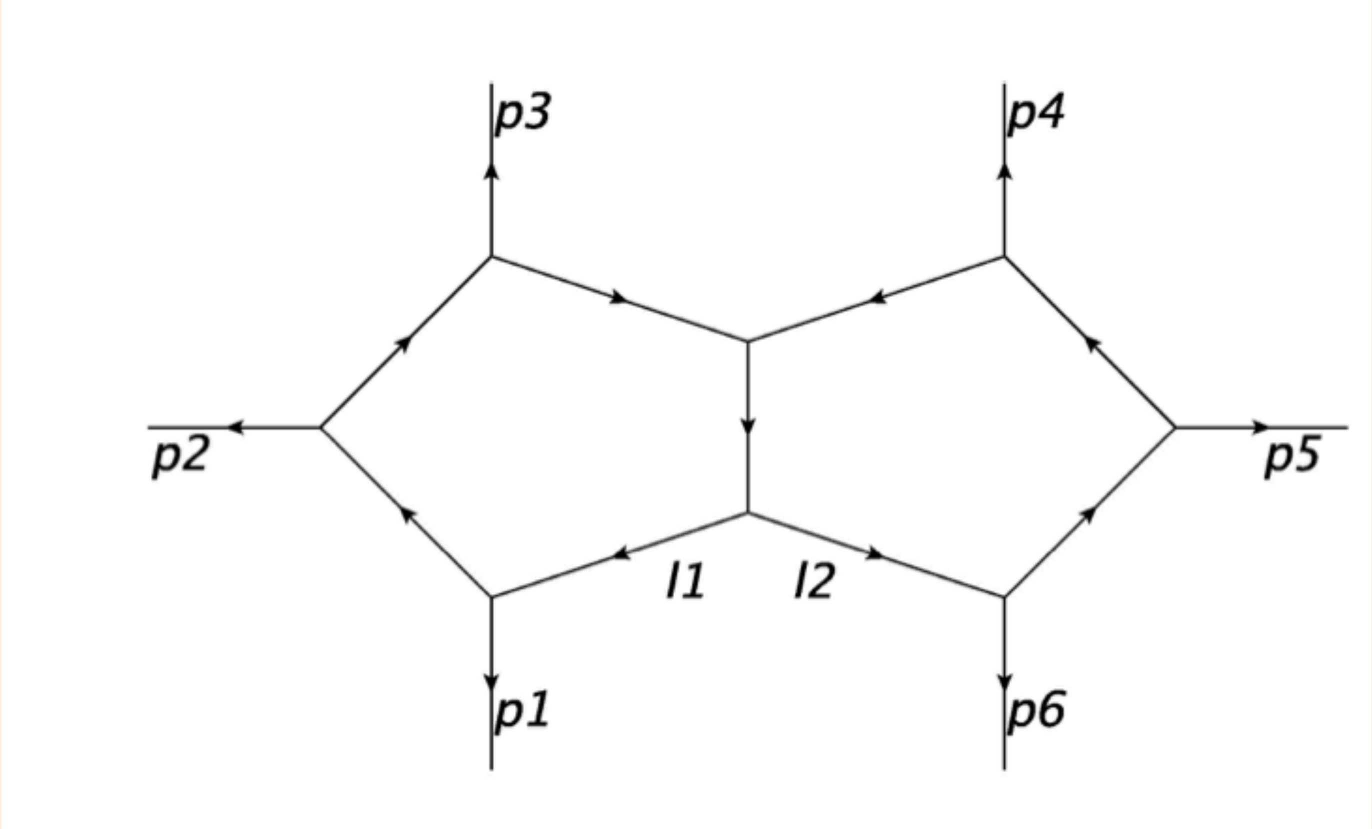
additional term added
from the canonical DE construction

parity odd

$$\Delta_6=\langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61]-\langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

$$-\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}}G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}$$

2loop 6point top sector, UT integrals



J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *2501.01847*

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9}$$ → evanescent

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9}$$ → evanescent

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9}$$ → evanescent, 6D weight-6 integral

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1 \dots D_9}$$

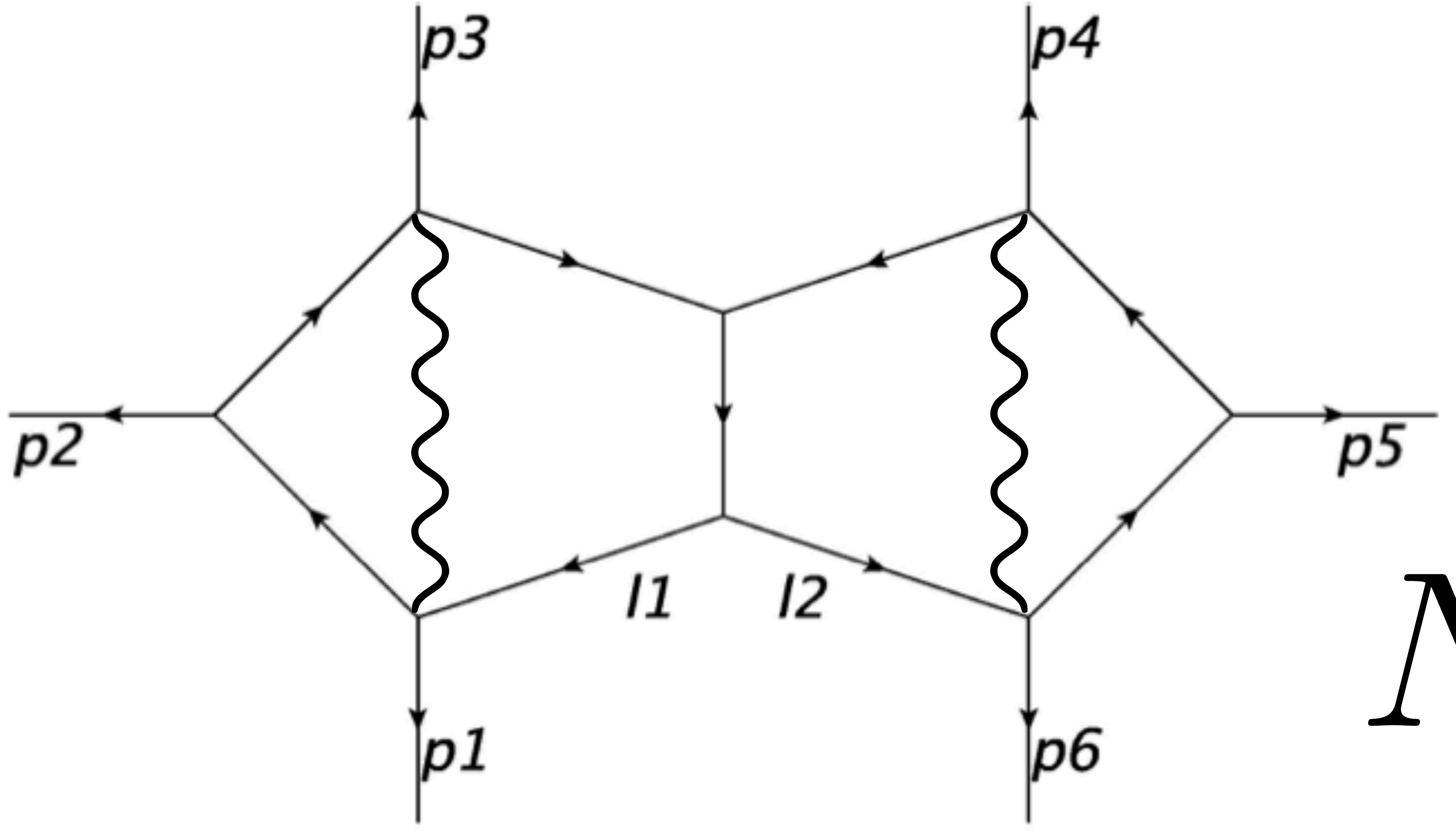
5 MIs (this sector)
267 MIs (whole family)

