第七届全国重味物理和量子色动力学研讨会 2025年4月18—22日 中国·南京

Recent progress in Femtoscopic studies

Li-Sheng Geng（耿立升）@ Beihang U.

(Image: CERN)

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- ☞ **Femtoscopic correlation functions (CFs)—general features**
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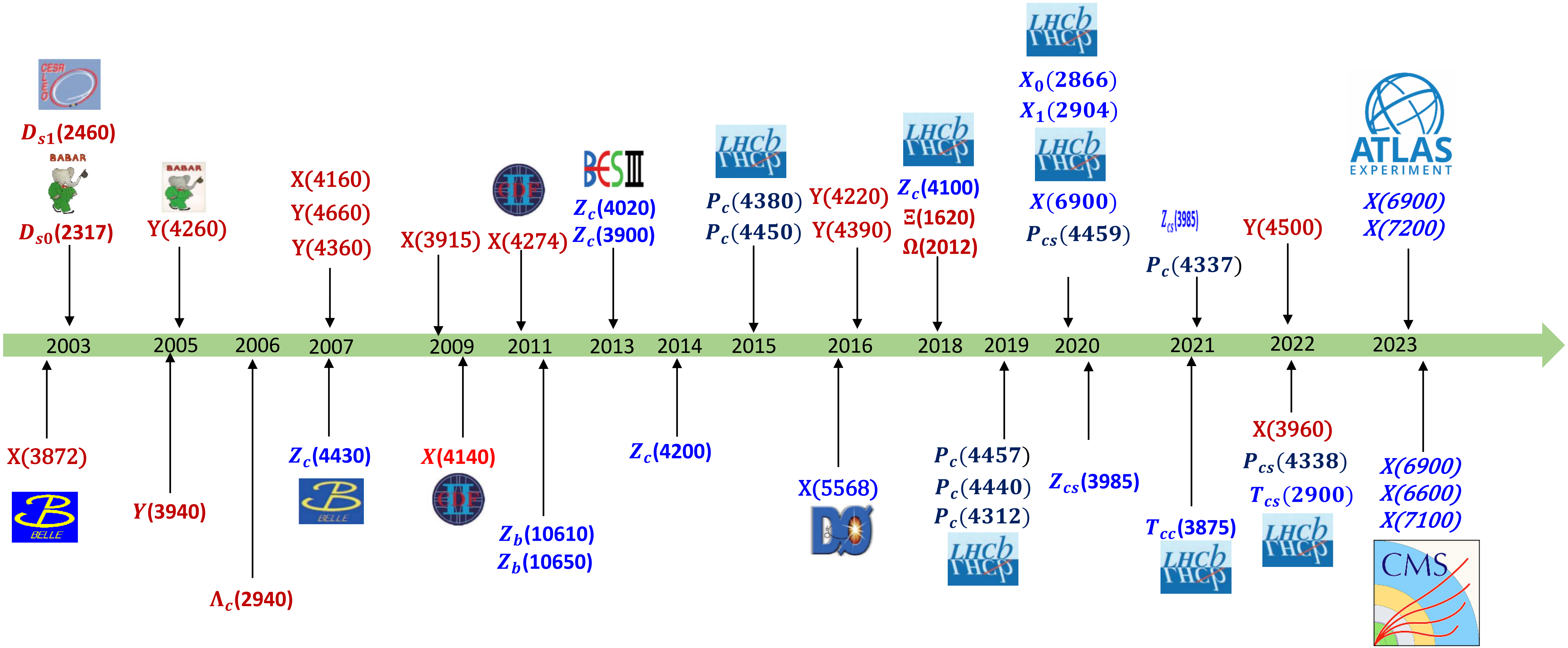
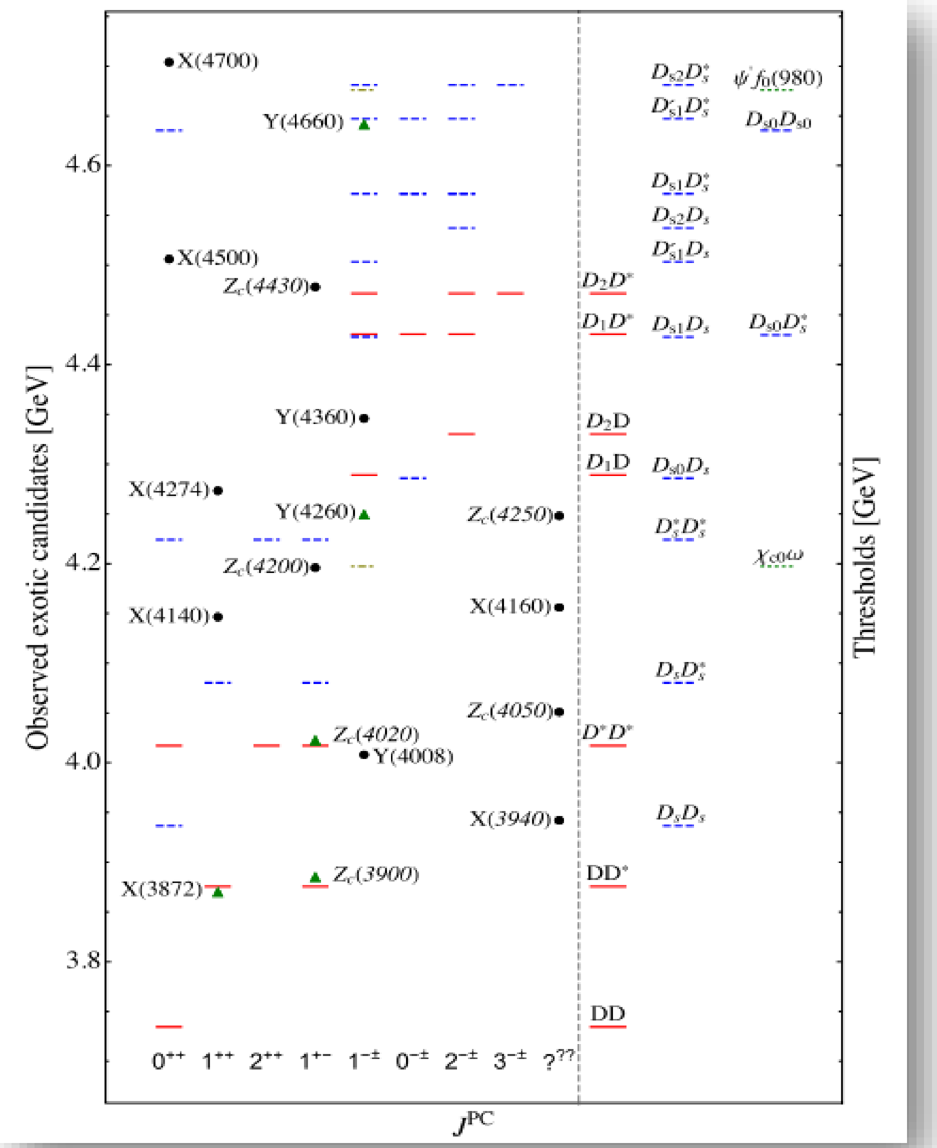
  - ➢ **J/Ψ- J/Ψ correlation functions with EFT potentials**
  - ➢ **J/Ψ-N and η_c-N correlation functions with lQCD phaseshifts**
- ☞ **Summary and outlook**

Many more exotic hadrons discovered



Most of them close to thresholds— hadronic molecules



Feng-Kun Guo, Christoph Hanhart, Ulf-G. Meißner, Qian Wang, Qiang Zhao, Bing-Song Zou. Rev.Mod.Phys. 90 (2018) 015004

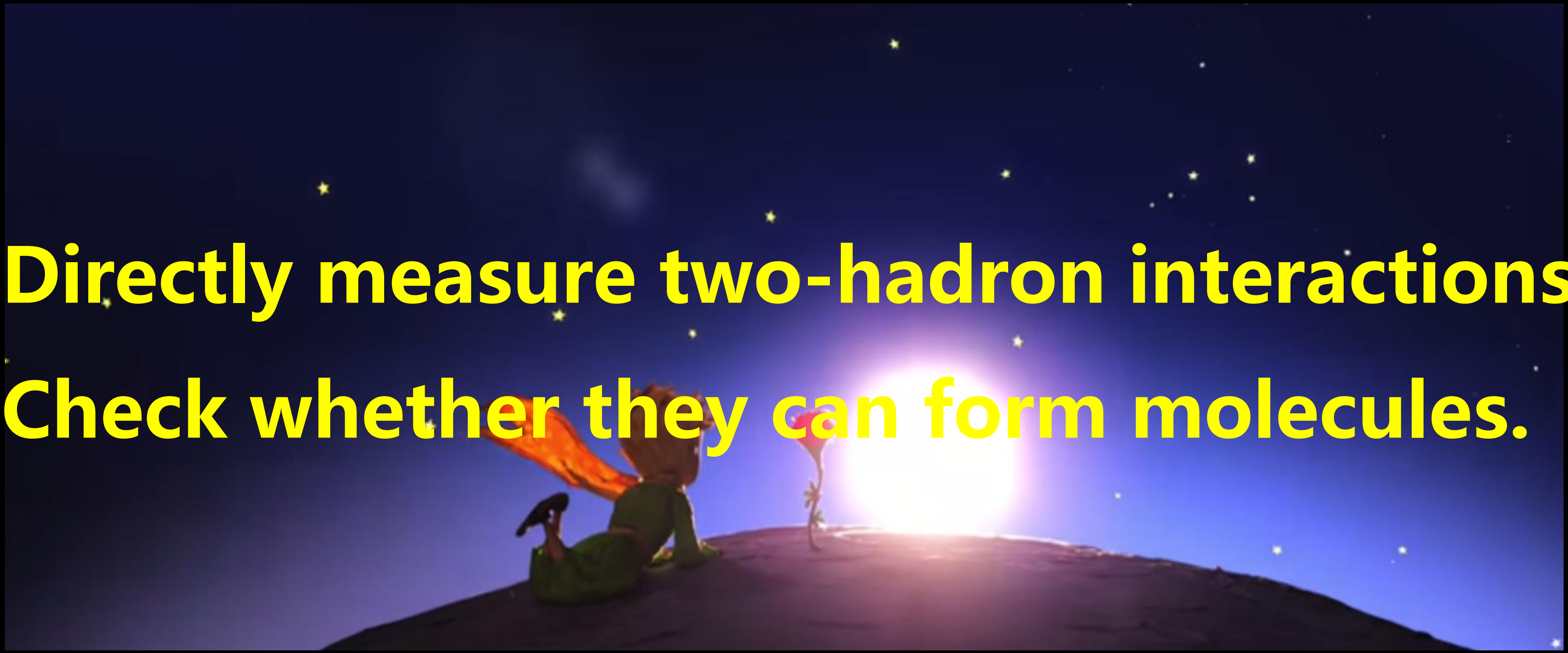
Richard F. Lebed, Ryan E. Mitchell, Eric S. Swanson, Prog.Part.Nucl.Phys. 93 (2017) 143

Atsushi Hosaka, Toru Iijima, Kenkichi Miyabayashi, Yoshihide Sakai , Shigehiro Yasui, PTEP 2016 (2016) 062C01

Hua-Xing Chen, Wei Chen, Xiang Liu Shi-Lin Zhu, Phys. Rept.639 (2016) 1

……

How to check the molecular picture?



Directly measure two-hadron interactions.

Check whether they can form molecules.

Physics Reports 1108 (2025) 1–108

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Physics Reports

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Three ways to decipher the nature of exotic hadrons: Multiplets, three-body hadronic molecules, and correlation functions

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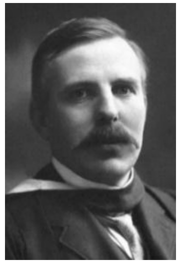
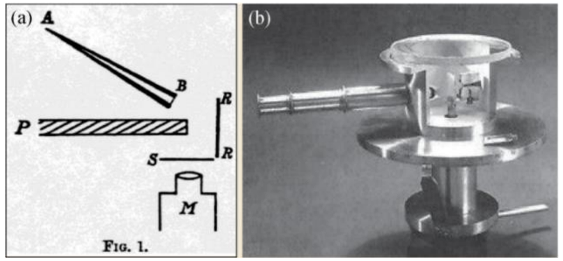
[h] *Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics, Chinese Academy of Sciences, Huizhou 516000, China*

New probe—femtoscopic correlation functions

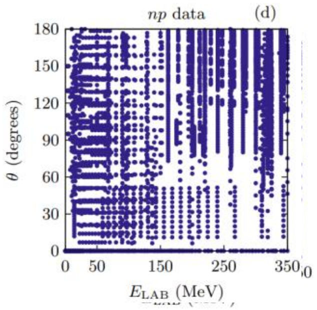
- **For stable hadrons, scattering experiments are extremely valuable in extracting their interactions**

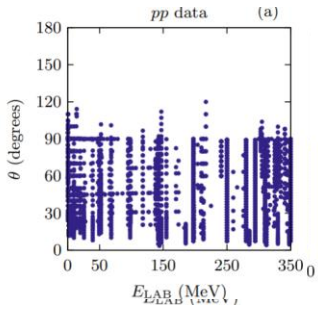


Ernest Rutherford



Rutherford Scattering Experiment





- **NN scattering，8125 data**
- Foundation of high-precision nuclear force

- **For unstable particles, direct scattering experiments are impossible!**

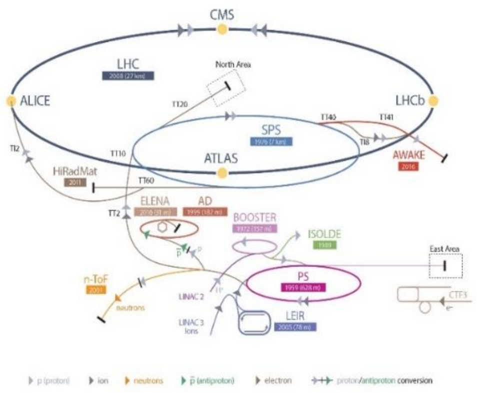
  - **Difficult to get large quantity of beam particles**
  - **No fixed targets available**

