Applications of conformal symmetry in QCD

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Conformal group on the light-cone

- **light-cone coordinate**

$$x^\mu = x_+ \bar{n}^\mu + x_- n^\mu + x_\perp^\mu \,, \qquad n^2 = \bar{n}^2 = 0$$

- **ultra-relativistic particles** [V. Braun, G. Korchemsky, D. Müller, (2003)]

$$\varphi(x) \mapsto \boxed{\varphi(zn) \equiv \varphi(z)} \quad \Longrightarrow \quad \text{full conformal group} \mapsto \boxed{SL(2,\mathbb{R}) \times \mathbf{E} \times \mathbf{H}}$$

◇ **collinear subgroup** $SL(2,\mathbb{R})$

- action

$$z \mapsto \tilde{z} = \frac{az+b}{cz+d} \,, \qquad a,b,c,d \in \mathbb{R} \,, \qquad ad - bc = 1 \,,$$

$$\varphi(z) \mapsto T^j \tilde{\varphi}(z) = \frac{1}{(cz+d)^{2j}} \varphi\left(\frac{az+b}{cz+d}\right) \,, \qquad \boxed{j = \frac{1}{2}(l+s) \,, \quad \text{conformal spin}}$$

- generators

$$S_-^i \equiv -\partial_{z_i} \,, \qquad S_+^i \equiv z_i^2 \partial_{z_i} + 2j z_i \,, \qquad S_0^i = z_i \partial_{z_i} + j \,,$$

$$[S_0^i, S_\pm^i] = \pm S_\pm^i \,, \qquad [S_+^i, S_-^i] = 2 S_0^i \,, \qquad SL(2,\mathbb{R}) \text{ algebra}$$

$$n-\text{particle:} \quad S_{\pm,0} = \sum_i S_{\pm,0}^i \,, \quad \widehat{\mathbb{C}} = S_0(S_0 - 1) + S_+ S_- \Leftarrow \text{Casimir operator}$$

Wilson-Fisher point

- **stay in non-integer dimensions $d = 4 - 2\epsilon$** [V. Braun, A. Manashov, (2013)]

$$\exists \epsilon \text{ for } \forall a_* , \quad \text{s.t.} \quad \boxed{\beta(a_*) = 0}$$

Wilson-Fisher fix point

scale invariance is restored! technically, $\bar{\beta}(a_*) \mapsto -\epsilon$

- **consider leading-twist (renormalized) light-ray operator**

$$\mathcal{O}_\ell(z_1, z_2) \equiv \bar{q}(z_1 n) \not{n} q(z_2 n)$$

admits conformal OPE

satisfying RGE

$$\left(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a)\right) \mathcal{O}_{\not{n}}(z_1, z_2) = 0$$

$$\beta(a_*) = 0 \implies \left(\mu\partial_\mu + \widetilde{\mathbb{H}}(a_*)\right) \mathcal{O}_{\not{n}}(z_1, z_2) = 0$$

$\widetilde{\mathbb{H}} \sim$ simple pole in Laurent ϵ expansion

- **$\widetilde{\mathbb{H}}$ is ϵ-independent in $\overline{\text{MS}}$-like schemes by construction**

$$\mathbb{H}(a, \beta) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + \cdots , \qquad \widetilde{\mathbb{H}}(a_*, \epsilon) = a_*\mathbb{H}^{(1)} + a_*^2\mathbb{H}^{(2)} + \cdots$$

QCD at critical point

- at the critical point with coupling a_*

$$[S_{\pm,0}(a_*), \mathbb{H}(a_*)] = 0$$

with $S_{\pm,0}(a_*)$ obeying $SL(2,R)$ algebra

$$[S_+(a_*), S_-(a_*)] = 2S_0(a_*)$$
$$[S_0(a_*), S_\pm(a_*)] = \pm S_\pm(a_*)$$

- $S_-(a_*) \leftrightarrow \mathbf{P}_+ \implies \boxed{S_-(a_*) = S_-}$, i.e., no quantum corrections!