245 letters in total
except the 6D ones

"evanescent": vanishing up to ϵ^0

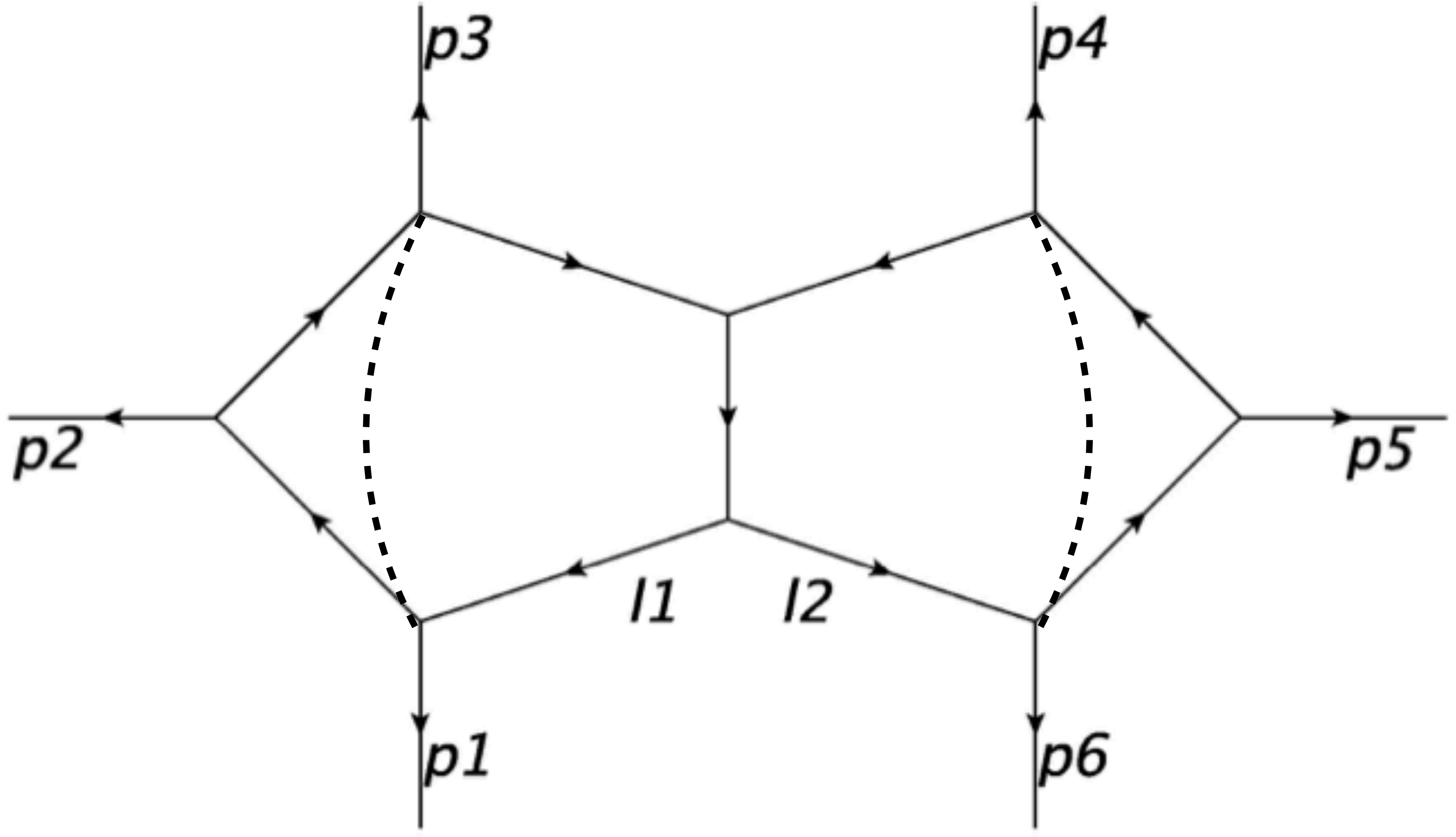
N_1, N_2, N_3 and N_4 are chiral numerators

Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010



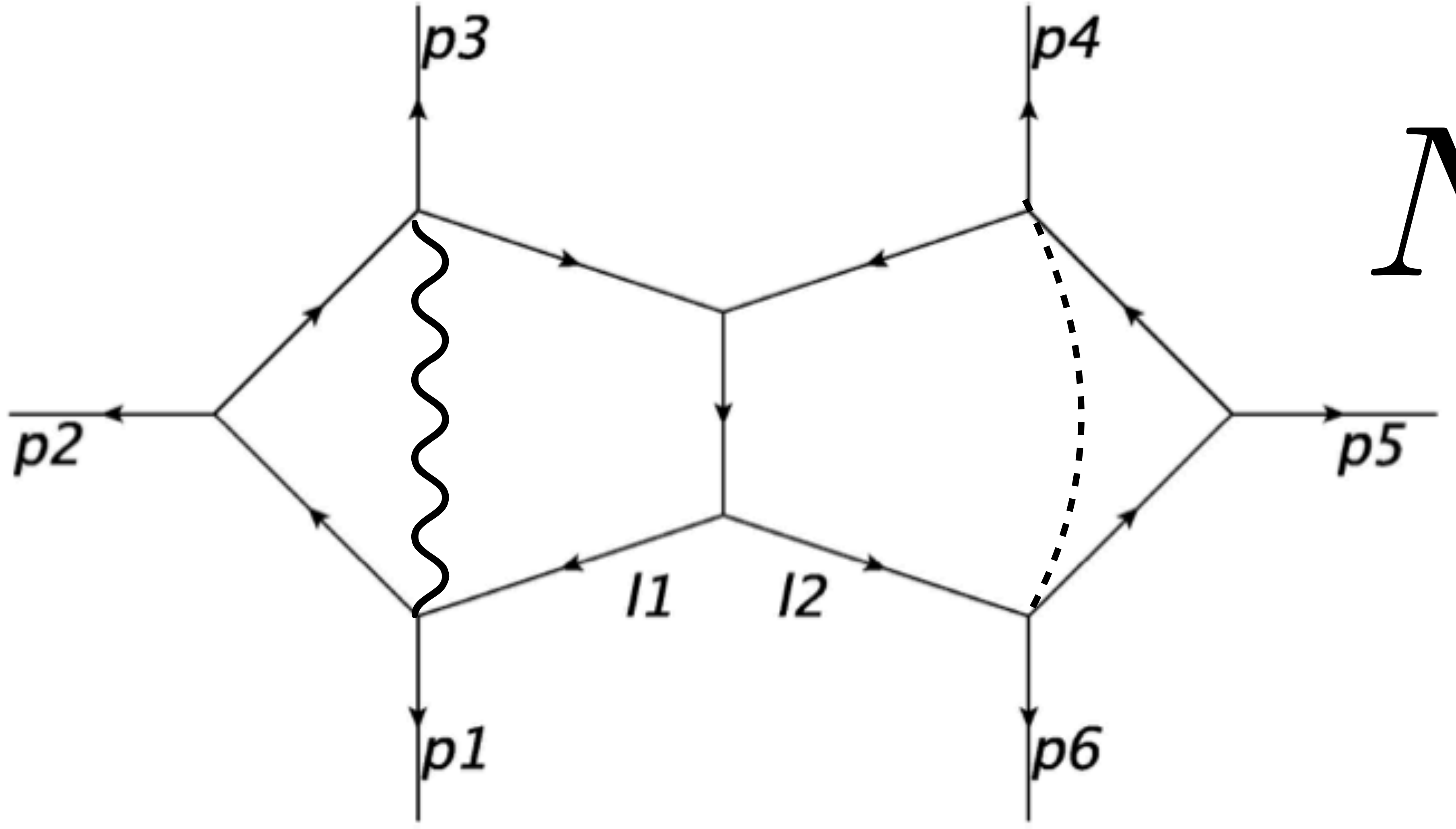
N_1

N_4



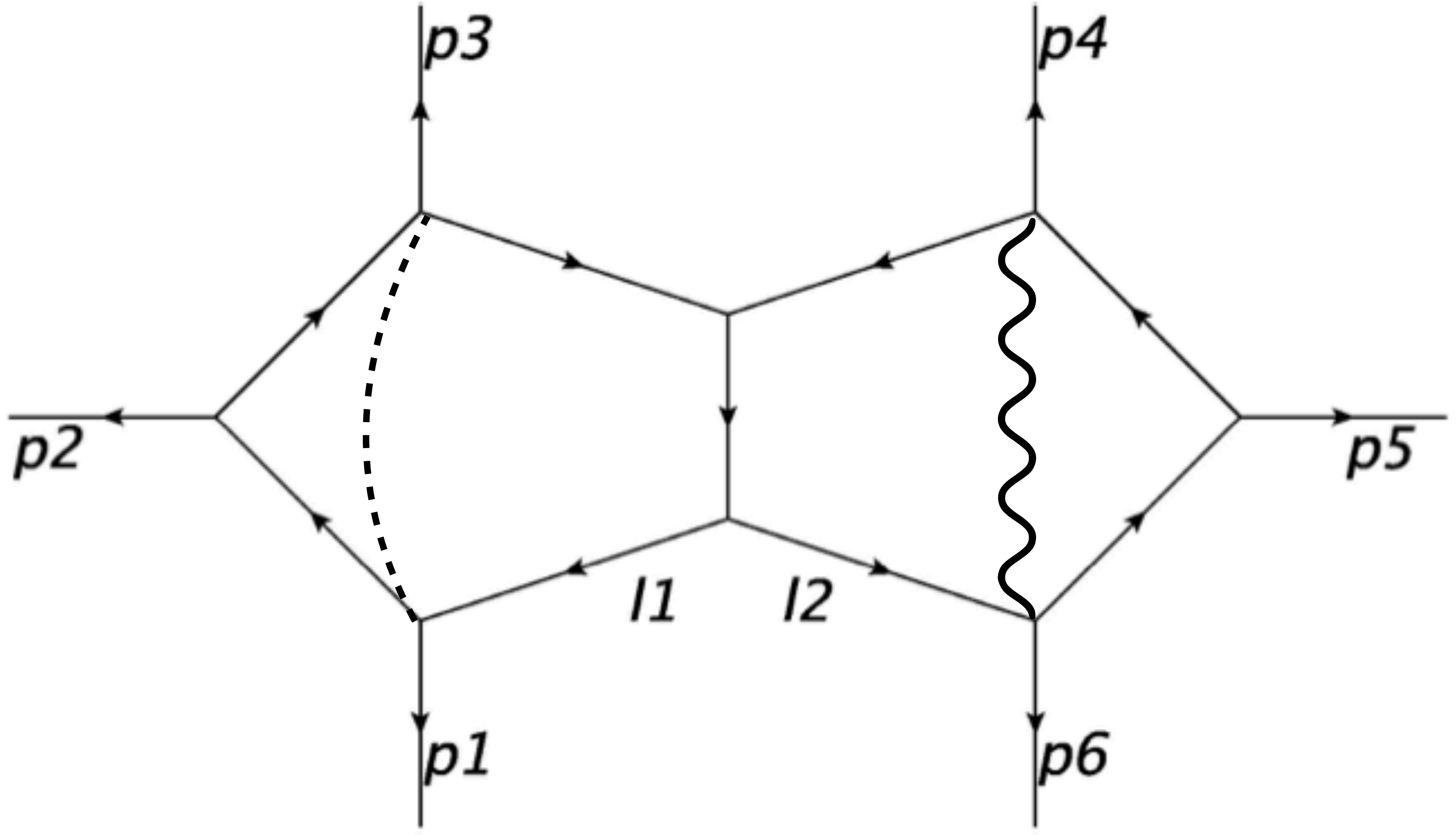
$$ut_2 = I[N_2 - N_3] = -2\tilde{\Omega}_{\text{odd}} + O(\epsilon)\,,$$
$$ut_5 = I[N_1 + N_4] = 2\Omega_{\text{even}} + O(\epsilon)$$