$\Lambda p \to \Lambda p$	$\Sigma^- p \to \Lambda n$	$\Sigma^+ p \to \Sigma^+ p$	$\Sigma^- p \to \Sigma^- p$	$\Sigma^- p \to \Sigma^0 n$
12	6	4	7	6

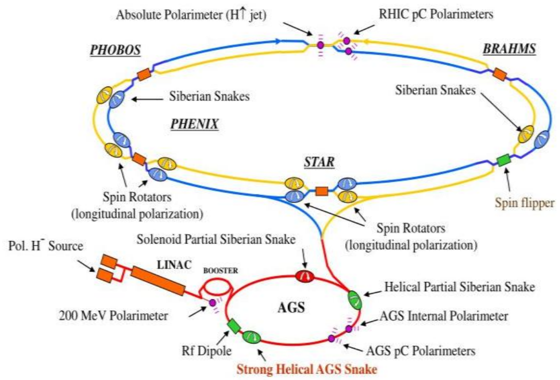
- **Hyperon-nucleon low energy scattering, 35 data**
- Hindering hyper-nuclear physics and neutron star studies

New probe—femtoscopic correlation functions

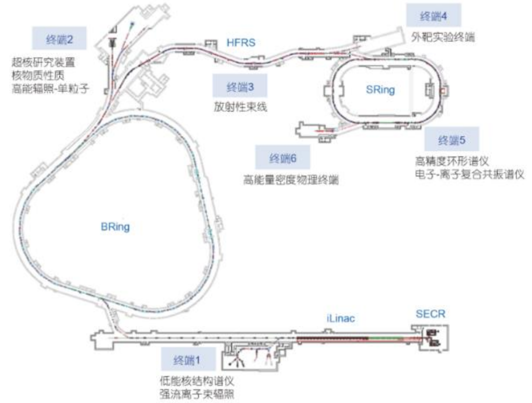
CERN — The Large Hadron Collider



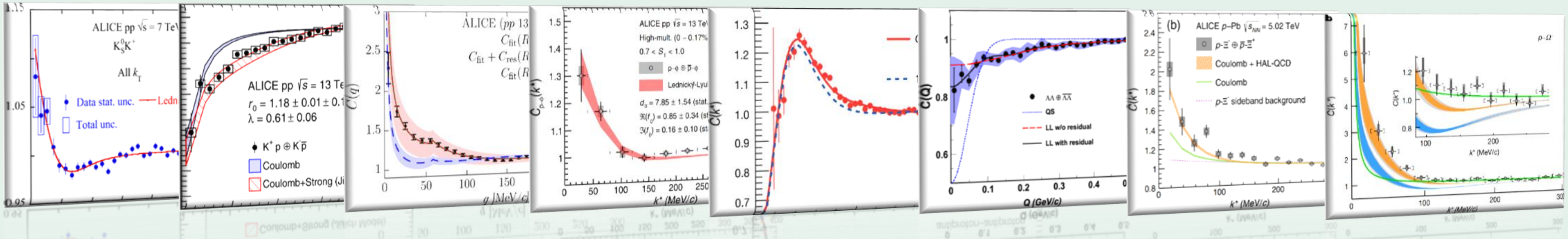
Brookhaven National Laboratory — RHIC Relativistic Heavy Ion Collider



HIAF 强流重离子加速器装置 High Intensity heavy-ion Accelerator Facility



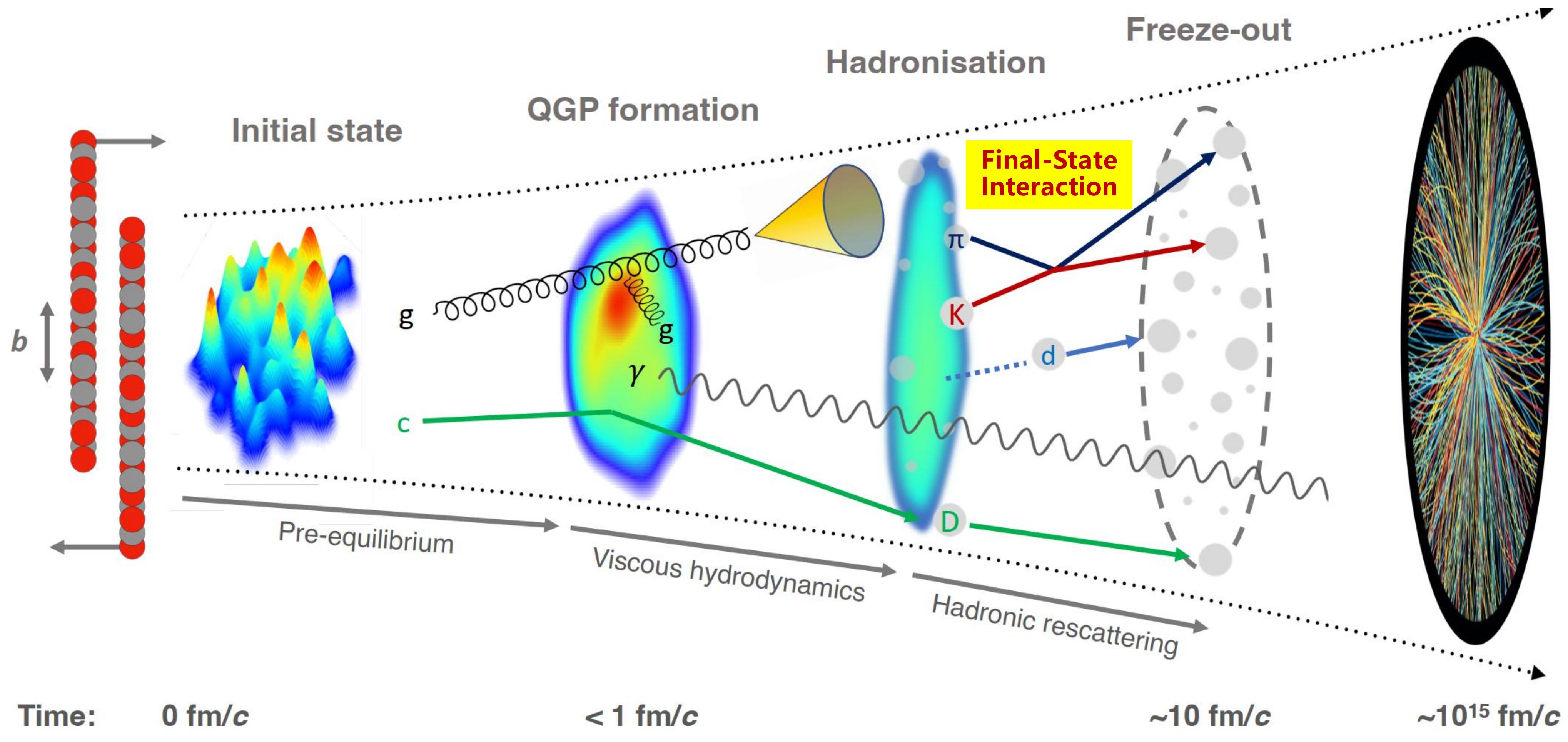
$K^0_S K^\pm$	$K^\pm p$	$K^- p$	ϕp	$\overline{p}\overline{p}$	$\Lambda\Lambda$	$\Xi^- p$	$\Omega^- p$



PLB 2019	PRL 2020	PRL 2020	PRL 2021	Nature 2015	PRL 2015	PRL 2019	Nature 2020

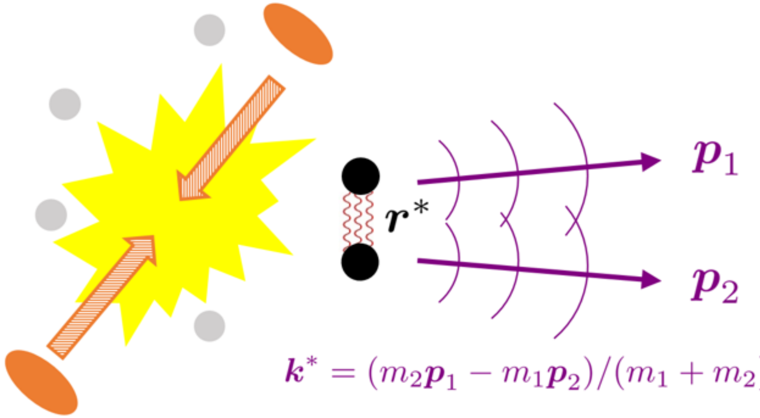
$K^- d$
$\Sigma^0 p$
$\Lambda\Xi$
$\Xi\Xi$
$\Omega\Omega$
pD
ppp
$pp\Lambda$

Abundant particles produced in AA, pA, and pp collisions



Femtoscopic correlation functions (CFs)

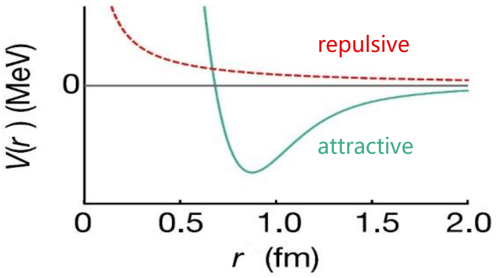
$$C(\boldsymbol{p}_1,\boldsymbol{p}_2) = \frac{\mathrm{P}(\boldsymbol{p}_1,\boldsymbol{p}_2)}{\mathrm{P}(\boldsymbol{p}_1)\cdot\mathrm{P}(\boldsymbol{p}_2)}$$



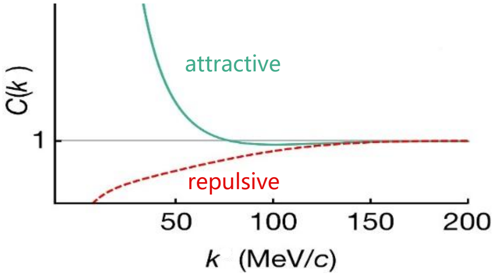
$\boldsymbol{k}^* = (m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2)/(m_1+m_2)$

Emission source $S_{12}(\boldsymbol{r}^*)$ **Two-particle wavefunction $\psi(\boldsymbol{k}^*,\boldsymbol{r}^*)$**

Interacting potential



Correlation function



Schrödinger equation

Two-particle wavefunction $\psi(\boldsymbol{k},\boldsymbol{r})$

Exp. measurement

mixed-event technique

$$C(k) = \xi(k)\frac{N_{\mathrm{same}}(k)}{N_{\mathrm{mixed}}(k)}$$

N_{same}: the same event distributions

N_{mixed}: the mixed event distributions

ξ: the corrections for experimental effects

Theo. description

Koonin–Pratt formula

S. E. Koonin, Phys. Lett. B 70 (1) (1977) 43
*A. Ohnishi, Nucl. Phys. A **954** (2016) 294*

$$C(k) = \int S_{12}(\boldsymbol{r})\;|\psi(\boldsymbol{k},\boldsymbol{r})|^2\;\mathrm{d}\boldsymbol{r}$$

spacial structure

final-state interactions

quantum statistics effects

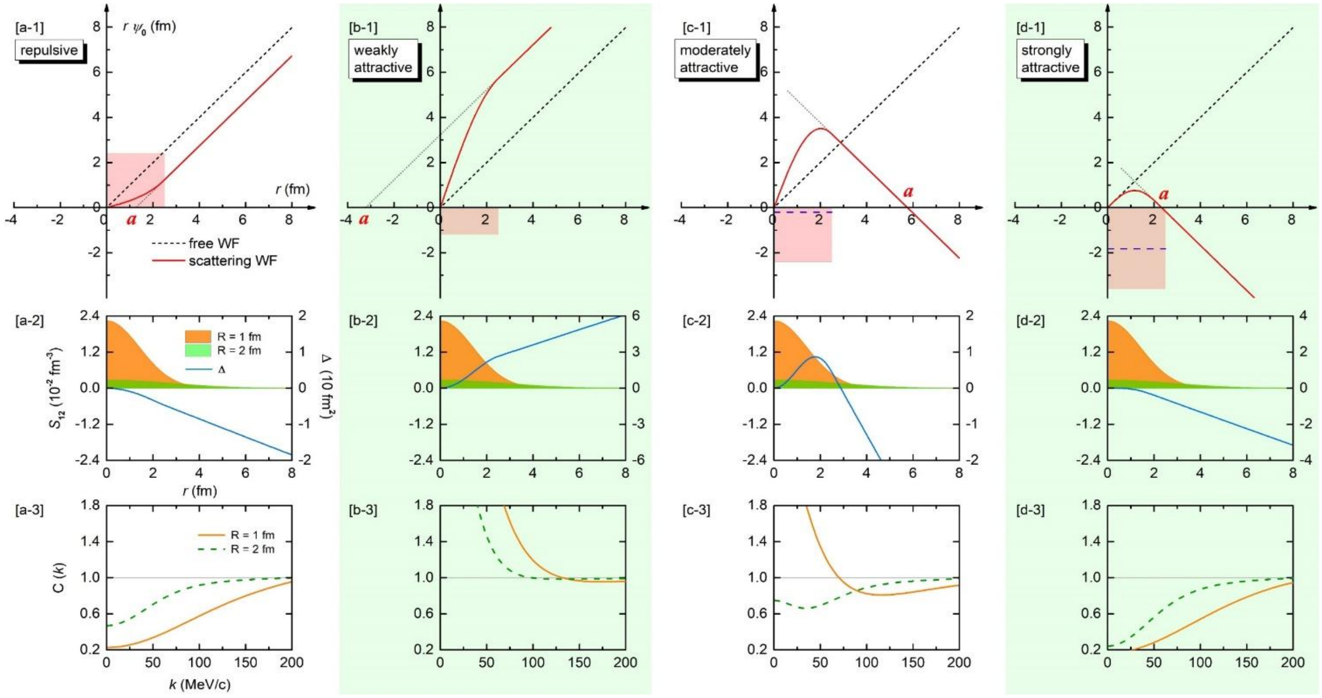
coupled-channel effects

Basic Properties

$C(k)$:
- **>1 if the interaction is attractive**
- **=1 if there is no interaction**
- **<1 if the interaction is repulsive**

