

Heavy Quark Masses: N³LO Relation between $\overline{\text{MS}}$ and RISMOM Scheme Away From Chiral Limit

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第七届全国重味物理与量子色动力学研讨会

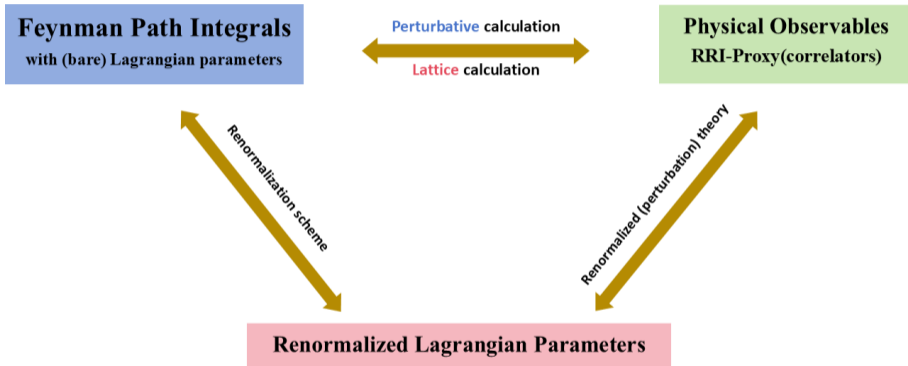
2025年4月18 - 4月22日, 南京

Based on on-going work with M. Niggetiedt



Regularization-Independent Renormalization: Why?

Quark masses are *fundamental* parameters of the Standard Model



Lattice-based approaches for determining quark masses:

- Lattice perturbation theory
- **Regularization-independent (non-perturbative) renormalization**
- Current-Current correlator method

Regularization-Independent Renormalization: Why?

Workflow for determining $\overline{\text{MS}}$ quark mass using Lattice with RI-renormalization:

- Determine a bare quark mass $m(a)$ in **Lattice** QCD (with spacing a) tuned **non-perturbatively** to reproduce the mass of a given hadron from experiment
- Compute **non-perturbatively** in **Lattice** QCD the mass renormalization factor Z_m^{RI} that converts $m(a)$ at each a to a mass defined in the RI-scheme at subtraction scale μ_s
- Compute **perturbatively** solely in *continuum* QCD in DR the mass conversion factor $Z_m^{\overline{\text{MS}}/\text{RI}}$ that transforms the RI-mass at μ_s to a $\overline{\text{MS}}$ mass $\overline{m}(\mu)$; Schematically,

$$\overline{m}(\mu) = Z_m^{\overline{\text{MS}}/\text{RI}}(\alpha_s(\mu), \mu, \mu_s) Z_m^{\text{RI}}(\mu_s, a) m(a)$$

RI-ren. is very effective in suppressing the discretization (lattice-cutoff) errors in extrapolation to the continuum limit $a \rightarrow 0$

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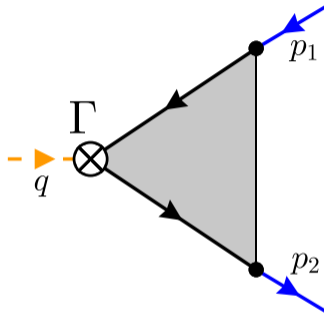
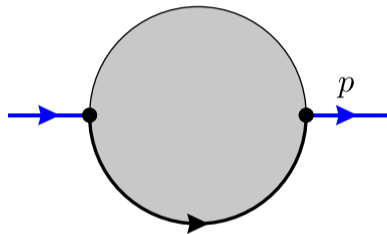
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Extension/Evolution: RIMOM \rightarrow RISMOM \rightarrow RI m SMOM

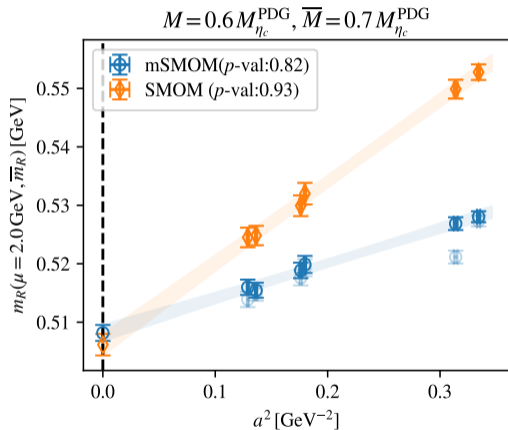
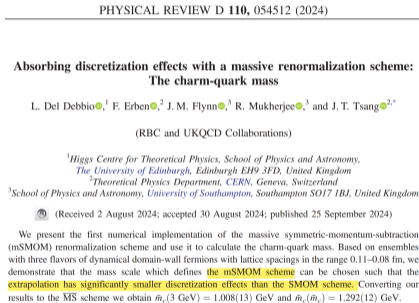
Family of Regularization Independent (massive, Symmetric) Momentum Subtraction schemes