chiral-numerator integrals are finite and calculated to weight-4, Dixon, Drummond, Henn 2011



N_2

N_3



plete canonical differential equation for 2l6p planar integrals

Use momentum twistor Variables

267 × 267 for double pentagon
202 × 202 for hexagon box

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

Brute-force IBP reduction doesn't work
use alphabet to fit the DE

Even letter, Odd letter and the more complicated …

Even letter $F(s)$ a polynomial in Mandelstam variables
or homogeneously linear in square roots

Conjecture: a Feynman integrals' even letters are all from Landau singularity?

Odd letter $$\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": ϵ_{ijkl}, Δ_6, , $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

ϵ_{ijkl} → pseudo scalar

Δ_6 → leading singularity hexagon

$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$ → Källin function from massive triangle diagrams

More complicated letter $$\frac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$$

Even letter, Odd letter and the more complicated ...

245 letters

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\dots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\dots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}$$
$$\dots$$
$$\frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}},$$
$$\dots \quad \dots$$
$$\frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6}$$

10 More complicated letters

$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \quad \dots$$

some letters not found by PLD.jl
Fevola, Mizera, Telen
or
BaikovLetter
Jiang, Liu, Xu, Yang,

but after our paper,
the recent package
SOFIA claims to find all letters
Correia, Giroux, Mizera

Then the canonical DE is derived analytically
after ~200 times of numeric IBP running

A new algorithm to search for odd letters

Odd letter $\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$

$$\boxed{P^2 - Q = c\prod_i W_i^{e_i}, \quad c \in \mathbb{Q}, \quad e_i \in \mathbb{N}}$$

Even letter

An observation (and conjecture) from Heller, von Manteuffel, Schabinger 2020

Algorithm to solve for e_i

Matijasic, J. Miczajka, to appear

Effortless

https://github.com/antonela-matijasic/Effortless

Boundary Values

Numeric boundary values

It is fine to use the package AMFlow to get ~100 digits as the boundary value for double-box, pentagon-triangle, hexagon-bubble diagrams

Liu, Wang, Ma, 2018
Liu, Ma 2022

Analytic boundary values

It is still possible to get *fully analytic* boundary values

$$X_0 : \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1, -1, -1, -1, -1, -1, -1, -1, -1\}$$

Solve the canonical DE on a curve starting with X_0 and require the finite solution
Some known integrals' boundary values

} analytic boundary value

boundary value for a point in the physical region also obtained

Boundary Values

Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$\epsilon^4 I_{\mathrm{db},1}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$\epsilon^4 I_{\mathrm{db},2}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$I_{\mathrm{db},3}(X_0) = I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0,$$

$$\epsilon^4 I_{\mathrm{db},6}(X_0) = -\left(\frac{\pi^4}{540} + \frac{4}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$\epsilon^4 I_{\mathrm{db},7}(X_0) = 0.$$