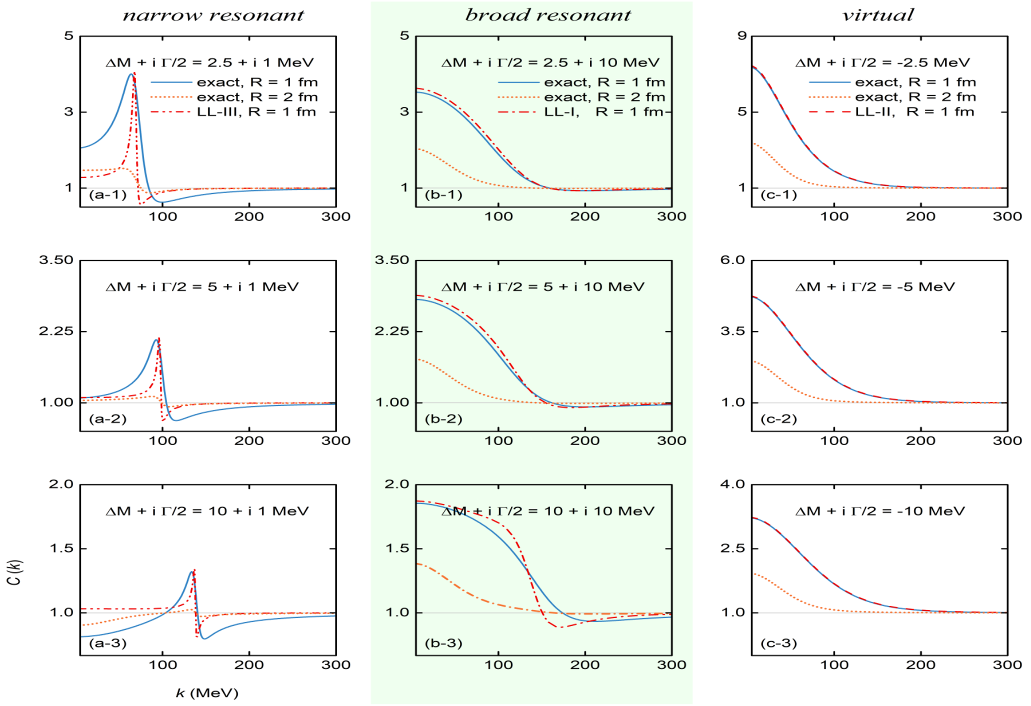
Reality is bit more complicated but understood

Repulsive, attractive, weakly bound, deeply bound



Zhi-Wei Liu, Jun-Xu Lu and LSG*, PRD107(2023)074019

Narrow resonance, broad resonance, virtual



Zhi-Wei Liu, Ming-Zhu Liu, Jun-Xu Lu and LSG*, 2404.18607

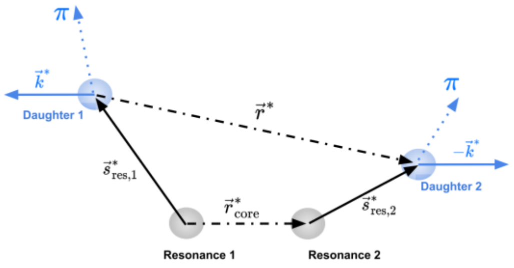
Constraining the source function

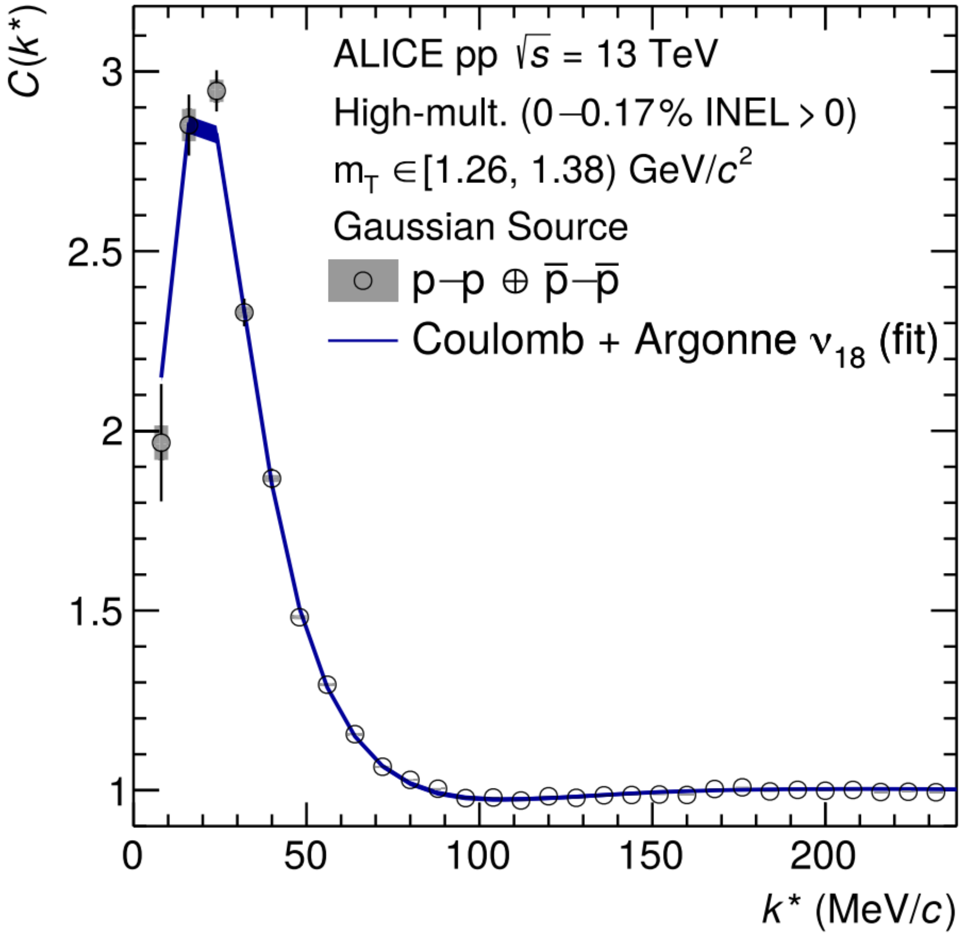
- **Using the well-known proton-proton interaction to calibrate the source**

$$C(k^*) = \int S\left(\vec{r}^*\right)\left|\psi\left(\vec{k}^*,\vec{r}^*\right)\right|^2 d^3\vec{r}^*$$

- **For CFs involving short-lived resonances ($c\tau \sim 1\text{fm}$)—Resonance Source Model**

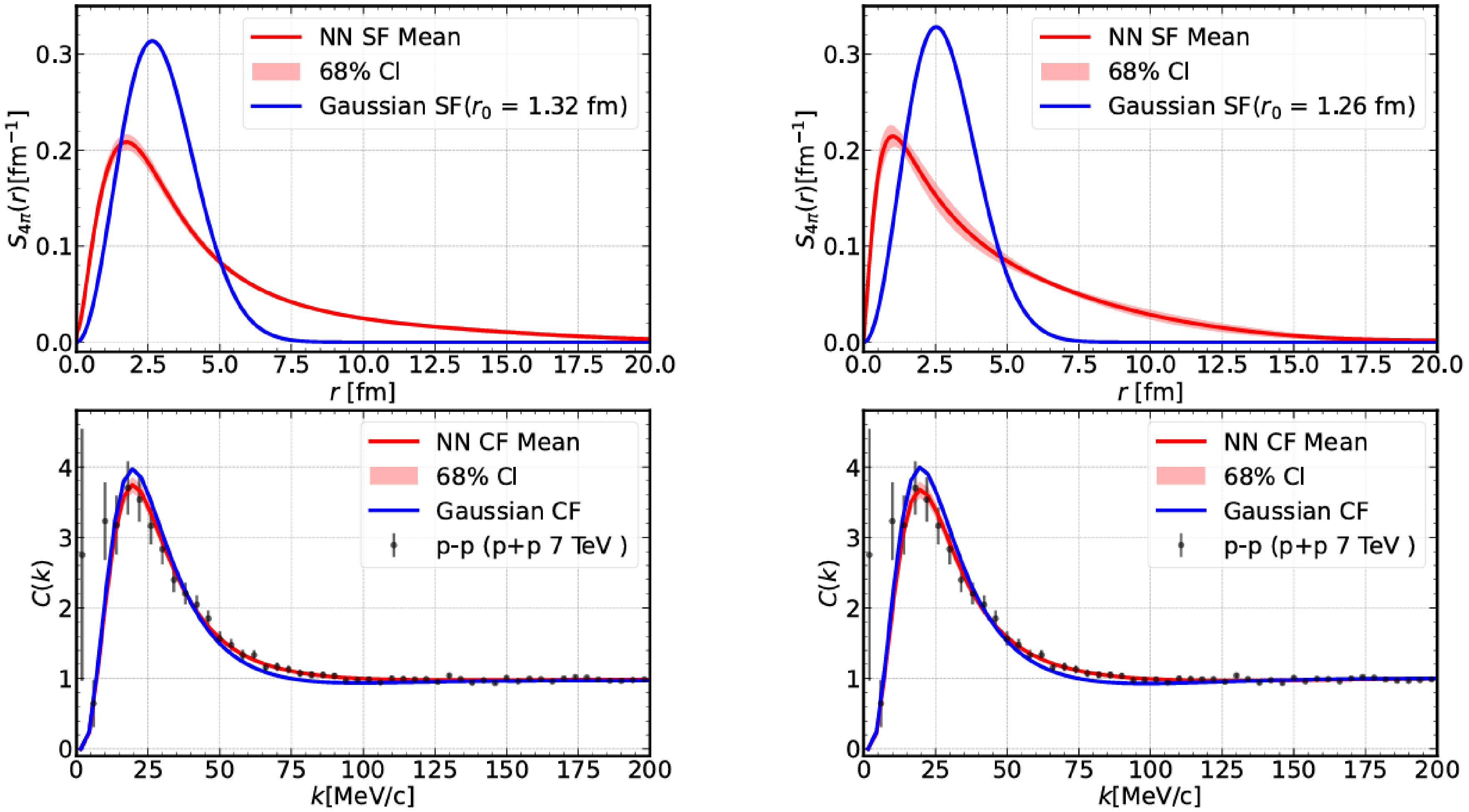
$$S(r^*) = \frac{1}{\left(4\pi r_{\text{core}}^2\right)^{3/2}} \exp\left(-\frac{r^{*2}}{4r_{\text{core}}^2}\right) \times \text{SLR}$$





ALICE, PLB, 811 (2020) 135849

Constraining the source function



Lingxiao Wang1 and Jiaxing Zhao, 2411.16343

Other domestic works

- **K-Lambda and N*(1535):** Si-Wei Liu, Ju-Jun Xie, 2503.22453
- **Sigma*(1430):** Hai-Peng Li, Chu-Wen Xiao, Wei-Hong Liang, Jia-Jun Wu, En Wang et al, 2409.05787
- **P-Omega:** Ye Yan, Qi Huang, Youchang Yang, Hongxia Huang, Jialun, 2408.15493
- **N*(1535):** Hai-Peng Li, Jing Song, Wei-Hong Liang, R. Molina, E. Oset, 2311.14365