- **RIMOM** [Martinelli et al. 95] : $p^2 = p_1^2 = p_2^2 = -\mu_s^2 < 0$ and $q = 0$
- **RISMOM** [Aoki 08; Sturm et al. 10] : $p^2 = p_1^2 = p_2^2 = q^2 = -\mu_s^2 < 0$ and $q = p_2 - p_1$
- **RI m SMOM** [Boyle, Debbio et al. 16] : same external kinematics as in RISMOM but at $m_Q^R \neq 0$

Extension/Evolution: RIMOM \rightarrow RISMOM \rightarrow RImSMOM

Additional discretization error in Lattice calculation for observables with heavy quarks (c, b): $\mathcal{O}(a^2 m_Q^2)$



- The RImSMOM scheme is claimed to be **useful** for reducing this kind of systematic errors.
- Alternative methods include *heavy-quark improved normalization scheme* [Hai-Yang Du et al. (CLQCD), 24].

Renormalization Conditions in RImSMOM

At the symmetric momentum configuration with $p^2 = q^2 = -\mu_s^2$ and **nonzero** renormalized quark propagator mass m_R [Boyle et al. 16, Debbio et al. 24] ,

$$Z_q : \frac{1}{12p^2} \text{Tr} \left[-iS_R(p)^{-1} \not{p} \right] = 1$$

$$Z_m : \frac{1}{12m_R} \left\{ \text{Tr} \left[S_R(p)^{-1} \right] + \frac{1}{2} \text{Tr} \left[(iq \cdot \Lambda_{A,R}) \gamma_5 \right] \right\} = 1$$

$$Z_V (= 1) : \frac{1}{12q^2} \text{Tr} \left[(q \cdot \Lambda_{V,R}) \not{q} \right] = 1$$

$$Z_A (= 1) : \frac{1}{12q^2} \text{Tr} \left[(q \cdot \Lambda_{A,R} + 2m_R \Lambda_{P,R}) \gamma_5 \not{q} \right] = 1$$

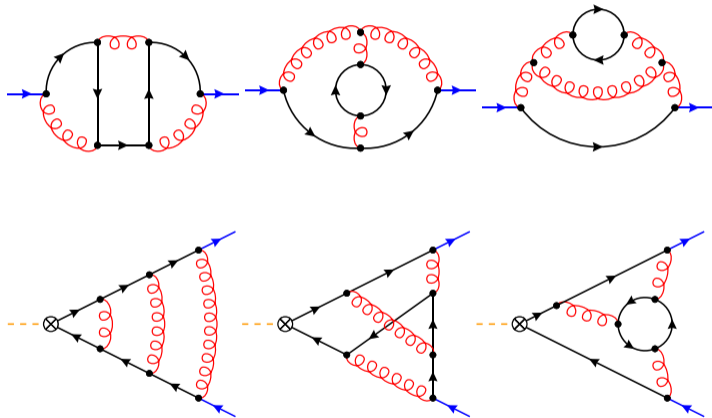
$$Z_P (= Z_m) : \frac{1}{12i} \text{Tr} \left[\Lambda_{P,R} \gamma_5 \right] = 1$$

$$Z_S (= Z_P?) : \frac{1}{12} \text{Tr} \left[\Lambda_{S,R} \right] + \frac{1}{6q^2} \text{Tr} \left[2m_R \Lambda_{P,R} \gamma_5 \not{q} \right] = 1$$

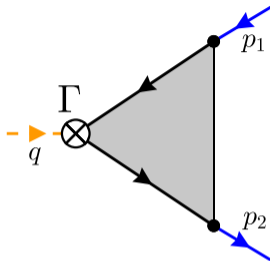
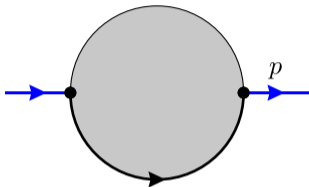
- $S_R(p)$ is the renormalized massive quark propagator
- $\Lambda_{\Gamma,R}$ is the renormalized amputated operator matrix element ($\Gamma = S, P, V, A$)