from the ordinary differential equation

spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Solution of canonical DE

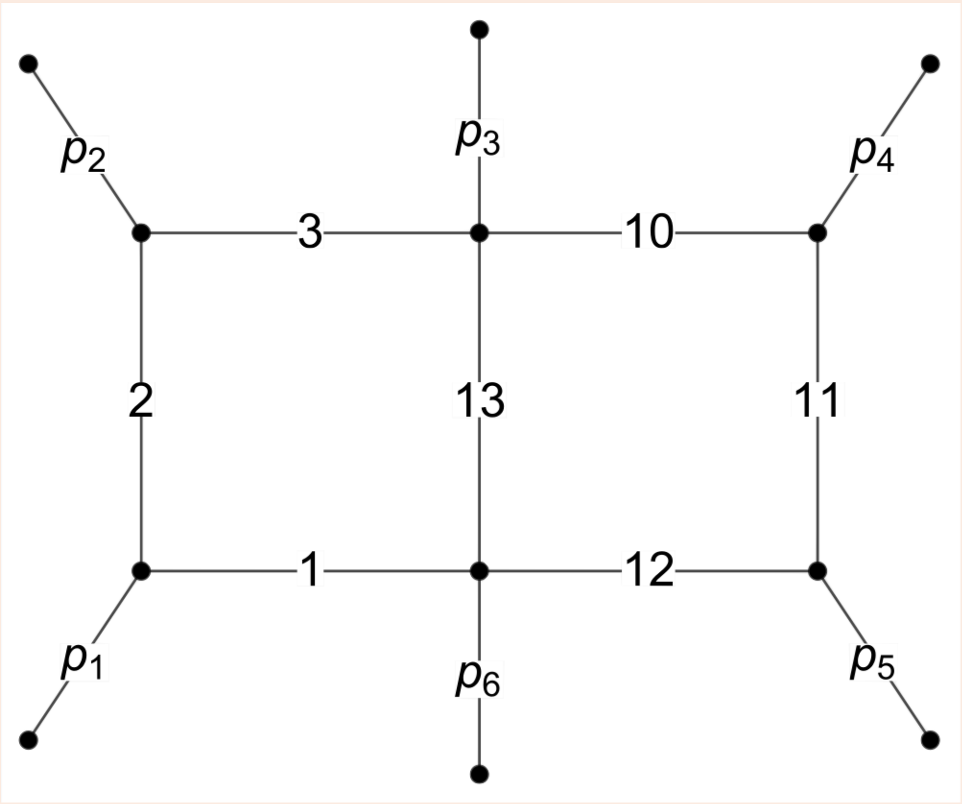
$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-1, weight-2

All in logarithm and classical poly-logarithm



$$\begin{aligned}
I^{(2)}_{\mathrm{db},1} = \; & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\
& \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\
& \log(-v_2)\log(-v_4) - \mathrm{Li}_2\left(1 - \frac{v_2 v_5}{v_7 v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\
& 2\mathrm{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\
& \mathrm{Li}_2\left(1 - \frac{v_3 v_6}{v_8 v_9}\right) - 2\mathrm{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\
& \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\
& \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\mathrm{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\
& \log^2(-v_5) - 2\mathrm{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\
& \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\
& \frac{\pi^2}{6}
\end{aligned}$$

Solution of canonical DE

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-3, weight-4

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1}\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2}\vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{f}^{(2)}(t)\right).\end{aligned}$$

one-fold integration

It takes minutes on a laptop to get 20 digits from our analytic solution

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *JHEP08(2024)027*

All planar 2loop 6point massless integrals calculated, up to ϵ^0

A counting of functions, with the dihedral symmetry

Weight	1	2	3	4
# Symbols	9	62	319	945
# One-loop squared symbols	9	59	221	428
# Two-loop five-point symbols	9	59	263	594
# Genuinely two-loop six-point symbols	0	0	3	45

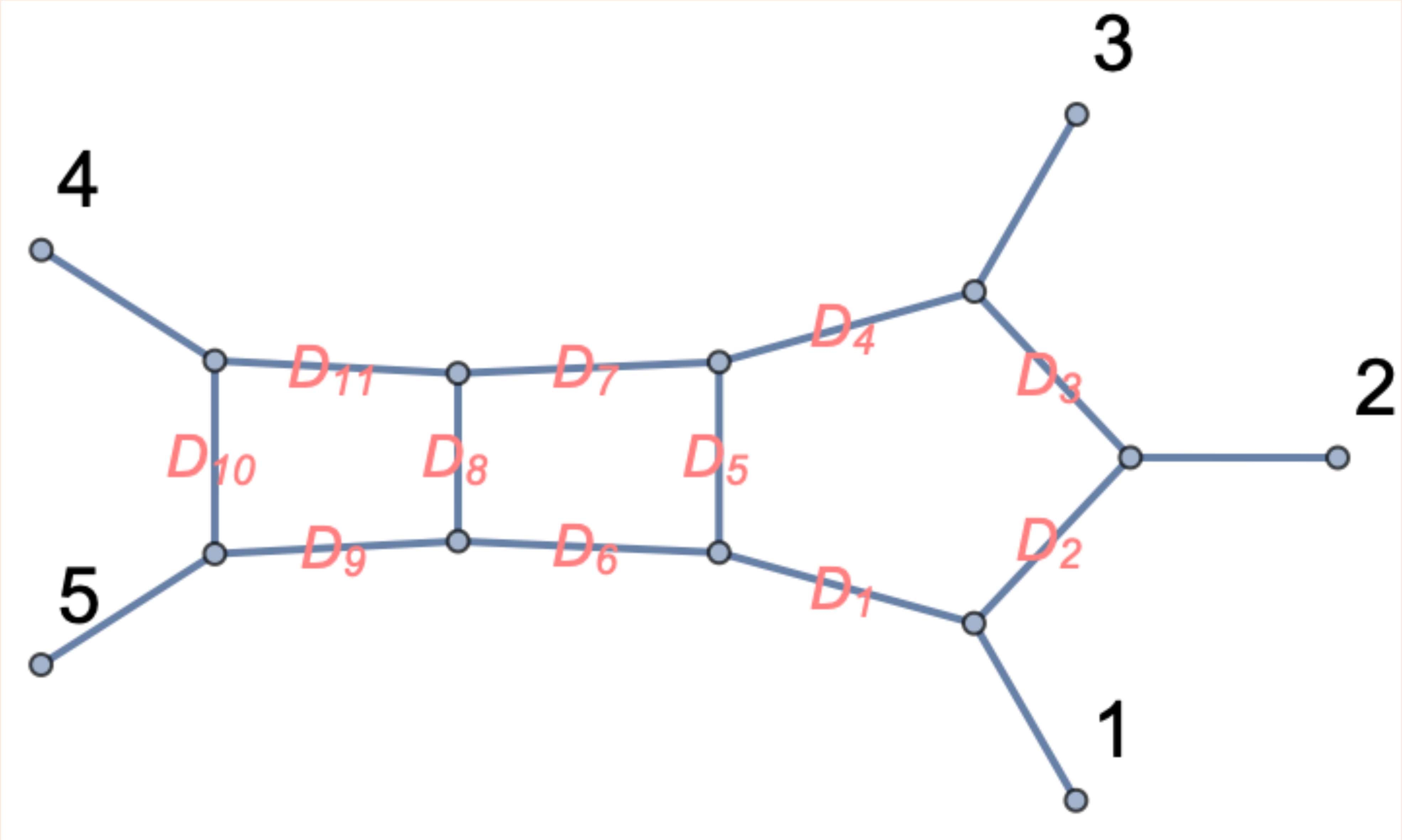
Surprisingly,
the number of genuinely two-loop six-point functions are very small ...
It is a good news for bootstrap

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847

3loop 5point Feynman integrals

from the request of John Ellis …

3loop 5point planar family



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, arXiv: 2411.18697

✓ UT basis found!