- however, $S_+(a_*) \sim \mathbf{K}_-$ and $S_0(a_*) \sim \mathbf{D}$ receive quantum corrections!

$$S_0(a_*) = S_0 - \epsilon + \frac{1}{2}\mathbb{H}(a_*)$$
$$S_+(a_*) = S_+^{(0)} + (z_1 + z_2)\Big(-\epsilon + \frac{1}{2}\mathbb{H}(a_*)\Big) + z_{12}\Delta(a_*)$$

$$z_{12} \equiv z_1 - z_2$$

- $\Delta(a_*)$ conformal anomaly: nontrivial information require diagrammatic calculations

What's the advantage?

an example

- $\mathcal{O}_\ell$ is $SL(2)$ invariant classically [Bukhvostov, Frolov, Kuraev, Lipatov (1985)]

  $SL(2)$ invariance $\Longrightarrow [S_{0,\pm}, \mathbb{H}^{(1)}] = 0 \Longrightarrow \mathbb{H}^{(1)}_{\bar{q}q} = h(\widehat{\mathbb{C}})$

$$\widehat{\mathbb{C}} = S_+ S_- + S_0(S_0 - 1)$$

quadratic Casimir operator

- z_{12}^N is the eigenfunction of $\mathbb{H}(a)$ to all orders (translational inv. + local OPE)

$$\mathbb{H}^{(1)} z_{12}^N = \gamma_N^{(1)} z_{12}^N \,,$$

forward lim. $\mapsto$ splitting function

$$\Downarrow$$

$$\mathbb{H}^{(1)} = 2C_F \left[\psi(\widehat{J}+1) + \psi(\widehat{J}-1) - 2\gamma_E - \frac{3}{2}\right] \qquad \widehat{\mathbb{C}} = \widehat{J}(\widehat{J}-1) \qquad \sim C_N^{3/2}$$

DGLAP+ERBL+GPD evolution

What's the advantage?

- recipe for obtaining nonforward $\mathbb{H}^{(\ell)}$
  - compute $\Delta^{(\ell-1)}$ diagrammatically (for $\mathbb{H}^{(1)}$, $\Delta^{(0)} = 0$, trivial)
  - find anomalous dimension $\gamma_N^{(\ell)}$ from forward kinematics (available upto high orders, e.g, splitting functions)
  - from the nontrivial identities

$$[S_+(a_*), \mathbb{H}(a_*)] = 0$$

obtain a series of commutation relations to the desired order in a, e.g.,

$$[S_+^{(0)}, \mathbb{H}^{(1)}] = 0\,, \quad [S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, \Delta S_+^{(1)}]\,, \quad \cdots$$

  - write $\mathbb{H}^{(\ell)} = \mathbb{H}^{(\ell)}_{\text{inv.}} + \mathbb{H}^{(\ell)}_{\text{ninv}}$ with $[S_+^{(0)}, \mathbb{H}^{(\ell)}_{\text{inv}}] = 0$, solve $\mathbb{H}^{(\ell)}_{\text{ninv}}$ from comm. relations, finally obtain $\mathbb{H}_{\text{inv}}$ by matching (e.g., moments of splitting function)

$$\mathbb{H}^{(\ell)} z_{12}^N = \gamma_N^{(\ell)} z_{12}^N$$

- Evolution kernel for genuine higher-twist operators also obtainable form CFT

[V. Braun, A. Manashov (2010); YJ, A. Belitsky (2014)]

Conformal techniques for computing coefficient functions

- **conformal techniques are also applicable for computing Wilson coefficient functions in exclusive processes: LP, (kinematic) NLP, NNLP ...**
  - $\Delta^{(\ell)}$ **needed for ℓ-loop coefficient function, but universal**
  - **forward limit helps to fix boundary condition**

Example: $\mathbb{T}_{\mu\nu}(x_1,x_2) = T\{j_\mu(x_1)j_\nu(x_2)\}\,,\quad j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$

[V. Braun, A. Manashov, S. Moch, J. Schoenleber (2020, 2021), V. Braun, YJ, A. Manashov (2021,2023,2025)]

$$T\{j_\mu(x_1)j_\nu(x_2)\} = \sum_N C_N^{\mu\nu,\mu_1\cdots\mu_N}(x,\partial)\,\underbrace{\mathcal{O}^N_{\mu_1\cdots\mu_N}}_{l.t.} + \text{genuine higher-twist contributions}$$

- **Δ + forward limit fix local coefficient $\mapsto$ resum = "standard CF" in factorization theorem for exclusive processes $C \star O_{l.t.} \overset{\text{F.T.}}{\mapsto} \mathcal{C} \otimes \mathcal{O}_{l.t.}$**
  - $\int d^4x_1 d^4x_2\, \mathrm{e}^{-i(q\cdot x_1 - q'\cdot x_2)} \langle p'|\, j^{\mu\nu}(x_1) j^\nu(x_2)\,|p\rangle$ **constructible by "$\Delta + \mathrm{DIS}$"**
  - **at tree level** $C_V^{\mu\nu}(x,\partial) = \dfrac{g^{\mu\nu}}{(x_1-x_2)^2} + O(\partial)$