Generate Feynman Diagram Representations for Form Factors

- Feynman diagrams generated using DiaGen
- Lorentz and Dirac algebra done using FORM
- Full ζ -dependence in form factors kept at **off-shell kinematics** (non-singlet type only)



IBP Reduction and Evaluation of Master Integrals

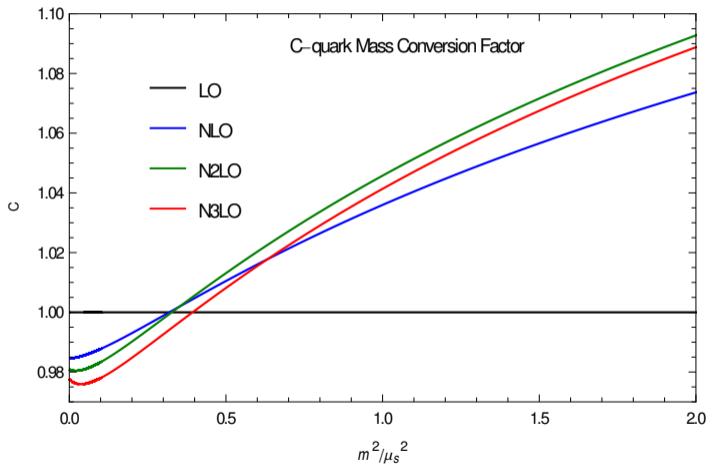


- Optimize the master basis simply by minimizing the size of DE (“trial-and-error”, \rightarrow 280 MB)
- Full reduction to this basis with symbolic ϵ and $m_s \equiv m^2 / \mu_s^2$ takes (Kira+FireFly) “several weeks” on a machine with \sim 200 CPUs and 2T RAM
- Due to **off-shellness**, number of masters **1115 more than doubled** compared to the cutting-edge 3-loop on-shell heavy-quark form factors [Fael et al. 2022]

- ▶ **Off-shell momenta** + **Massive** propagator \Rightarrow **(Diagrammatic) Large-Mass Expansion**
- ▶ Piecewise (generalized) power-series expansion using DE (DESolver utility from AMFlow [Liu Ma 22])
- ▶ Boundary conditions obtained using AMFlow **modified** to take **directly** external ready-to-use DE (avoiding the **most time-consuming** step of setting up DE with symbolic m_s , **critical @ 3-loop**)

Results for $C_{\overline{\text{MS}}/\text{RImSMOM}}(\mu_s, m_R, \mu, \zeta) @ 3\text{-Loop in QCD}$

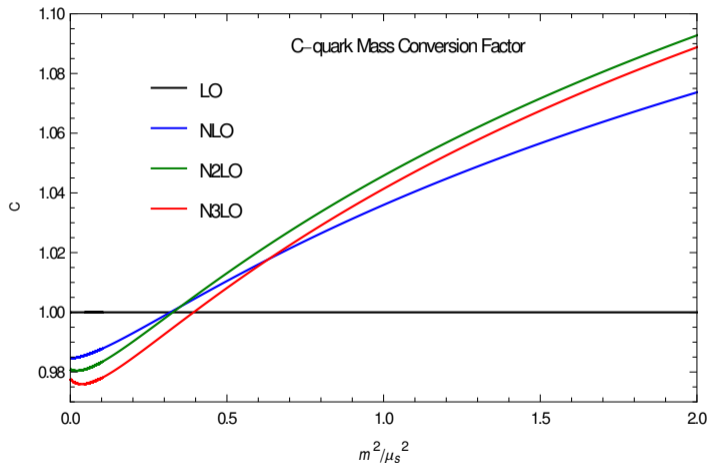
- Extending the previous result by **two more loop order!**
- The size of $\mathcal{O}(\alpha_s^3)$ is **comparable** to $\mathcal{O}(\alpha_s^2)$ in the typical scenario of m_c application
- A window where $C_{\overline{\text{MS}}/\text{RImSMOM}}$ is **smaller** than $C_{\overline{\text{MS}}/\text{RISMOM}}$, **good for reducing systematic uncertainties**
- $C_{\overline{\text{MS}}/\text{RImSMOM}}$ depends on ζ , and ζ renormalization is needed for $\zeta \neq 0$
- $C_{\overline{\text{MS}}/\text{RImSMOM}}(\zeta = 0)$'s dependence on μ satisfies the same RGE as $\overline{\text{MS}}$ -mass