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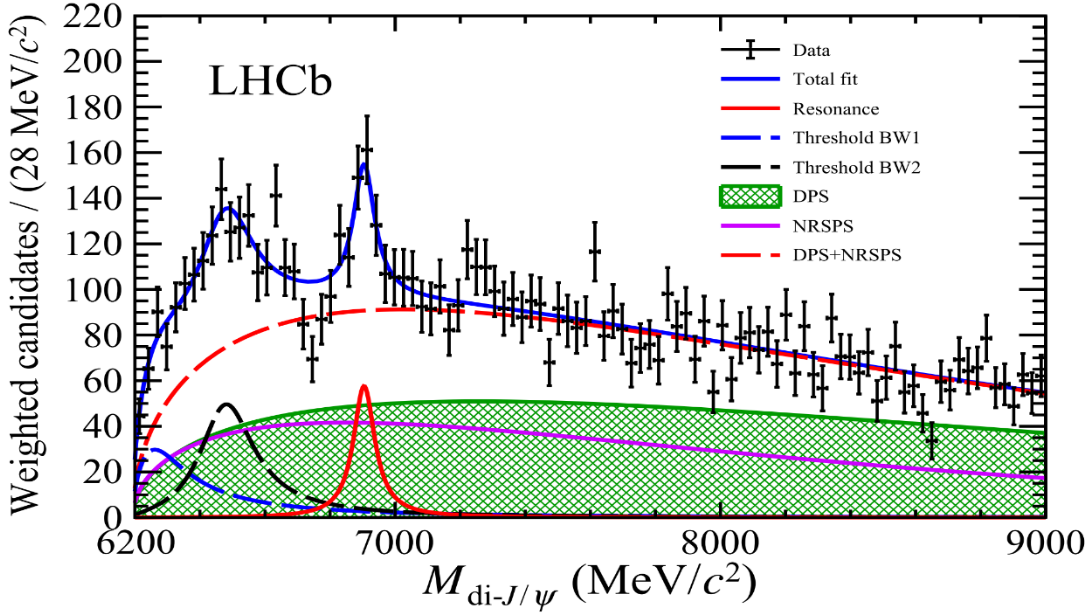
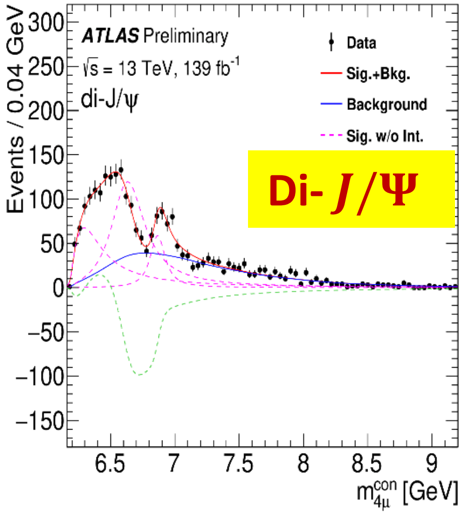
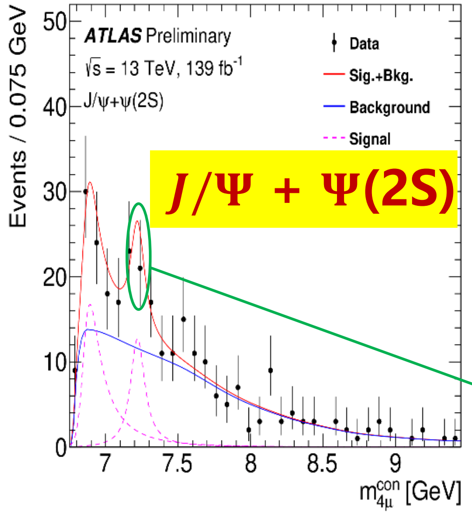
Tetracharm states — LHCb & ATLAS







Science Bulletin 65 (2020) 1983–1993

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Article

Observation of structure in the J/ψ-pair mass spectrum

LHCb collaboration [1]

- **A narrow structure around 6.9 GeV — X(6900)**
- **A broad structure just above twice the J/Ψ mass**

PHYSICAL REVIEW LETTERS **131,** 151902 (2023)

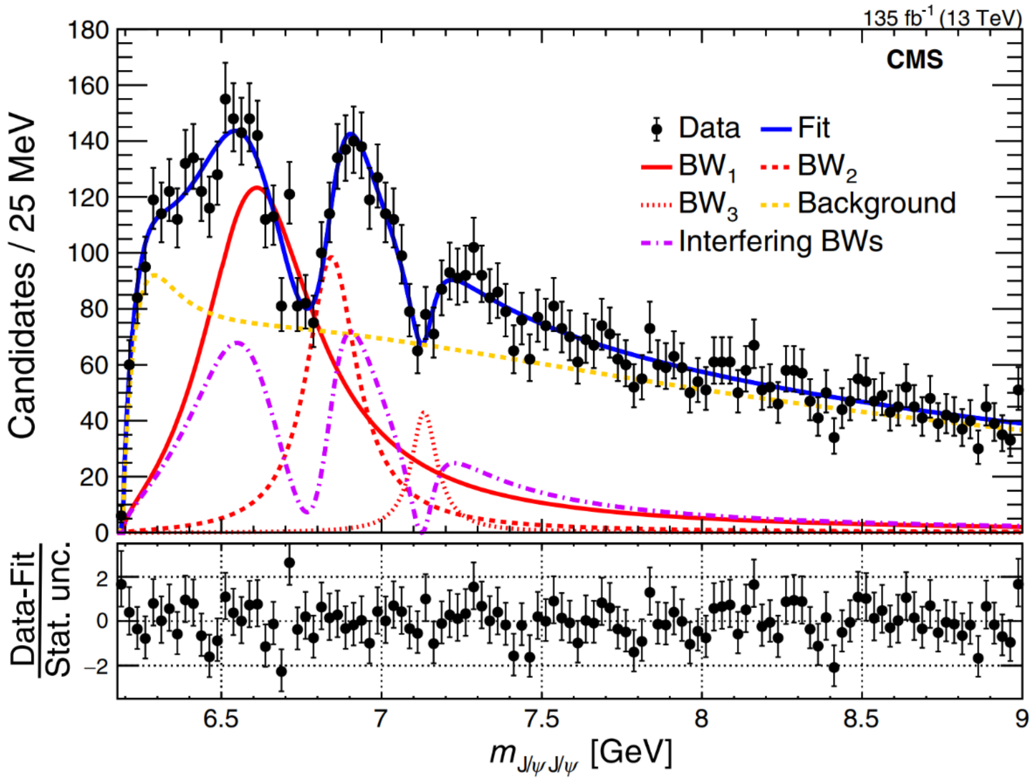
Editors' Suggestion

Observation of an Excess of Dicharmonium Events in the Four-Muon Final State with the ATLAS Detector

G. Aad *et al.*[*]
(ATLAS Collaboration)

- **X(6900) and a broad structure at threshold**
- **A resonance structure at about 7.2 GeV**

Tetracharm states — CMS



PHYSICAL REVIEW LETTERS **132,** 111901 (2024)

Editors' Suggestion

New Structures in the $J/\psi J/\psi$ Mass Spectrum in Proton-Proton Collisions at $\sqrt{s} = 13$ TeV

A. Hayrapetyan *et al.*[*]
(CMS Collaboration)

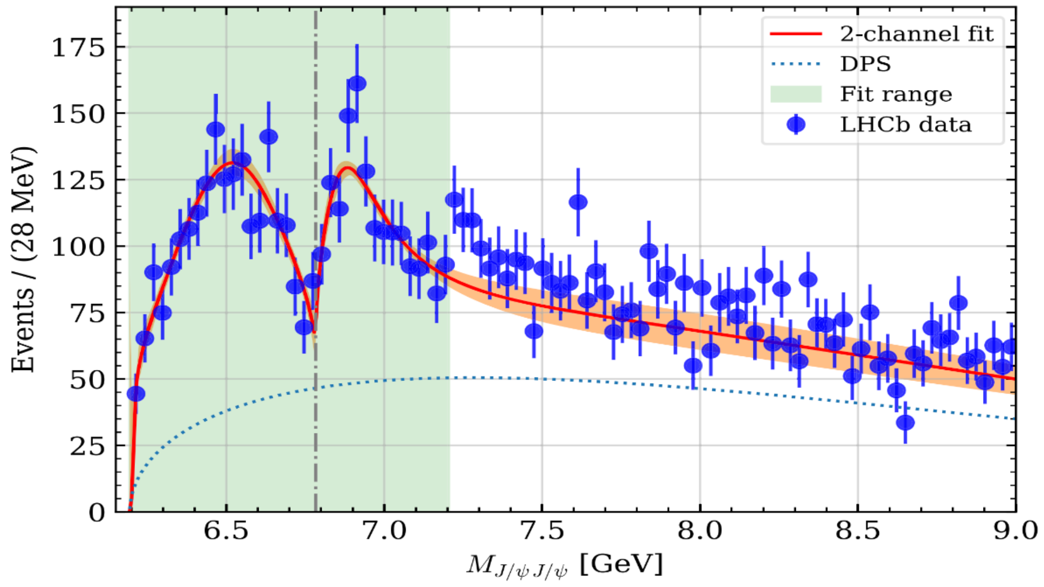
(Received 12 June 2023; revised 7 December 2023; accepted 31 January 2024; published 15 March 2024)

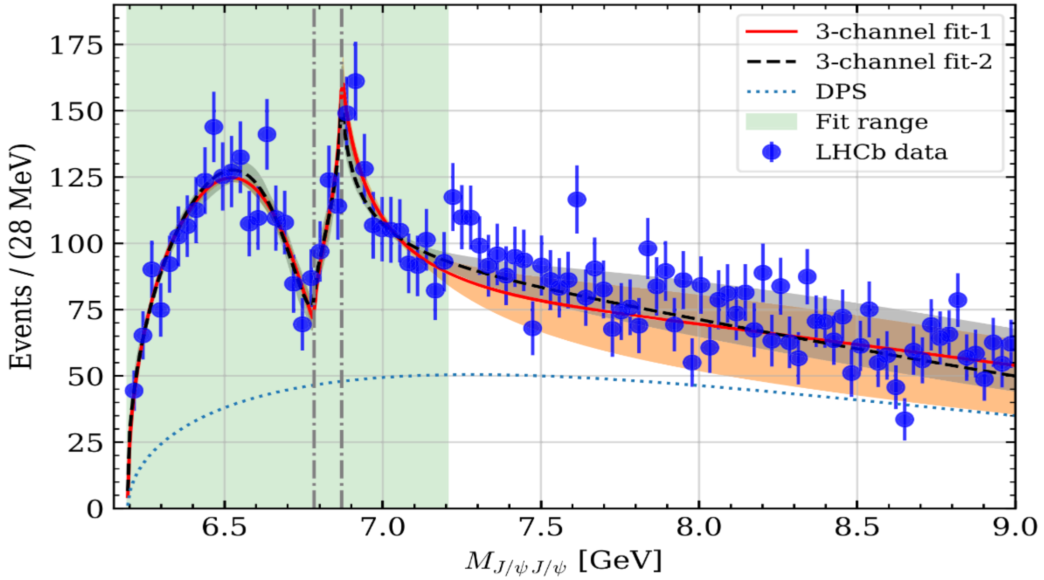
- **X(6900) structure**
- **Two new structure X(6600) and X(7100)**

broad structure

		BW$_1$	BW$_2$	BW$_3$
Interference	m (MeV)	6638^{+43+16}_{-38-31}	6847^{+44+48}_{-28-20}	7134^{+48+41}_{-25-15}
	Γ (MeV)	$440^{+230+110}_{-200-240}$	191^{+66+25}_{-49-17}	97^{+40+29}_{-29-26}

Tetracharm states — near $J/\Psi J/\Psi$ threshold





PHYSICAL REVIEW LETTERS **126,** 132001 (2021)

Coupled-Channel Interpretation of the LHCb Double-J/ψ Spectrum and Hints of a New State Near the $J/\psi J/\psi$ Threshold

Xiang-Kun Dong,[1,2] Vadim Baru,[3,4,5] Feng-Kun Guo,[1,2,*] Christoph Hanhart,[6] and Alexey Nefediev[5,7]

- **Two coupled-channel:** $J/\Psi J/\Psi$, $J/\Psi\Psi(2S)$

$$V_{2\text{ch}}(E) = \begin{pmatrix} a_1 + b_1 k_1^2 & c \\ c & a_2 + b_2 k_2^2 \end{pmatrix}$$

→ near $J/\Psi J/\Psi$ threshold: **virtual state** / **resonance** / **bound state**

- **Three coupled-channel:** $J/\Psi J/\Psi$, $J/\Psi\Psi(2S)$, $J/\Psi\Psi(3770)$

$$V_{3\text{ch}}(E) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

→ near $J/\Psi J/\Psi$ threshold: **virtual state** / **bound state**

How to decipher the nature of possible state near the J/Ψ-pair threshold ?