Leading-twist distribution amplitude in HQET

Definition

[A. Grozin, M. Neubert (1997)]

$$\langle 0|\big[\bar{q}(zn)\, \not{n}[zn,0]\gamma_5 h_v(0)\big]_R|\bar{B}(v)\rangle = iF_B(\mu)\;\Phi_+(z,\mu)$$

- v_μ is the heavy quark velocity
- n_μ is the light-like vector, $n^2=0$, such that $n\cdot v=1$
- The twist-2 LCDA $\Phi_+(z-i0,\mu)$ is an analytic function of z in the lower half-plane

Fourier transform

$$\phi_+(\omega,\mu) = \int_{-\infty}^{\infty}\frac{dz}{2\pi}\, e^{i\omega z}\Phi_+(z-i0,\mu)\,, \qquad \Phi_+(z,\mu) = \int_0^{\infty} d\omega\, e^{-i\omega z}\phi_+(\omega,\mu)\,.$$

- $\omega > 0$ is the $(2\times)$ light quark energy in the b–quark rest frame
- $h_v(0)=[0,\infty v]$, so $\bar{q}(zn)\not{n}[zn,0]\gamma_5 h_v(0) = \bar{q}(zn)\not{n}[zn,0]\gamma_5[0,\infty v]$, $\mathcal{L}_{m=0}$ **is applicable**
- **operator is not conformal invariant**

One-loop evolution of leading twist DA

- **RGE** $$\left(\mu\frac{\partial}{\partial\mu}+\beta(a)\frac{\partial}{\partial a}+\mathcal{H}(a)\right)\Phi_+(z,\mu)=0$$

- **One-loop Lange-Neubert (LN) kernel**

$$\mathcal{H}^{(1)}\Phi_+(z,\mu)=4C_F\left\{[\ln(i\tilde{\mu}z)+1/2]\Phi_+(z,\mu)+\int_0^1 du\frac{\bar{u}}{u}\left[\Phi_+(z,\mu)-\Phi_+(\bar{u}z,\mu)\right]\right\}$$

where $\widetilde{\mu}=\mathrm{e}^{\gamma_E}\mu_{\overline{\mathrm{MS}}}$ and $\bar{u}=1-u$. [B. Lange, M. Neubert (2003); V. Braun, D. Ivanov, G. Korchemsky (2004)]

- **Solution to one-loop RGE** [G. Bell, T. Feldmann, Y.-M. Wang, M. W. Y. Yip (2013); V. Braun, A. Manashov (2014)]

$$\Phi_+(z,\mu)=-\frac{1}{z^2}\int_0^\infty ds\, s\, e^{is/z}\,\eta_+(s,\mu)\,,$$

$$\eta_+(s,\mu)=R(s,\mu,\mu_0)\eta_+(s,\mu_0)\,,\qquad R(s,\mu,\mu_0)\propto s^{\frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}}$$

a direct consequence of

$$[S_+,\mathcal{H}^{(1)}]=0\,,\qquad [S_0,\mathcal{H}^{(1)}]=\Gamma^{(1)}_{\mathrm{cusp}}\,,\qquad\Longrightarrow\qquad \mathcal{H}^{(1)}=\Gamma_{\mathrm{cusp}}\ln\left(i\tilde{\mu}\mathrm{e}^{\gamma_E}S_+\right)+\Gamma_+$$

One-loop evolution of higher-twist DAs

$$-2iF(\mu)\Phi_3(z_1,z_2,\mu) = \langle 0|\,\bar{q}(z_1)gG_{\mu\nu}(z_2)n^\nu\sigma^{\mu\rho}n_\rho\gamma_5 h_v(0)\,|\bar{B}(v)\rangle$$

- $\mathbb{H}$ **is sum of integral operators; becomes integrable at large** N_c.

$$\boxed{[\mathbb{Q}_1,\mathbb{Q}_2] = \left[\mathbb{Q}_1,\mathcal{H}^{(1)}_{\Phi_3}\right] = \left[\mathbb{Q}_2,\mathcal{H}^{(1)}_{\Phi_3}\right] = 0}$$

Two conserved charges

explicitly [V. Braun, A. Manashov, N. Offen (2015)]

$$\begin{aligned} \mathbb{Q}_1 &= i(S_q^+ + S_g^+)\,, \\ \mathbb{Q}_2 &= \frac{9}{4}iS_g^+ - iS_g^+\left(S_g^+S_q^- + S_g^0S_q^0\right) - iS_g^0\left(S_q^0S_g^+ - S_g^0S_q^+\right) \end{aligned}$$