Baikov analysis
Gram determinant

Canonical differential equation complicated ?

We use NeatIBP to find the differential equation

hard to integrate to weight-6?

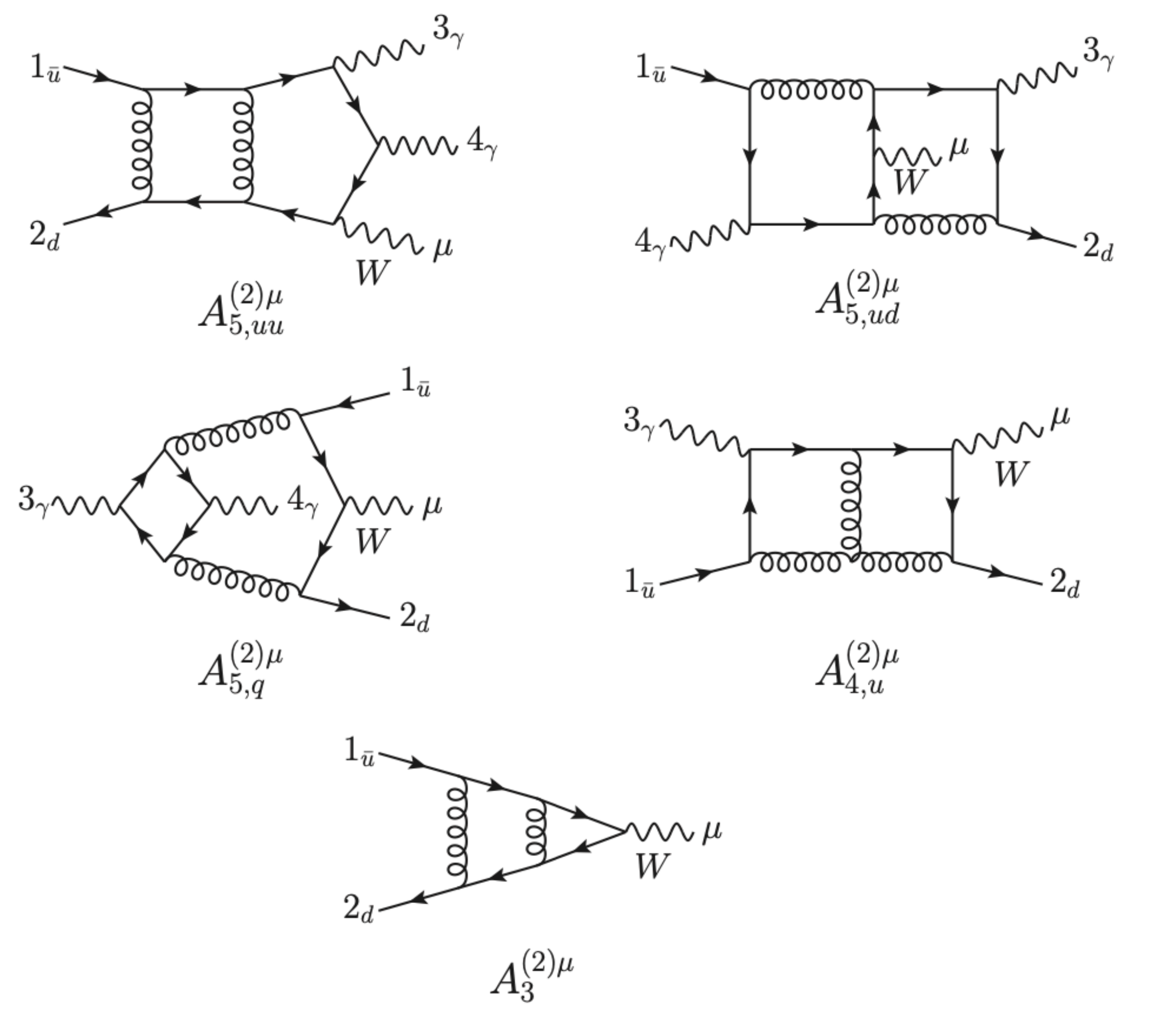
A novel one-fold representation

Using NeatIBP

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

Using algebraic geometry (module intersection) to find short IBP system,
2 or 3 orders of magnitudes shorter than that from Laporta algorithm.

Example: NNLO QCD correction to W + 2 photon production,



Badger, Hartanto, Wu, *YZ*, Zoia, JHEP12(2024)221

Planar diagram:
NeatIBP+ Finiteflow 8 times faster than Finiteflow itself, with 1/3 RAM usage

non-Planar diagram:
NeatIBP+ Finiteflow works
Finiteflow itself does not provide the result

A novel representation of iterative integrals

$$\mathbf{I}^{(n+2)}(x) = \mathbf{I}^{(n+2)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t} \mathbf{I}^{(n+1)}(x_0)\mathrm{d}t + \int_0^1 (\tilde{A}(1) - \tilde{A}(t)) \frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t} \mathbf{I}^{(n)}(t)\mathrm{d}t.$$