$n_l = 3, n_h = 1, \mu = \mu_s = 2 \text{ GeV } (\alpha_s = 0.3), \zeta = 0$ (Landau gauge)

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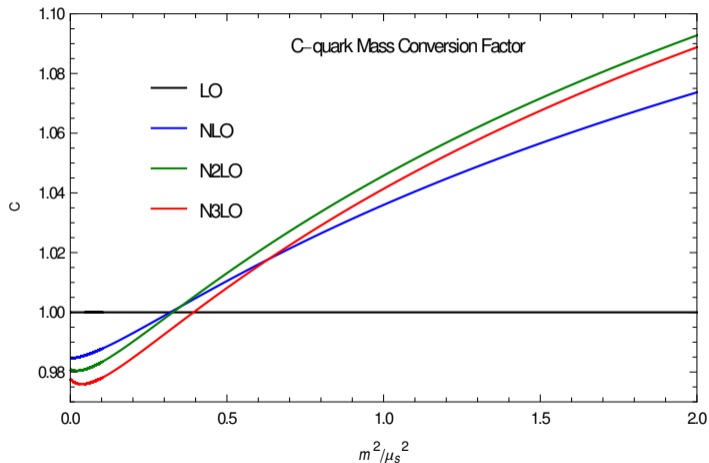
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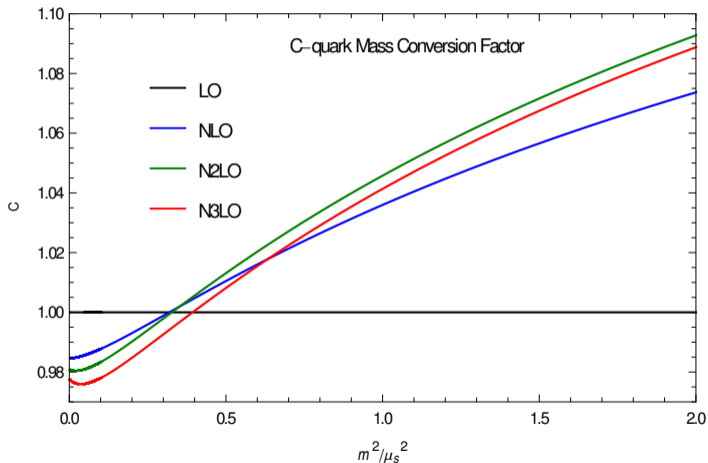
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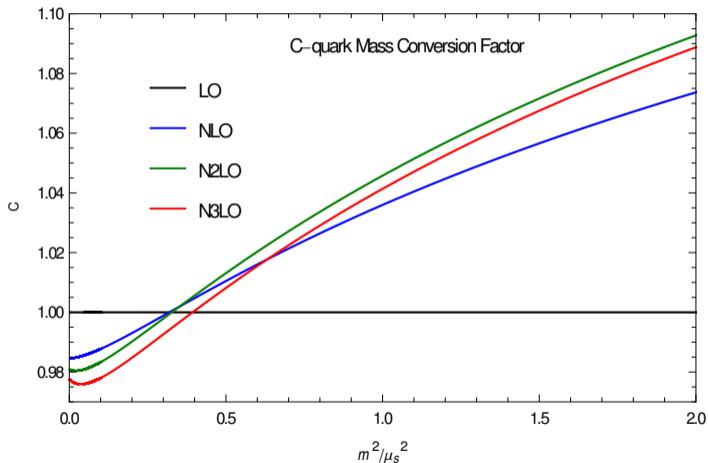
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RI(m)SMOM Conditions in DR **Re-interpreted** in a **Weaker** Sense

The original RI(m)SMOM conditions interpreted as **exact equations to all orders in ϵ** in Dimensional Regularization:

$$\begin{aligned} \frac{1}{12p^2} \text{Tr} \left[-iS_R(p)^{-1} \not{p} \right] &= 1, & Z_q \\ \frac{1}{12i} \text{Tr} [\Lambda_{P,R} \gamma_5] &= 1, & Z_P = Z_m \end{aligned}$$

$$\text{solved for } Z \equiv 1 + \sum_{i=1}^3 \sum_{j=-i}^{3-i} Z_{ij}(\mu_s, m_R, \mu, \xi) \alpha_s^i \epsilon^j + \mathcal{O}(\alpha_s^4)$$