- **Two DOF in** $\Phi^{(1)}_{\Phi_3} \implies \mathcal{H}^{(1)}_{\Phi_3}$ **and** $\{\mathbb{Q}_1,\mathbb{Q}_2\}$ **share the same eigenfunction.**
- **Integrability of RGE** $\Leftrightarrow$ **Integrable spin chains** [V. Braun, YJ, A. Manashov (2018)]
- $z_2 \to \infty$ **recovers evolution properties for off-diagonal DY soft function at NLP**

[M. Beneke, YJ, E. Sünderhauf, X. Wang (2025)]

One-loop evolution of higher-twist DAs

- **Solving for eigenfunction of $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ leads to (complete orthonormal basis)**

$$\phi_-(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu) + \int_0^\infty ds\, J_0(2\sqrt{\omega s})\, \eta_3^{(0)}(s,\mu)$$

$$\phi_3(\underline{\omega},\mu) = \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\,\underline{\omega}) + \frac{1}{2}\int_{-\infty}^\infty dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\,\underline{\omega})\Big],$$

where $Y_3^{(0)}(s\,|\,\underline{\omega}) = Y_3(s, x = i/2\,|\,\underline{\omega})$ and

$$Y_3(s,x\,|\,\underline{\omega}) = -\int_0^1 du\, \sqrt{us\omega_1}\, J_1(2\sqrt{us\omega_1})\, \omega_2\, J_2(2\sqrt{\bar{u}s\omega_2})\, {}_2F_1\left(\begin{matrix} -\frac{1}{2}-ix, -\frac{1}{2}+ix \\ 2\end{matrix}\,\Big|\, -\frac{u}{\bar{u}}\right)$$

Solving RGE for $\phi_3(\underline{\omega},\mu)$ up to $1/N_c^2$ gives

$$\eta_3(s,x,\mu) = L^{\gamma_3(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_3(s,x,\mu_0)$$

$$\eta_3^{(0)}(s,\mu) = L^{N_c/\beta_0} R(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)$$

where $L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$ and $\gamma_3(x) = N_c[\psi(3/2+ix) + \psi(3/2-ix) + 2\gamma_E]$.

- $1/N_c^2 \sim \mathcal{O}(10^{-1})$ **taken perturbatively**

One-loop evolution of higher-twist DAs

- **RGEs for twist-4 DAs are also integrable** [V. Braun, YJ, A. Manashov (2017)]

  Light fields mixing: kernels of 2×2 matrices. [V. Braun, A. Manashov, J. Rohrwild (2009); YJ, A. Belitsky (2014)]

  Three conserved charges $\{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}$

$$\Phi_4(\underline{\omega}) = \frac{1}{2}\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;1}^{(+)}(s,x\,|\underline{\omega})\,,$$

$$(\Psi_4 + \widetilde{\Psi}_4)(\underline{\omega}) = -\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;2}^{(+)}(s,x\,|\underline{\omega})\,,$$

$$(\Psi_4 - \widetilde{\Psi}_4)(\underline{\omega}) = 2\int_0^\infty \frac{ds}{s}\left(-\frac{\partial}{\partial\omega_2}\right)\left\{\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\underline{\omega})\right\}$$
$$- \int_0^\infty ds \int_{-\infty}^\infty dx\, \varkappa_4^{(-)}(s,x,\mu)\, Z_{4;2}^{(-)}(s,x\,|\underline{\omega})\,,$$

$$\eta_4^{(+)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_4^{(+)}(s,x,\mu_0)$$
$$\varkappa_4^{(-)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \varkappa_4^{(-)}(s,x,\mu_0)$$

- **Redundant operators are traded for others using EOMs and Lorentz symmetry.**

Conformal symmetry of heavy kernels

what about higher-loops??

$$\mathcal{H} = \Gamma_{\text{cusp}}(a)\ln(i\bar{\mu}S_+) + \Gamma_+(a)$$ **to all orders** **why?**

Exact conformal symmetry in $d = 4 - 2\epsilon$ at the critical point $\beta(a_*) = 0$

$$(1) \quad \left[S_+^{\text{full}}, \mathcal{H}(a_*)\right] = 0$$

Conformal generators receive quantum corrections:

$$S_+^{(0)} = z^2\partial_z + 2z \mapsto S_+^{\text{full}}(a_*) = S_+^{(0)} + z\left[-\epsilon + \Delta(a_*)\right],$$
$$S_0^{(0)} = z\partial_z + 1 \mapsto S_0^{\text{full}}(a_*) = S_0^{(0)} - \epsilon + \mathcal{H}(a_*)$$