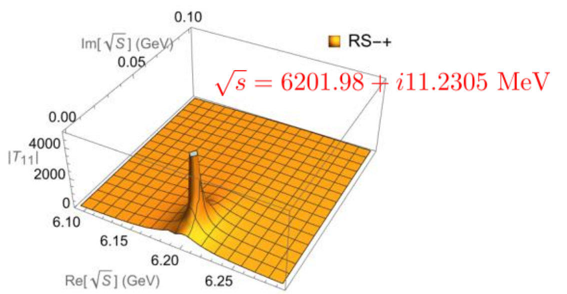
General potential from EFTs

□ Strength of potential from double J/Ψ spectrum

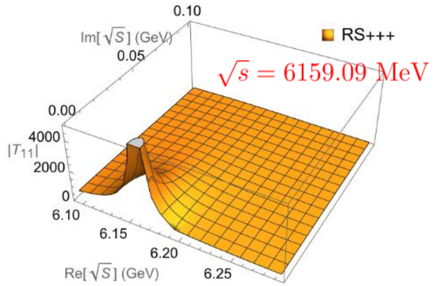
Parameters	$\bar{a}_1$ [GeV^{-2}]	$\bar{a}_2$ [GeV^{-2}]	$\bar{c}$ [GeV^{-2}]	$\bar{b}_1$ [GeV^{-4}]	$\bar{b}_2$ [GeV^{-4}]
Fit	$0.2^{+0.6}_{-0.5}$	-4.2 ± 0.7	$2.94^{+0.36}_{-0.29}$	$-1.8^{+0.4}_{-0.5}$	-7.1 ± 0.4

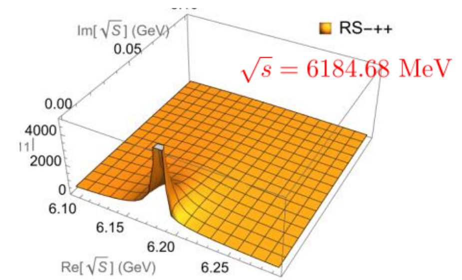
Parameters	$\bar{a}_{11}$ [GeV^{-2}]	$\bar{a}_{12}$ [GeV^{-2}]	$\bar{a}_{13}$ [GeV^{-2}]	$\bar{a}_{22}$ [GeV^{-2}]	$\bar{a}_{23}$ [GeV^{-2}]	$\bar{a}_{33}$ [GeV^{-2}]
Fit 1	$6.0^{+2.2}_{-1.6}$	$10.3^{+3.4}_{-2.8}$	$-0.2^{+1.9}_{-1.3}$	13^{+5}_{-4}	$-2.6^{+2.4}_{-1.3}$	$-2.3^{+1.5}_{-1.1}$
Fit 2	$7.8^{+3.4}_{-2.0}$	16 ± 4	$0.9^{+2.3}_{-2.5}$	26^{+12}_{-6}	-3^{+4}_{-5}	$-2.5^{+2.1}_{-1.0}$

X(6200)



Resonant





Bound

Virtual

Taken from X.-K. Dong, V. Baru, F.-K. Guo, C. Hanhart, and A. Nefediev, Phys. Rev. Lett. 126, 132001 (2021)

□ Identical neutral mesons (J=1) correlation function

Spatial part of wave function

$$\Psi_E(r,k) = \frac{1}{\sqrt{2}}[\Psi(r)+\Psi(-r)] = \sqrt{2}\left[\cos(\boldsymbol{k}\cdot\boldsymbol{r}) - j_0(kr) + \psi_0(r,k)\right]$$

Symmetric (for 1S0, 5S2)

$$\Psi_O(r,k) = \frac{1}{\sqrt{2}}[\Psi(r)-\Psi(-r)] = \sqrt{2}i\sin(\boldsymbol{k}\cdot\boldsymbol{r})$$

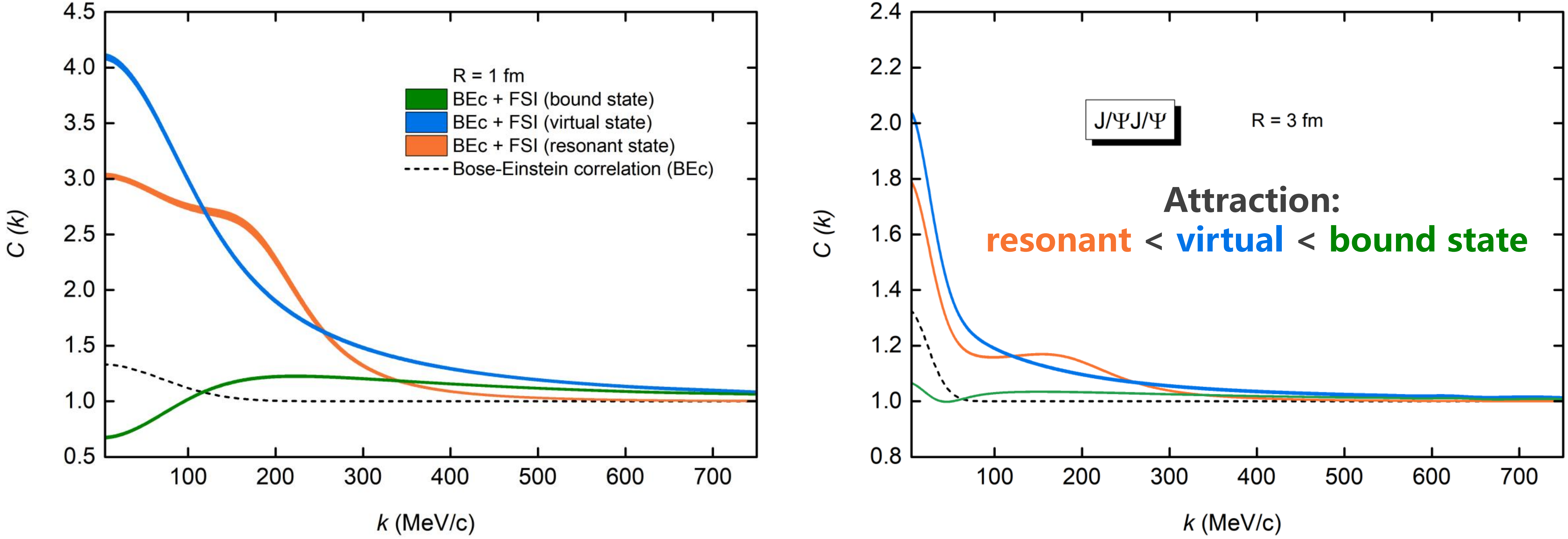
Anti-symmetric (for 3S1)

$$C(k) = 1 + \frac{1}{3}e^{-4k^2R^2} + \frac{4}{3}\int_0^\infty 4\pi r^2 \mathrm{d}r\ S_{12}(r)\ \left[|\psi_0(r,k)|^2 - |j_0(kr)|^2\right]$$

Quantum Statistic (QS) effect

Final State Interaction (FSI) effect

Preliminary J/Ψ-pair CFs



Typical low-momentum features of different pole positions

- BEc + bound state < BEc
- BEc + resonant state > BEc
- BEc + virtual state > BEc

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Charmonium-nucleon Interaction

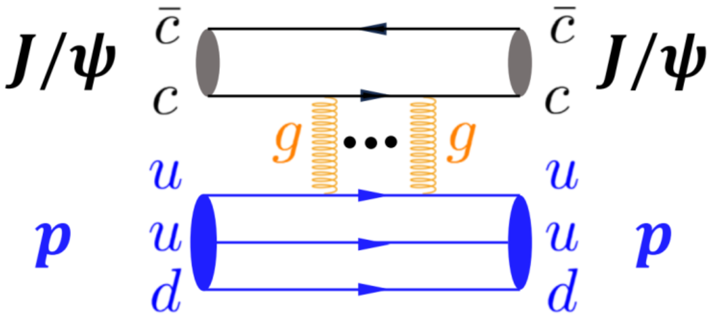
□ OZI suppression

- Predominant multiple-gluon exchanges at low energies

S. Okubo, Phys. Lett. 5, 165 (1963)

G. Zweig, "An SU(3) model for strong interaction symmetry and its breaking. Version 2," DEVELOPMENTS IN THE QUARK THEORY OF HADRONS. VOL. 1. 1964 - 1978

J. Iizuka, Prog. Theor. Phys. Suppl. 37, 21 (1966)



Gluon exchanges

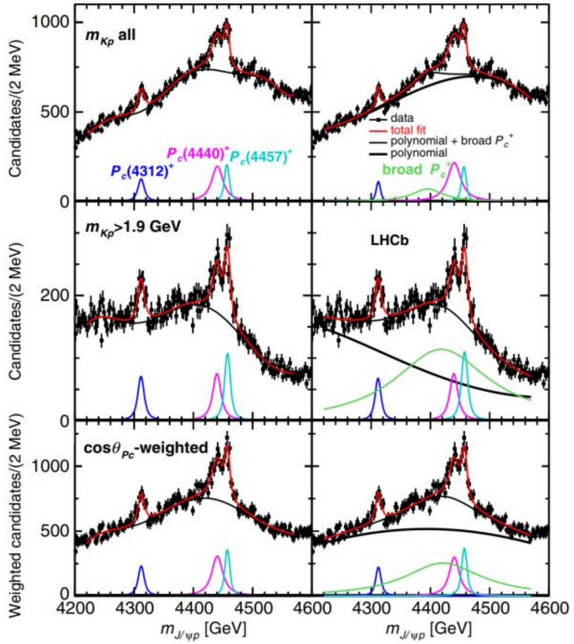
Taken from Bing Wu's report on "East Asian Workshop on Exotic Hadrons 2024"

□ Nucleon mass origin

- Gluon trace anomaly contribution to the nucleon mass

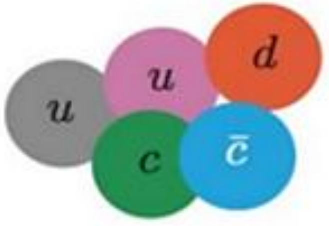
Y. Hatta et al., JHEP 12 (2018) 008

X.-D. Ji, Phys. Rev. Lett. 74, 1071 (1995)



□ Hidden-charm pentaquark states

- P_c states observed by LHCb — $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$



LHCb Collaboration, PRL 115 (2015) 072001
LHCb Collaboration, PRL 122 (2019) 222001

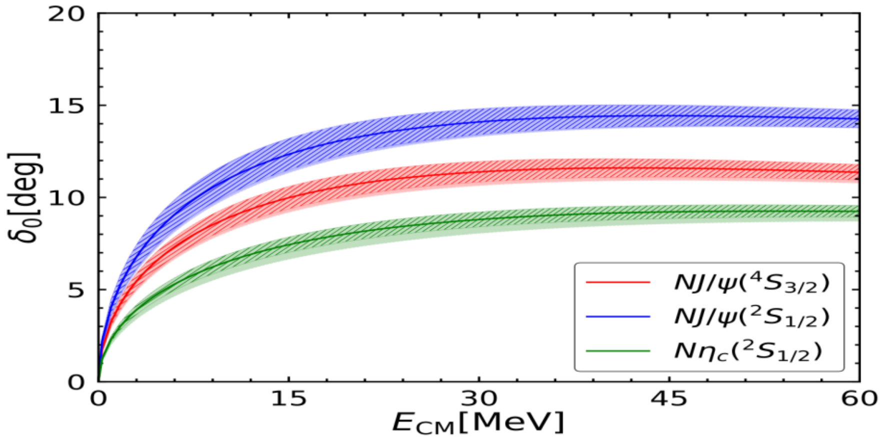
□ In-medium properties of charmonia

A. Sibirtsev and M. B. Voloshin, Phys. Rev. D 71, 076005 (2005)

N. Brambilla et al., Eur. Phys. J. C 74, 2981 (2014)

Charmonium-nucleon phase shifts from HAL QCD

◻ Phase shifts for J/Ψ-N and η_c-N



channel	a_0 [fm]	$r_{\rm eff}$ [fm]
$N\text{-}J/\psi(^4S_{3/2})$	$0.30(2)\left(^{+0}_{-2}\right)$	$3.25(12)\left(^{+6}_{-9}\right)$
$N\text{-}J/\psi(^2S_{1/2})$	$0.38(4)\left(^{+0}_{-3}\right)$	$2.66(21)\left(^{+0}_{-10}\right)$
$N\eta_c(^2S_{1/2})$	$0.21(2)\left(^{+0}_{-1}\right)$	$3.65(19)\left(^{+0}_{-6}\right)$

Lyu, Takumi Doi , Tetsuo Hatsuda, and Takuya Sugiura, Phys. Lett. B 860 (2025) 139178

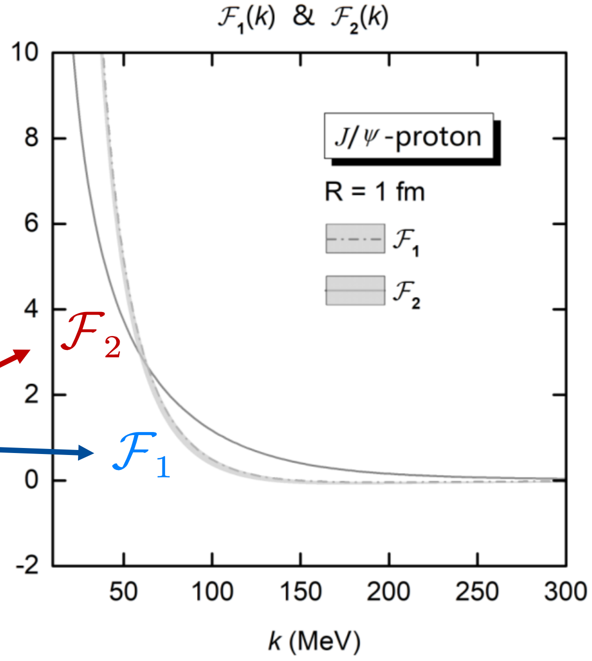
◻ Phase shifts to correlation functions

$$C(k) = 1 + \int_0^\infty d^3r S_{12}\left[\left|j_0 + T\tilde{G}\right|^2 - |j_0|^2\right]$$

$$= 1 + \mathcal{F}_1 \sin^2\delta + \mathcal{F}_2 \sin\delta\cos\delta$$

$$\mathcal{F}_1 = \int_0^\infty d^3r S_{12}\left[\left(\mathrm{Re}\tilde{G}/\rho\right)^2 - |j_0|^2\right]$$