We find that the following **weak variant**

$$\begin{aligned} \frac{1}{12p^2} \text{Tr} \left[-iS_R(p)^{-1} \not{p} \right] \Big|_{\epsilon \rightarrow 0} &= 1, & \tilde{Z}_q \\ \frac{1}{12i} \text{Tr} [\Lambda_{P,R} \gamma_5] \Big|_{\epsilon \rightarrow 0} &= 1, & \tilde{Z}_P = \tilde{Z}_m \end{aligned}$$

$$\text{solved for } \tilde{Z} \equiv 1 + \sum_{i=1}^3 \sum_{j=-i}^0 Z_{ij}(\mu_s, m_R, \mu, \xi) \alpha_s^i \epsilon^j + \mathcal{O}(\alpha_s^4)$$

lead to **different**, albeit **simpler**, Z_m , but the **same** $C_{\overline{\text{MS}}/\text{RI(m)SMOM}}(\mu_s, m_R, \mu, \xi) = Z_m^{\text{RI(m)SMOM}} / \tilde{Z}_m!$

$C_{\overline{\text{MS}}/\text{RIImSMOM}}(\mu_s, m_R, \mu, \zeta)$ **Re-obtained in an Acrobatic Way**

$$\frac{1}{12p^2} \text{Tr} \left[-iS_R(p)^{-1} \not{p} \right] |_{\epsilon \rightarrow 0} = 1, \quad \tilde{Z}_q$$
$$\frac{1}{12i} \text{Tr} [\Lambda_{P,R} \gamma_5] |_{\epsilon \rightarrow 0} = 1, \quad \tilde{Z}_P = \tilde{Z}_m$$

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- **The normal way:**

insert *Laurent ϵ -expansions* of $S(p)$ and Λ_P and extract **exact ϵ -free algebraic equations** for Z_{ij} (truncated precisely to ϵ^0), solved exactly.

- **An *acrobatic* way:**

insert $S(p)$ and Λ_P evaluated at numerical samples of ϵ [Liu, Ma 19, 22], **every single algebraic equation so-extracted for Z_{ij} is incorrect(!)**, miraculously, **the same**

$C_{\overline{\text{MS}}/\text{RIImSMOM}}(\mu_s, m_R, \mu, \zeta)$ **is restored by extrapolation in $\epsilon \rightarrow 0$** (conceptually similar as extrapolation in a in Lattice)

RImSMOM's statement on $Z_S = Z_P(Z_m)$ **Revised**

- RISMOM

$$\frac{1}{12} \text{Tr} [\Lambda_{S,R}] \Big|_{m \rightarrow 0} = 1$$

$Z_S = Z_P$ holds in this *chiral* limit $m \rightarrow 0$

- RImSMOM

$$\frac{1}{12} \text{Tr} [\Lambda_{S,R}] + \frac{1}{6q^2} \text{Tr} [2m_R \Lambda_{P,R} \gamma_5 \not{q}] = 1$$

- We observe, however,

$$Z_S \neq Z_P$$

in general (accidentally equal in Feynman-gauge @ 1-loop), but **approach each other again in the *chiral* limit $m \rightarrow 0$**

- Alternatively, we **suggest** to simply take

$$Z_S = Z_P$$

in the Scalar-Operator renormalization away from chiral limit (completely detached from the others), as part of the **definition of the revised RImSMOM prescription**.

Summary and Outlook

- ☑ We present the **first 3-loop result for mass conversion factor** $C_{\overline{\text{MS}}/\text{RImSMOM}}(\mu/\mu_s, m_R/\mu_s)$ away from chiral limit (extending the previous result by two more loop orders)
- ☑ The $\mathcal{O}(\alpha_s^3)$ correction is quite **sizeable** in the typical scenario of m_c -determination, but there exists a window where $C_{\overline{\text{MS}}/\text{RImSMOM}}$ is **smaller** than RISMOM counterpart, **good for reducing systematic uncertainties**
- ☑ We have provided an alternative interpretation of the original RI(m)SMOM conditions in DR in a **weaker sense** (holding just in 4-dimensional limit rather than exactly in D dimensions)
- ☑ Furthermore, when solving the **weaker variant** of the RImSMOM conditions, an exact explicit truncation to ϵ^0 is shown to be not necessary.
- ☑ The original RImSMOM's claim on scalar-operator renormalization shall be **revised**.
- Extend to tensor operators, the effect of a second mass via the singlet-type diagrams...

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