Conformal anomaly: $\Delta(a_*) = a_*\Delta^{(1)} + a_*^2\Delta^{(2)} + \dots$

from (1) and $SL(2)$ algebra $\Longrightarrow$ $(2) \quad [z\partial_z, S_+^{\text{full}}(a_*)] = S_+^{\text{full}}(a_*)$

$\ln\mu z$ enters $\mathcal{H}$ only linearly with coefficient Γ_{cusp} [G. Korchemsky, A. Radyushkin (1992)]

$$(3) \quad [z\partial_z, \mathcal{H}(a_*)] = \Gamma_{\text{cusp}}(a_*)$$

$$(1) \Longrightarrow \mathcal{H}(a_*) = f(S_+^{\text{full}}(a_*)) \overset{(2),(3)}{\Longrightarrow} zf'(z) = \Gamma_{\text{cusp}}(a_*) \Longrightarrow \mathcal{H} = \Gamma_{\text{cusp}}\ln(i\bar{\mu}S_+) + \Gamma_+$$

Conformal generators at one-loop

- **Two-loop evolution of twist-2 DA** [V. Braun, YJ, A. Manashov (2019)]

$$\mathcal{H}_h^{(2)}(a_*) = \Gamma^{(2)}_{\text{cusp}}(a_*)\ln(i\bar{\mu}S_+^{(1)}(a_*)) + \Gamma_+^{(2)}(a_*)\,,$$
$$S_+^{(1)}(a_*) = S_+^{(0)} + z\big(-\epsilon(a_*) + a_*\Delta^{(1)}\big)$$

$$\bar{\mu} = \widetilde{\mu}e^{\gamma_E} = \mu_{\overline{\text{MS}}}\, e^{2\gamma_E}$$

$$\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$$

One-loop conformal anomaly *four one-loop diagrams* obtainable from $\Delta_l^{(\ell)}$

$$\Delta^{(1)}\mathcal{O}(z) = C_F\left\{3\mathcal{O}(z) + 2\int_0^1 d\alpha\left(\frac{2\bar{\alpha}}{\alpha} + \ln\alpha\right)[\mathcal{O}(z) - \mathcal{O}(\bar{\alpha}z)]\right\}$$

- **The scheme-dependent constant $\Gamma_+^{(2)}(a)$ is reconstructable from other AMs**

Two-loop kernel in position space

- **Integral representation for $\mathcal{H}$ is usually preferred**

Ansatz

$$\mathcal{H}(a)\mathcal{O}(z) = \Gamma_{\text{cusp}}(a)\left[\ln(i\widetilde{\mu}z)\mathcal{O}(z) + \int_0^1 d\alpha\,\frac{\bar{\alpha}}{\alpha}\big(1+h(a,\alpha)\big)\big(\mathcal{O}(z)-\mathcal{O}(\bar{\alpha}z)\big)\right] + \gamma_+(a)\mathcal{O}(z)$$

- $\Delta^{(1)}$ **and** $\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$ **dictate** $h(a,\alpha)$ *going to Mellin space*

$$h(a,\alpha) = a\ln\bar{\alpha}\left\{\beta_0 - 2C_F\left(\frac{3}{2} + \ln\frac{\alpha}{\bar{\alpha}} + \frac{\ln\alpha}{\bar{\alpha}}\right)\right\} + O(a^2)$$

- γ_+ **requires additional calculation, obtainable from other known** γ**'s** *scheme-dependent*

$$\gamma_+^{\overline{\text{MS}}}(a) = -aC_F + a^2 C_F\left\{4C_F\left[\frac{21}{8} + \frac{\pi^2}{3} - 6\zeta_3\right] + C_A\left[\frac{83}{9} - \frac{2\pi^2}{3} - 6\zeta_3\right] + \beta_0\left[\frac{35}{18} - \frac{\pi^2}{6}\right]\right\} + \ldots$$

- **Three-loop obtained** $\mathcal{H}_h^{(3)}(a)$**, to appear soon** [YJ, X. Wang]

- **Conformal symmetry allows relating** $\mathcal{H}_h$ **to kernel governing soft function for** $h \to \gamma\gamma$ **at NLP to all orders** [M. Beneke, YJ, X. Wang (2024)]

Conclusion and Outlook

Conclusion

- conformal symmetry is extremely powerful for loop calculations in nonforward kinematics (DVCS, meson production/decays):
  one-loop less for kernel, less diagram for CF; universal Δ
- applicable to calculating kernel and solving RGE in HQET
- allows to relate different evolution kernels from symmetry analysis

Outlook for future work

- Two-loop CFs for heavy meson decays from conformal symmetry
- one-loop (kinematic) higher-power/twist corrections