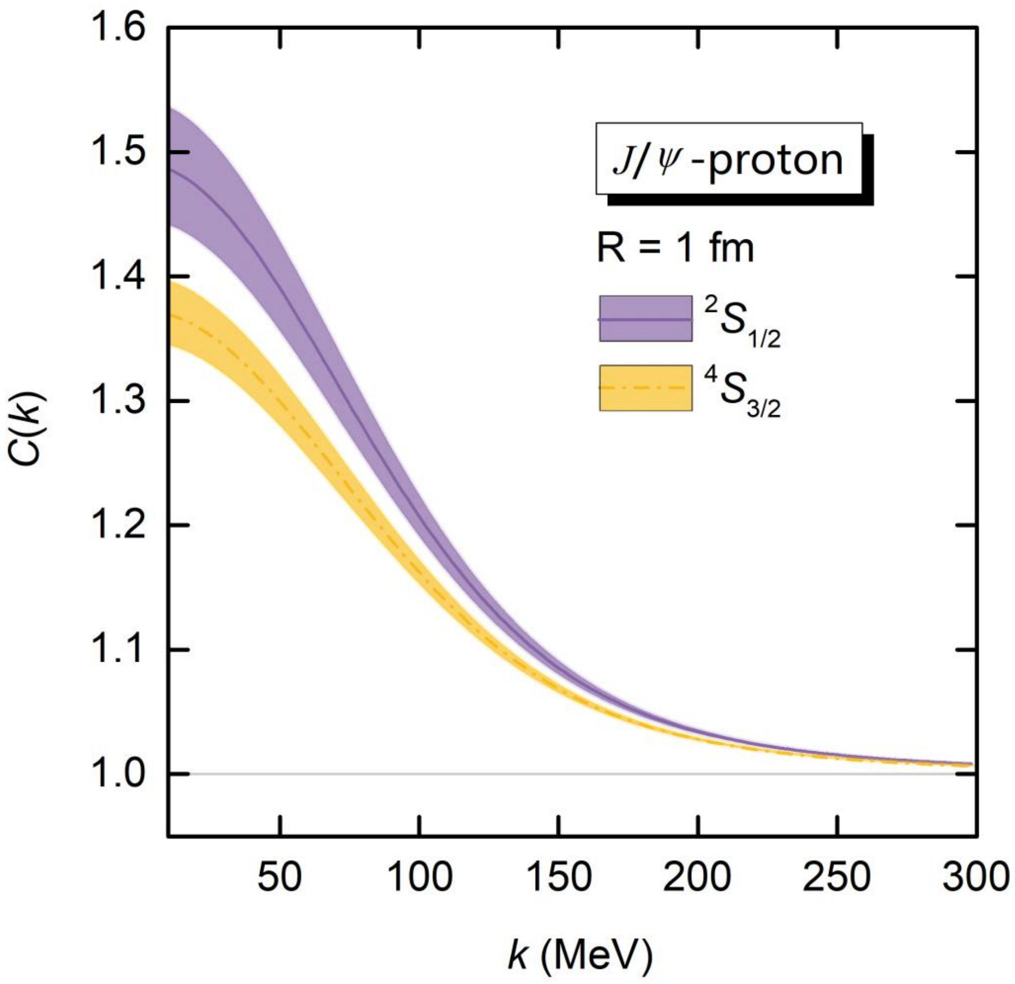
$$\mathcal{F}_2 = -\int_0^\infty d^3r S_{12} 2 j_0 \mathrm{Re}\tilde{G}/\rho$$

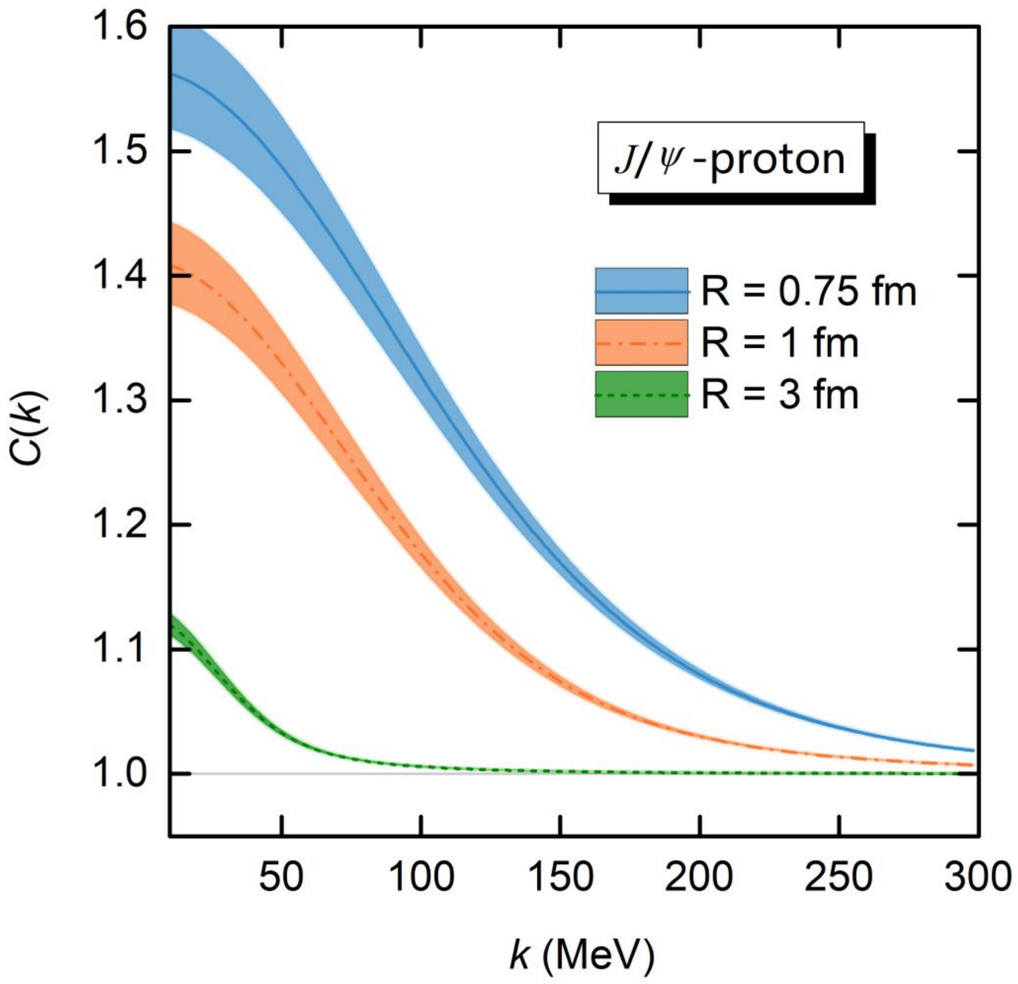


evaluated with gaussian source

J/Ψ-proton CFs

*Zhi-Wei Liu, Duo-Lun Ge, Jun-Xu Lu, Ming-Zhu Liu and **LSG*,** 2504.04853*



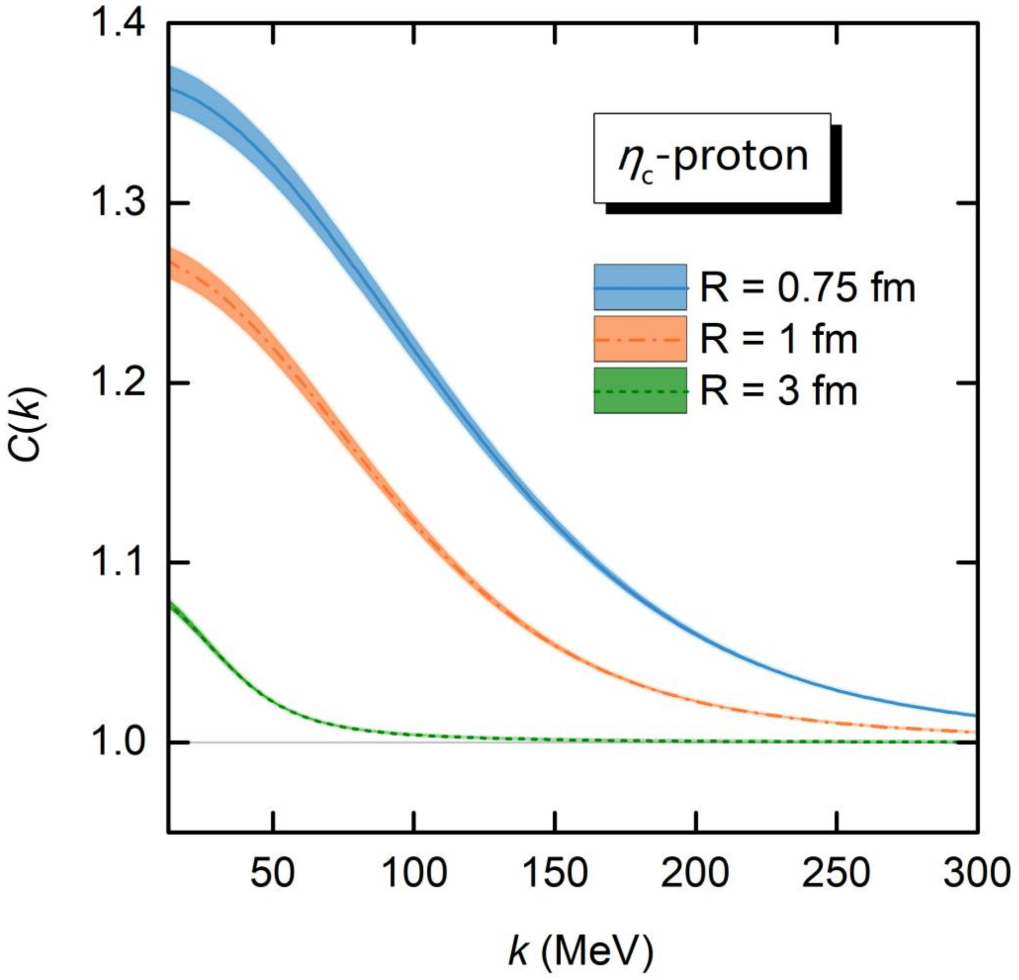


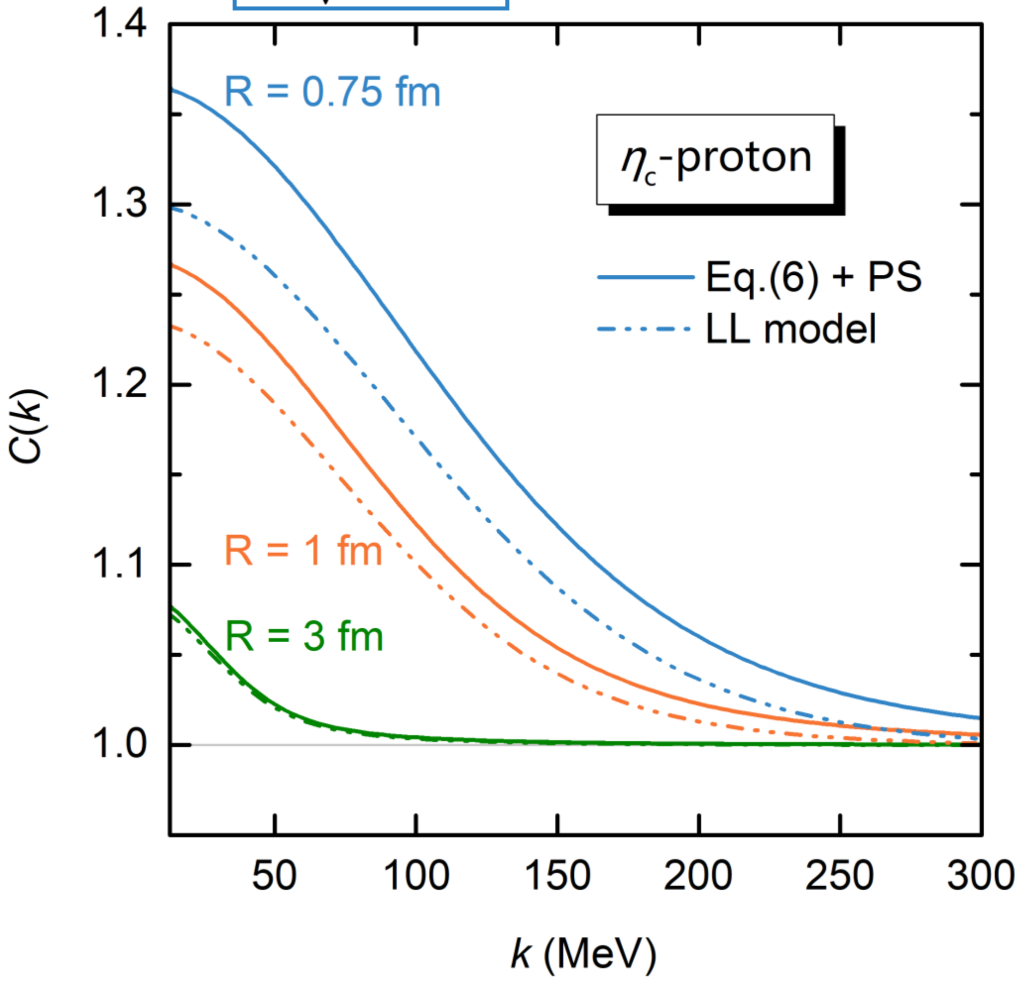
Attraction: $^4S_{3/2}$ < $^2S_{1/2}$

$$C_{\rm avg}(k) = 1/3\ C_{^2S_{1/2}}(k) + 2/3\ C_{^4S_{3/2}}(k)$$

η_c-N CFs

$$C_{\rm LL}(k) = 1 + \frac{|f(k)|^2}{2R^2} + \frac{2{\rm Re}f(k)}{\sqrt{\pi}R}F_1(x) - \frac{{\rm Im}f(k)}{R}F_2(x) - \frac{|f(k)|^2}{4\sqrt{\pi}R^3}r_{\rm eff}$$