Weight +2

A novel formula

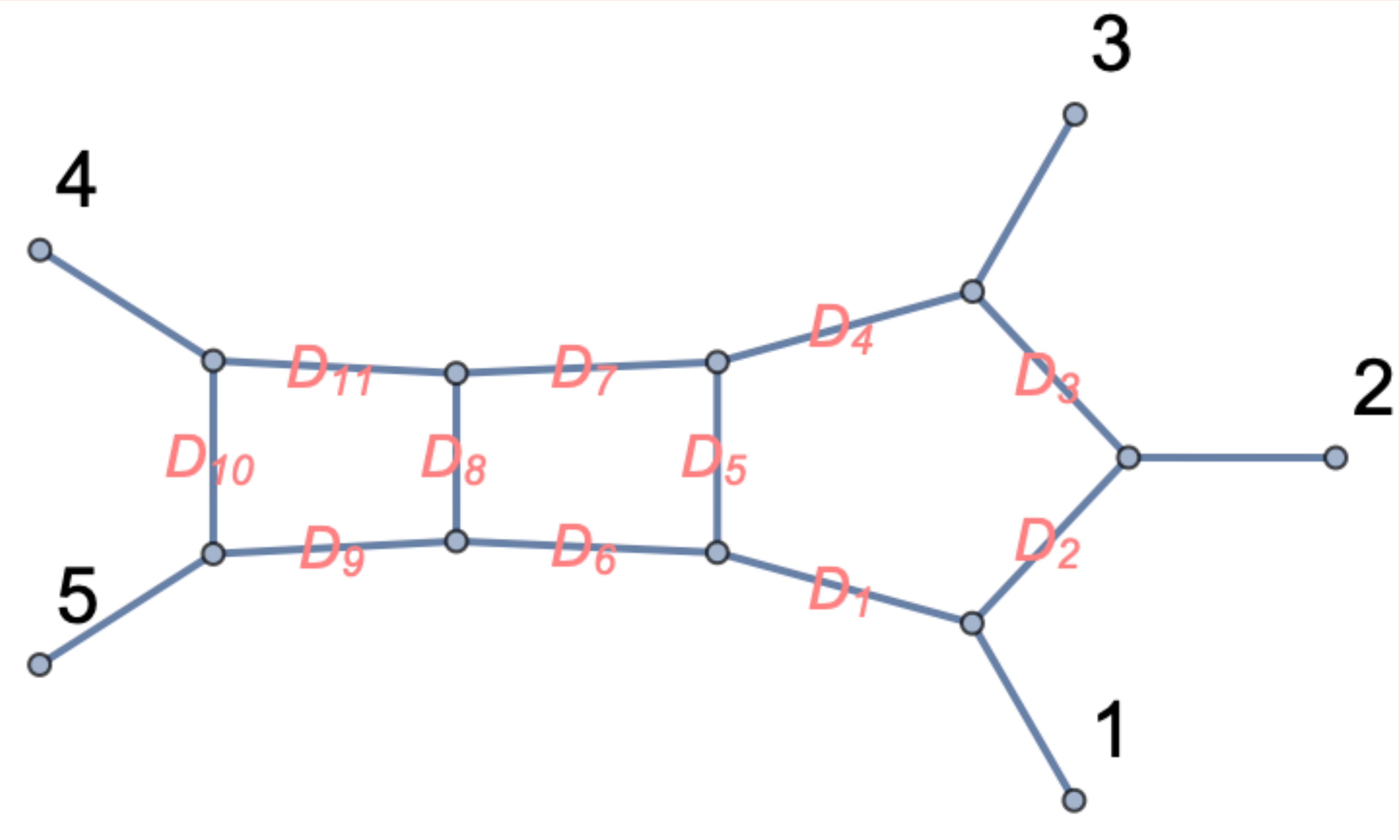
$$\mathrm{d}\tilde{B} = (\mathrm{d}\tilde{A})\tilde{A},$$

$\tilde{B}$ exists due to Poincare lemma

$$\begin{aligned}\mathbf{I}^{(n+3)}(x) =& \mathbf{I}^{(n+3)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n+2)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(1) - \tilde{A}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n+1)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(t) - \tilde{A}(1)\right) \tilde{A}(t) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n)}(t)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{B}(1) - \tilde{B}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n)}(t)\mathrm{d}t.\end{aligned}$$

Weight +3

What we achieved



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

✔ UT basis found!

✔ Canonical differential equation found with NeatIBP

31 letters …

All boundary values up to weight-6 are obtained by spurious pole analysis as GPL values

✔ First 3loop 5point integral family evaluated

weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration

Liu, Matijasic, Miczajka, Xu, Xu, YZ, *arXiv: 2411.18697*

Summary and Outlook

Analytic computation of all 2loop 6point planar massless integrals is done
The first computation on 3loop 5point family is done;

very interesting to see the 2 to 3 NNNLO infrared subtraction

NeatIBP, Effortless and novel representation for iterative integrals
make multi-leg multi-scale Feynman integrals analytic computation easier

AI for finding UT integrals?
Better algorithm to find boundary values?
Elliptic, hyperelliptic, Calabi-Yau

from **Michelangelo's Genesis**

Thank you

Side plot: 2loop N=4 spacelike splitting amplitude

Space-like collinear: generalized factorization (factorization violation)