Attraction: η_c-N < J/Ψ-N

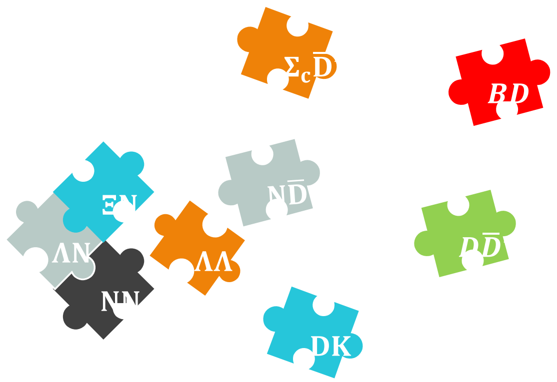
LL model: not suitable for small source

Summary and outlook

- **Femtoscopy offers high-precision tests of the strong interaction between pairs of (un)stable particles and can be valuable to decipher the nature of the many exotic hadrons discovered so far.**
  - ✓ J/Ψ - J/Ψ correlation functions can be used to decipher the nature of possible $X(6200)$
  - ✓ J/Ψ -proton and η_c -proton correlation functions can unveil valuable insights into charmonium-nucleon interaction

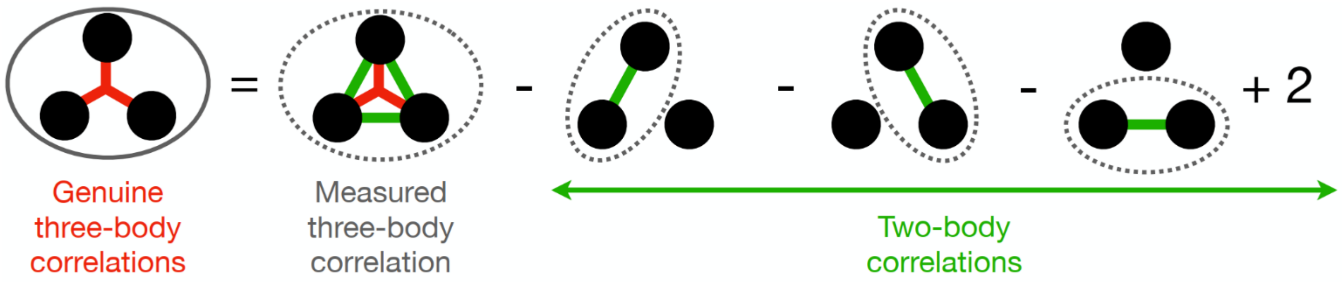
- **Many two-hadron correlations involving s, c, b quarks studied**

DD^*, *I. Vidana, A. Feijoo, M. Albaladejo, J. Nieves, and E. Oset, PLB 846 (2023) 138201*
DD^*, *Y. Kamiya, T. Hyodo, and A. Ohnishi, EPJA 58 (2022) 131*
BB^*, *A. Feijoo, L. R. Dai, L. M. Abreu, and E. Oset, PRD 109 (2024) 016014*
BD, *H.P. Li, J.Y. Yi, C.W. Xiao, D.L. Yao, W.H. Liang, and E. Oset, CPC (2024)*
……



Summary and outlook

- Precise measurement of many two-body correlation functions relevant to understanding the nature of many exotic hadrons
- Measurement of three-particle correlations — genuine three-body effects

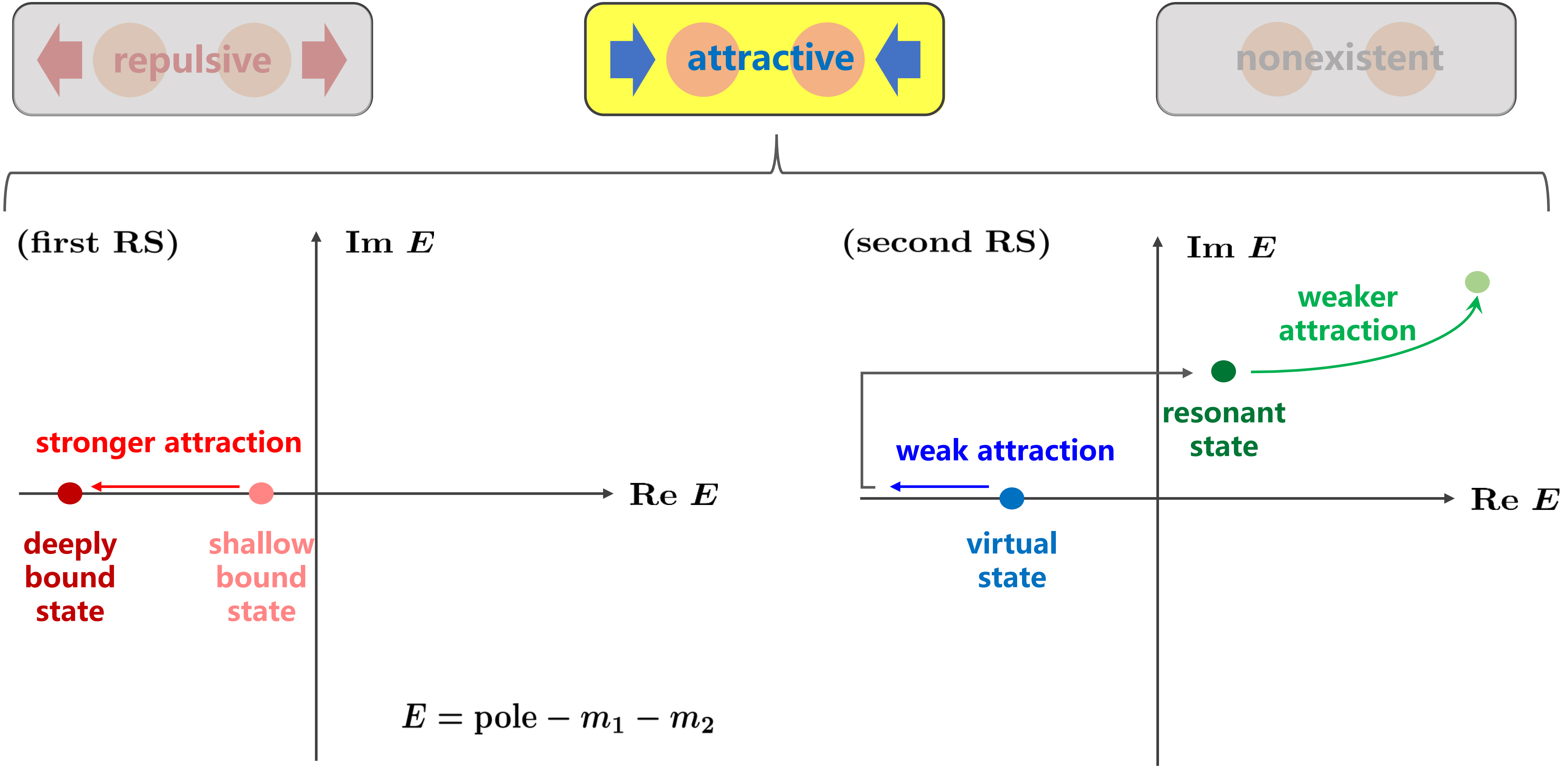


ppp, ppΛ, ALICE Collaboration, Eur. Phys. J. A 59 (2023) 145
$ppK^{\pm}$, ALICE Collaboration, Eur. Phys. J. A 59 (2023) 298
ppp, A. Kievsky and et al., Phys. Rev. C 109 (2024) 034006

Thanks a lot for your attention!

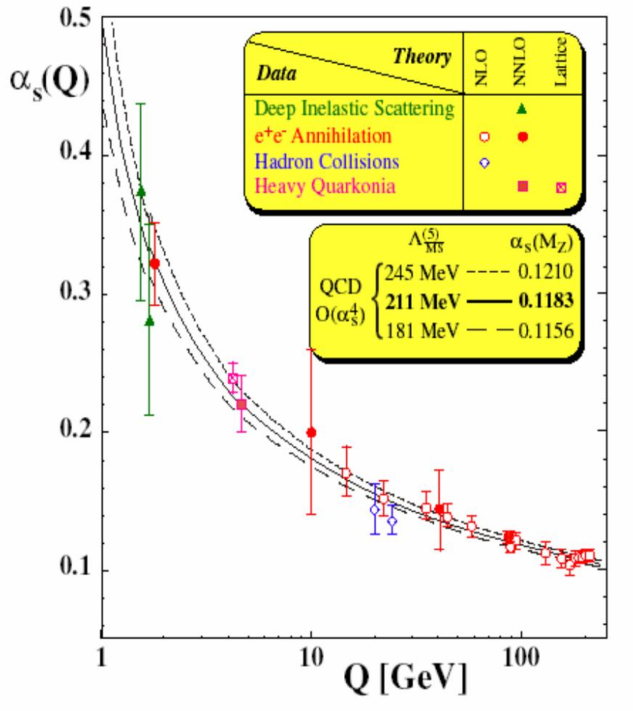
(Image: CERN)

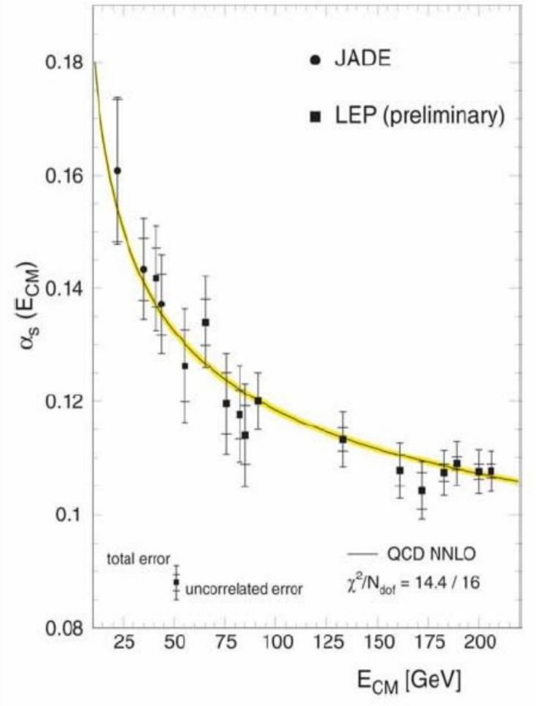
Classification of hadron-hadron interactions



Better understanding of the nonperturbative strong interaction

Asymptotic Freedom





The Nobel Prize in Physics 2004

Photo from the Nobel Foundation archive.	Photo from the Nobel Foundation archive.	Photo from the Nobel Foundation archive.
David J. Gross	H. David Politzer	Frank Wilczek
Prize share: 1/3	Prize share: 1/3	Prize share: 1/3

The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek "for the discovery of asymptotic freedom in the theory of the strong interaction"

CMI

Millennium Problems

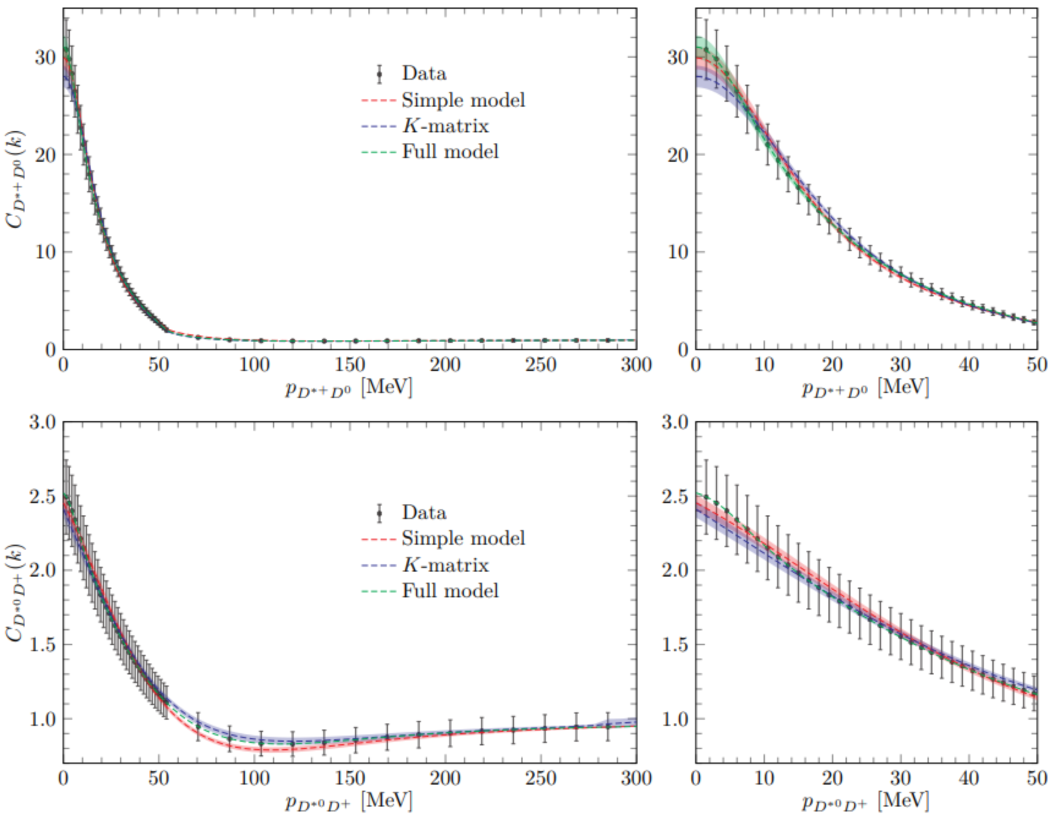
Color Confinement

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Summary and outlook

- One can also perform inverse studies and extract hadron-hadron interaction from the exp. CF data



Inverse problem in femtoscopic correlation functions: The Tcc(3875)+ state,
Albaladejo , Feijoo , Vidaña , Nieves , and Oset, 2307.09873

$$C_{D^0D^{*+}}(p_{D^0}) = 1 + 4\pi\,\theta(\Lambda - p_{D^0}) \int_0^\infty dr r^2 S_{12}(r) \times \left\{ \left| j_0(p_{D^0}r) + T_{11}(s)\widetilde{G}_1(r;s) \right|^2 + \left| T_{12}(s)\widetilde{G}_2(r;s) \right|^2 - j_0^2(p_{D^0}r) \right\} , \quad (1)$$

$$C_{D^+D^{*0}}(p_{D^+}) = 1 + 4\pi\,\theta(\Lambda - p_{D^+}) \int_0^\infty dr r^2 S_{12}(r) \times \left\{ \left| j_0(p_{D^+}r) + T_{22}(s)\widetilde{G}_2(r;s) \right|^2 + \left| T_{12}(s)\widetilde{G}_1(r;s) \right|^2 - j_0^2(p_{D^+}r) \right\} , \quad (2)$$

Femtoscopic correlation functions (CFs)

Koonin–Pratt (KP) formula

*S. E. Koonin, Phys. Lett. B **70** (1) (1977) 43*
*A. Ohnishi, Nucl. Phys. A **954** (2016) 294*

$$C(k) = \int S_{12}(r)\ |\Psi(\boldsymbol{r},\boldsymbol{k})|^2\ \mathrm{d}\boldsymbol{r}$$

Only S-waves

$$C(k) \simeq 1 + \int_0^{\infty} 4\pi r^2 \mathrm{d}r\ S_{12}(r)\ \left[|\psi_0(r,k)|^2 - |j_0(kr)|^2\right]$$

- **Common static and spherical Gaussian source**

$$S_{12}(r) = \exp[-r^2/(4R^2)]/(2\sqrt{\pi}R)^3$$

- **Scattering wave function**
  - **the Schrödinger equation**

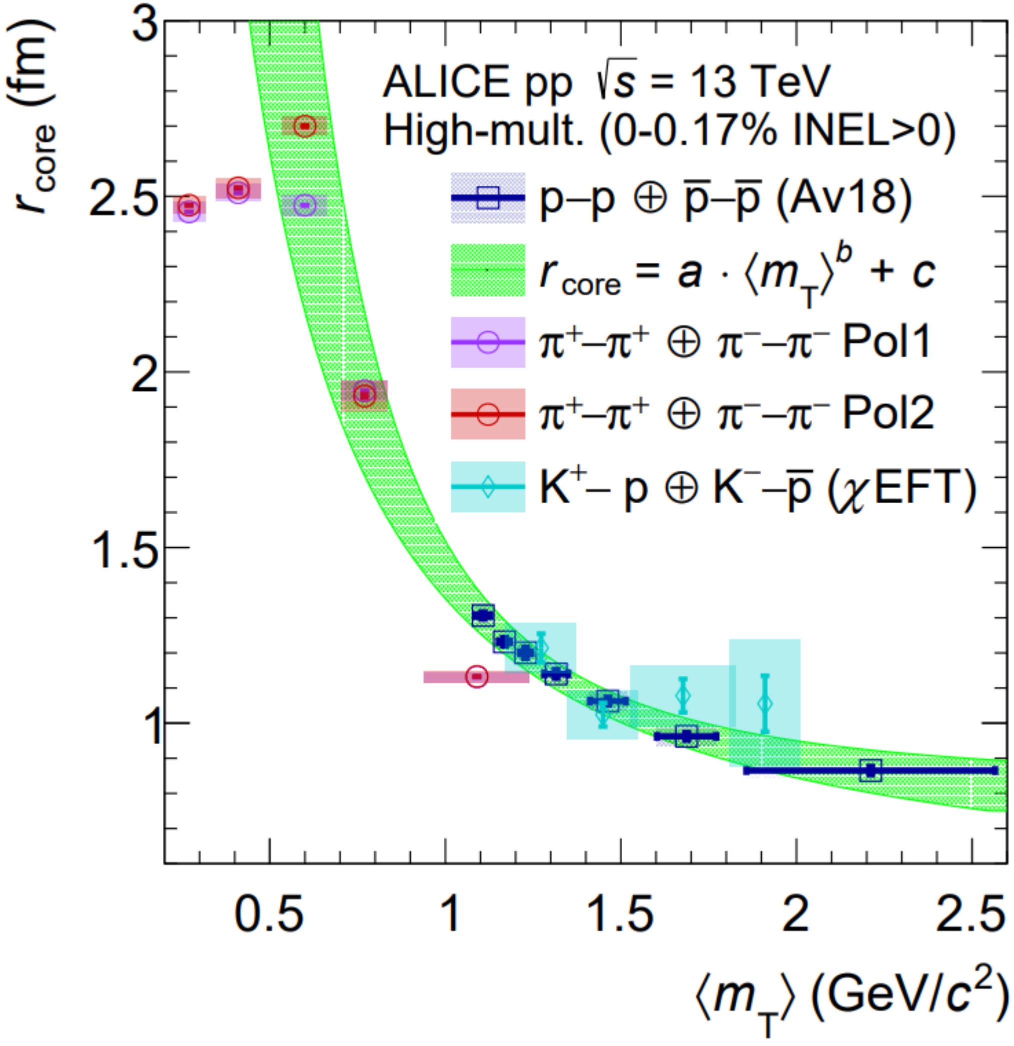
$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi = E\psi$$

  - **the Lippmann-Schwinger equation**

$$T = V + VGT \quad \Longrightarrow \quad |\psi\rangle = |\phi\rangle + GT|\phi\rangle$$

Constraining the source function

2311.14527v1.

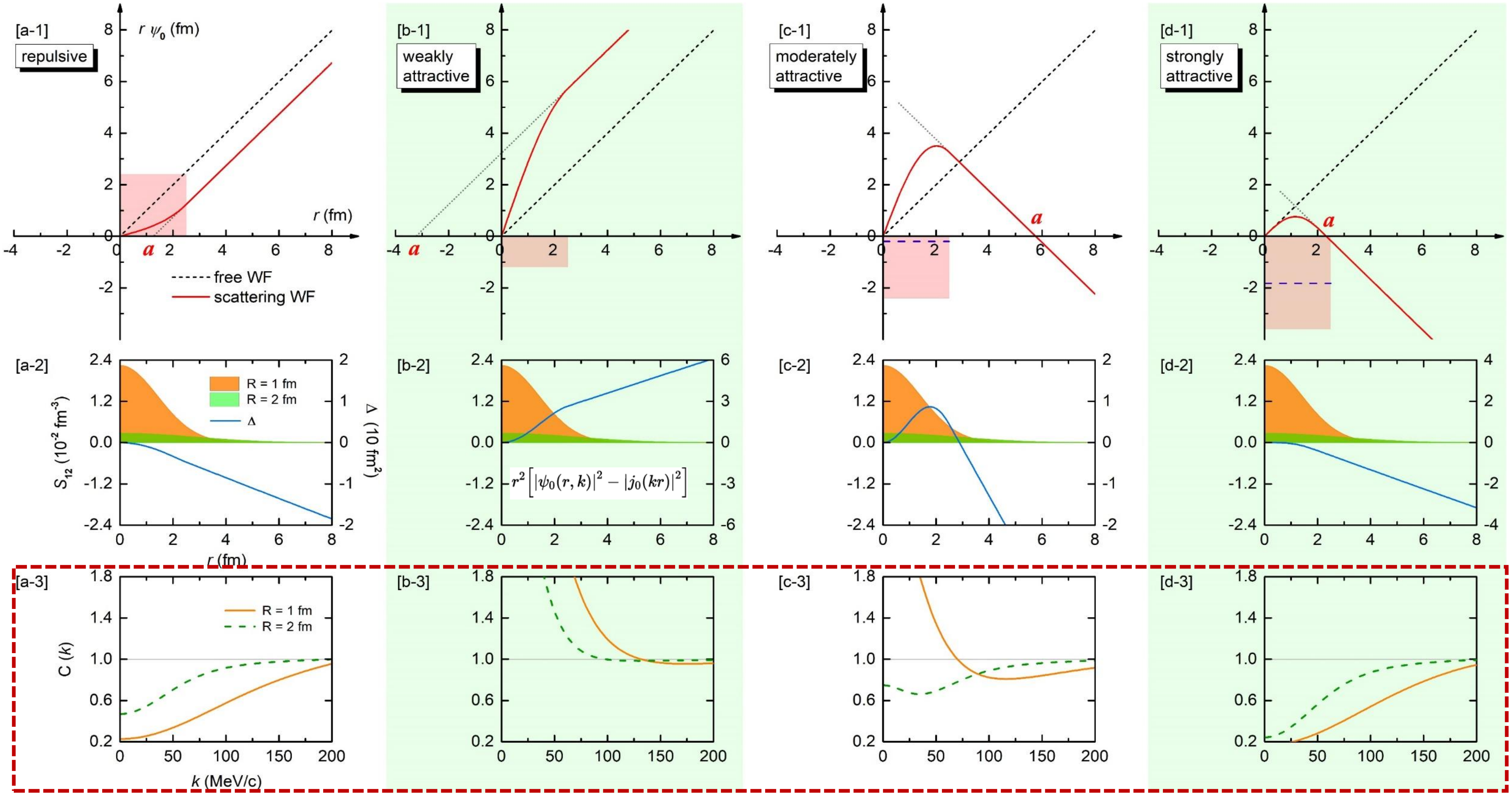


Convention dependent, see, Ref.[2504.08631]

… "providing compelling evidence for the presence of a common emission source for all hadrons in small collision systems at the LHC …"

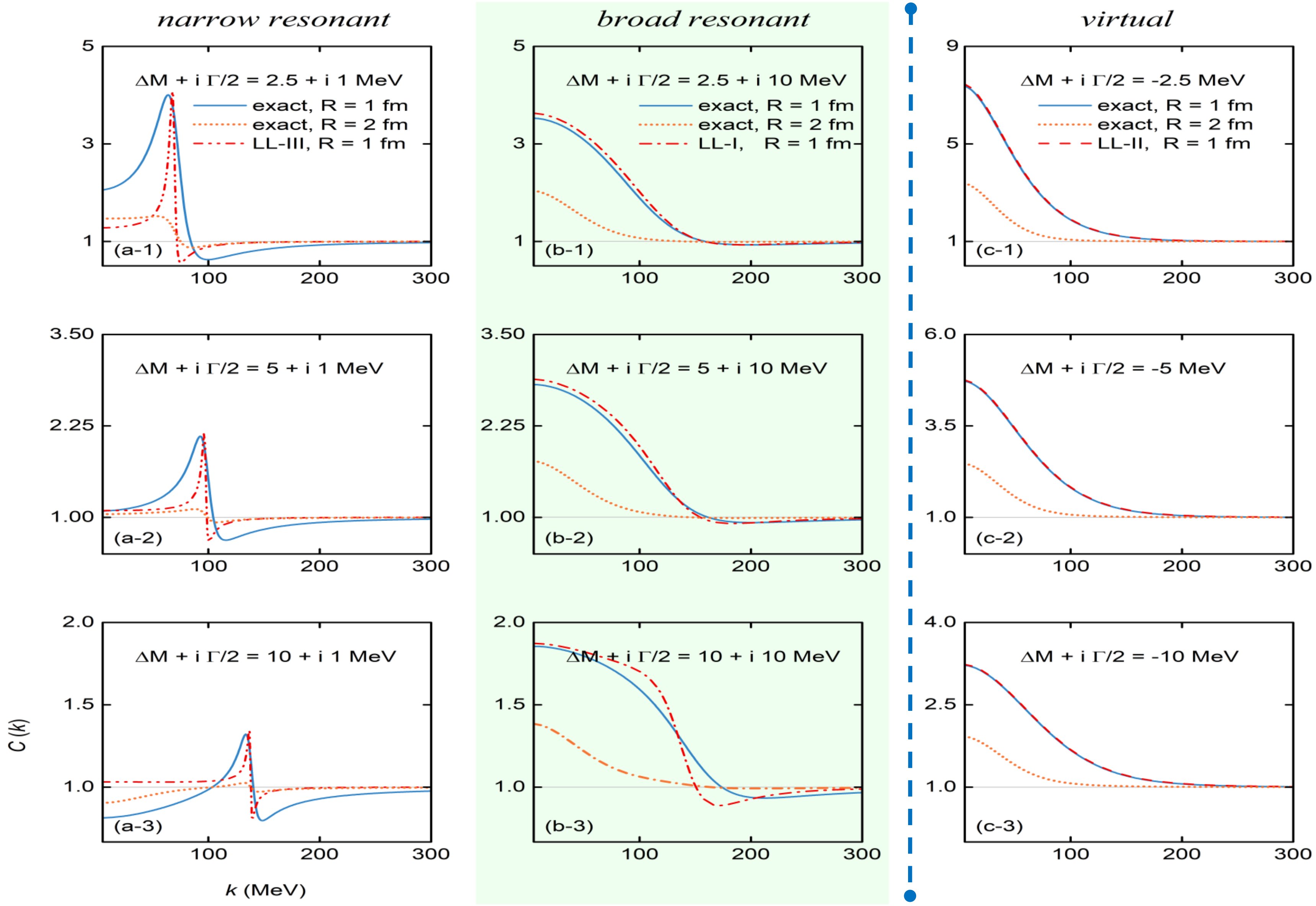
CFs in the presence of bound states

*Zhi-Wei Liu, Jun-Xu Lu and **LSG***, PRD 107, 074019 (2023)*



CFs in the presence of resonant and virtual states

*Zhi-Wei Liu, Ming-Zhu Liu, Jun-Xu Lu and **LSG*,** 2404.18607*



$$V = a + b \cdot k^2$$

$$T = \frac{V}{1 - V \cdot G}$$

VS

$$f = \frac{1}{-\frac{1}{a_0} + \frac{r_{\rm eff}}{2} k^2 - ik}$$