Collinear particles one incoming one outgoing $p_a \cdot p_b < 0$

Tree

$$|\mathcal{M}^{(0)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle$$

One loop

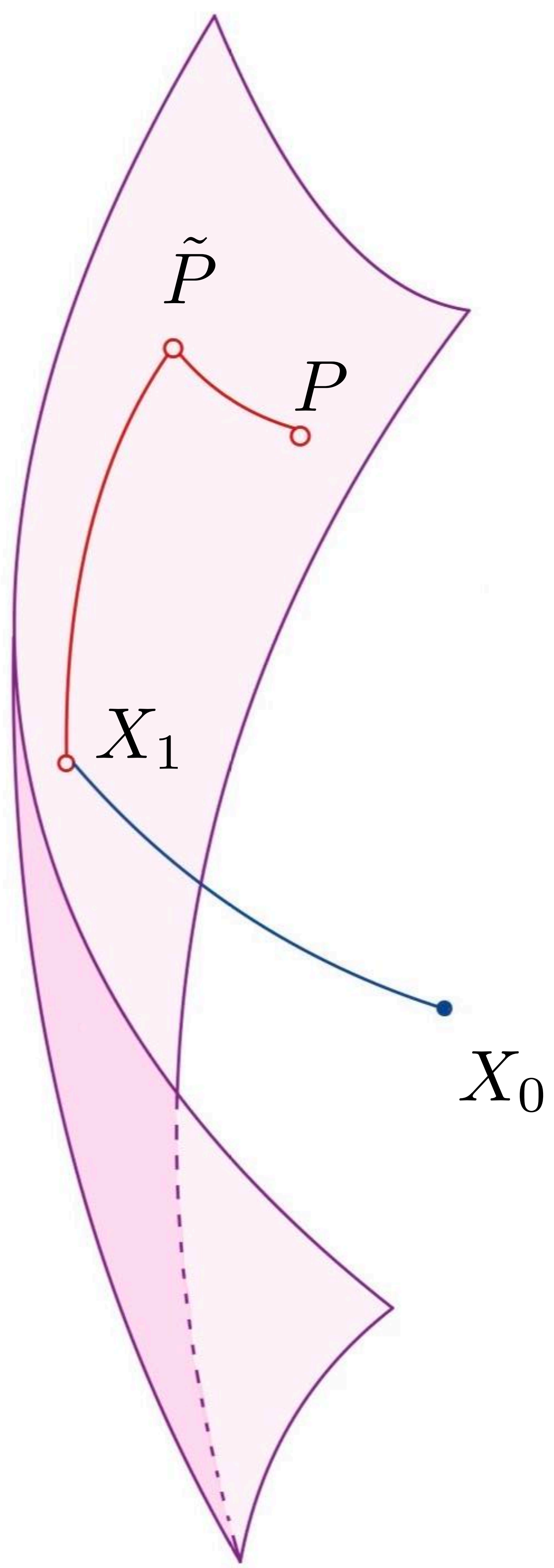
$$|\mathcal{M}^{(1)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(1)}(p_a, p_b; P; p_1, \ldots, \widehat{p_a}, \ldots \widehat{p_b}, \ldots p_n) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(1)}(P, \ldots, p_n)\rangle \; .$$

$$\mathbf{SP}^{(1)}(p_a, p_b; P; p_1, \ldots, \widehat{p_a}, \ldots \widehat{p_b}, \ldots p_n) \supset \frac{2}{\epsilon} \sum_{j \neq a,b} \mathbf{T}_b \cdot \mathbf{T}_j f(\epsilon, z_2 - i s_{j,b} 0^+)$$

Two-loop $\mathbf{Sp}^{(2)}$ was not completely known at that time ...

Catani, de Florian, Rodrigo, 2012

Side plot: 2loop N=4 spacelike splitting amplitude



Solve canonical DE for 2loop 5point master integrals

Sotnikov, Chicherin 2020

$$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{3, -1, 1, 1, -1\}$$

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{\frac{4}{\lambda^2+1}, \frac{-4\lambda^2}{\lambda^2+1}, 1, \frac{2-2\lambda^2}{\lambda^2+1}, -1\}$$

$$X_1: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{4, -4\delta^2, 1, 2, -1\}$$

$$P: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{sz, -4\delta^2, (1-z)xs, s, xs + c\delta\}$$

29040 master integrals solved in terms of s, z, x, δ, y in terms of GPL functions up to weight 4.

With the integrand given in *Carrasco and Johansson 2012*, the 2loop N=4 (planar + nonplanar) 5-point SYM amplitude is obtained in spacelike collinear region as GPL→ HPL → Li functions.

Henn, Ma, Xu, Yan, YZ, Zhu, arXiv 2406.14604

Side plot: 2loop N=4 spacelike splitting amplitude

Henn, Ma, Xu, Yan, YZ, Zhu, arXiv 2406.14604

in memorial of Stefano Catani (1958-2024)

$$\mathbf{Sp}^{(2)} = \left[\frac{\mu^2 z}{s_{ab}(1-z)}\right]^{2\epsilon} \Big\{ 4N_c^2\, \bar{r}_S^{(2)}(z+i0)$$

dipole
$$+ N_c\, \mathbf{T}_a \cdot \mathbf{T}_{\text{in}}\, (2\pi i) \left[c_2(\epsilon)\frac{1}{\epsilon^3} + c_1^2(\epsilon)\left(-\frac{2}{\epsilon^2}\ln z + \frac{2}{\epsilon}\ln z \ln\left(\frac{z}{z-1}\right) \underline{-2\,\text{Li}_3\left(1-\frac{1}{z}\right) - \ln(z)\ln^2\left(\frac{z}{z-1}\right)}\right)\right]$$

tripole
$$+ \sum_{I\in\text{outgoing}} [\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}, \mathbf{T}_a\cdot\mathbf{T}_I]\,(2\pi i)\left[\left(\frac{1}{2\epsilon^2} - \underline{\frac{1}{2}\zeta_2}\right)(\ln|z_I|^2 + i\pi) + \underline{\frac{1}{6}\left(\ln^2\frac{z_I}{\bar{z}_I} + 4\pi^2\right)\ln\frac{z_I}{\bar{z}_I} + 2\zeta_3}\right]$$

$$+ \sum_{I\in\text{outgoing}} \{\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}, \mathbf{T}_a\cdot\mathbf{T}_I\}\,(2\pi^2)\left[\frac{1}{2\epsilon^2} - \underline{\frac{1}{2}\zeta_2}\right]\Big\}\,\mathbf{Sp}^{(0)}\,.$$

The ε-pole terms were given in *Catani, de Florian, Rodrigo 2012.*

We computed the finite part, from the fully analytic computation of 2loop 5point Feynman integrals.

Side plot: 2loop N=4 spacelike splitting amplitude

Time-like collinear: strict factorization

Collinear particles are both outgoing $p_a \cdot p_b > 0$, with $+ - - -$ signature

Tree $$|\mathcal{M}^{(0)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle$$

One loop $$|\mathcal{M}^{(1)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(1)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(1)}(P, \ldots, p_n)\rangle \; .$$

Two loop $$|\mathcal{M}^{(2)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(2)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(1)}(p_a, p_b; P) \; |\mathcal{M}^{(1)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(2)}(P, \ldots, p_n)\rangle \; .$$

Catani, Grazzini, 1999
Catani, de Florian, Rodrigo, 2012