Heavy Quark Masses: N^{3}LO Relation between $\overline{\mathrm{MS}}$ and RISMOM Scheme Away From Chiral Limit

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第七届全国重味物理与量子色动力学研讨会

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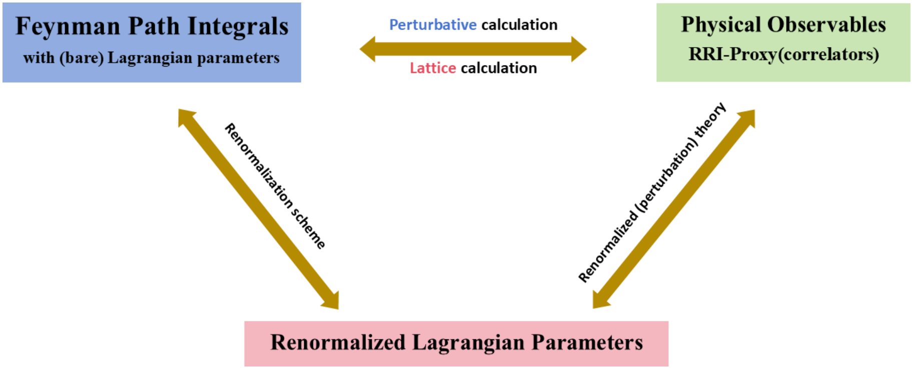
Based on on-going work with **M. Niggetiedt**

Regularization-Independent Renormalization: Why?

Quark masses are *fundamental* parameters of the Standard Model



Lattice-based approaches for determining quark masses:

- Lattice perturbation theory
- Regularization-independent (non-perturbative) renormalization
- Current-Current correlator method

Regularization-Independent Renormalization: Why?

Workflow for determining $\overline{\text{MS}}$ quark mass using Lattice with RI-renormalization:

- Determine a bare quark mass $m(a)$ in **Lattice** QCD (with spacing a) tuned non-perturbatively to reproduce the mass of a given hadron from experiment

- Compute non-perturbatively in **Lattice** QCD the mass renormalization factor Z_m^{RI} that converts $m(a)$ at each a to a mass defined in the RI-scheme at subtraction scale μ_s

- Compute perturbatively solely in *continuum* QCD in DR the mass conversion factor $Z_m^{\overline{\text{MS}}/\text{RI}}$ that transforms the RI-mass at μ_s to a $\overline{\text{MS}}$ mass $\overline{m}(\mu)$; Schematically,

$$\boxed{\overline{m}(\mu) = Z_m^{\overline{\text{MS}}/\text{RI}}(\alpha_s(\mu), \mu, \mu_s)\, Z_m^{\text{RI}}(\mu_s, a)\, m(a)}$$

RI-ren. is very effective in suppressing the discretization (lattice-cutoff) errors in extrapolation to the continuum limit $a \to 0$

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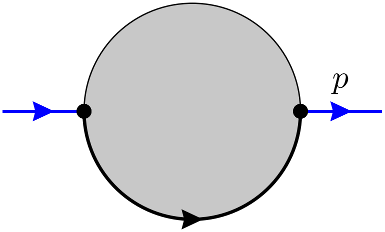
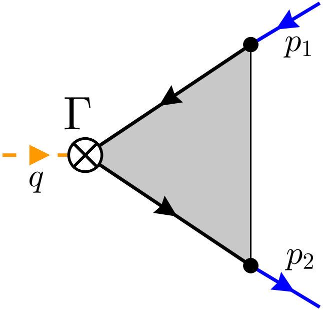
Extension/Evolution: RIMOM → RISMOM → RImSMOM

Family of **Regularization Independent (massive, Symmetric) Momentum Subtraction** schemes

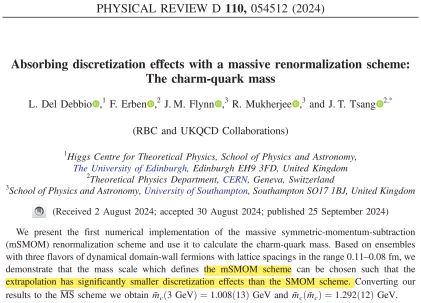




- **RIMOM** [Martinelli et al. 95] : $p^2 = p_1^2 = p_2^2 = -\mu_s^2 < 0$ and $q = 0$
- **RISMOM** [Aoki 08; Sturm et al. 10] : $p^2 = p_1^2 = p_2^2 = q^2 = -\mu_s^2 < 0$ and $q = p_2 - p_1$
- **RImSMOM** [Boyle, Debbio et al. 16] : same external kinematics as in RISMOM but at $m_Q^R \neq 0$

Extension/Evolution: RIMOM → RISMOM → RImSMOM

Additional discretization error in Lattice calculation for observables with **heavy quarks** ($c\,, b$): $\mathcal{O}(a^2\, m_Q^2)$



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Absorbing discretization effects with a massive renormalization scheme: The charm-quark mass

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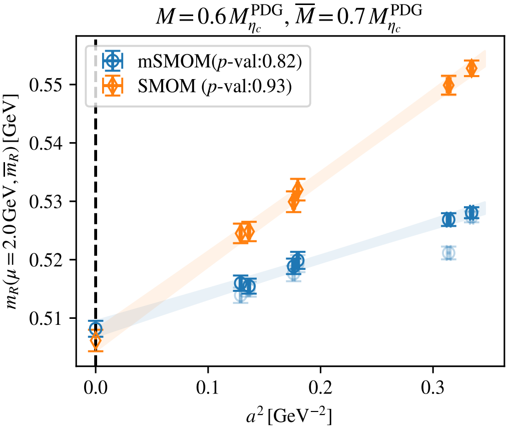
We present the first numerical implementation of the massive symmetric-momentum-subtraction (mSMOM) renormalization scheme and use it to calculate the charm-quark mass. Based on ensembles with three flavors of dynamical domain-wall fermions with lattice spacings in the range 0.11–0.08 fm, we demonstrate that the mass scale which defines the mSMOM scheme can be chosen such that the extrapolation has significantly smaller discretization effects than the SMOM scheme. Converting our results to the $\overline{\rm MS}$ scheme we obtain $\bar{m}_c(3\ \text{GeV}) = 1.008(13)$ GeV and $\bar{m}_c(\bar{m}_c) = 1.292(12)$ GeV.



- The RImSMOM scheme is claimed to be **useful** for reducing this kind of systematic errors.
- Alternative methods include *heavy-quark improved normalization scheme* [Hai-Yang Du et al. (**CLQCD**), 24] .

Renormalization Conditions in RImSMOM

At the symmetric momentum configuration with $p^2 = q^2 = -\mu_s^2$ and nonzero renormalized quark propagator mass m_R [Boyle et al. 16, Debbio et al. 24] ,

$$Z_q: \quad \frac{1}{12p^2} \mathrm{Tr}\left[-iS_R(p)^{-1}\not{p}\right] = 1$$

$$Z_m: \quad \frac{1}{12m_R}\left\{\mathrm{Tr}\left[S_R(p)^{-1}\right] + \frac{1}{2}\mathrm{Tr}\left[\left(iq \cdot \Lambda_{\mathrm{A},R}\right)\gamma_5\right]\right\} = 1$$

$$Z_V(=1): \quad \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{V},R}\right)\not{q}\right] = 1$$

$$Z_A(=1): \quad \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{A},R} + 2m_R\Lambda_{\mathrm{P},R}\right)\gamma_5\not{q}\right] = 1$$

$$Z_P(= Z_m): \quad \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R}\gamma_5\right] = 1$$

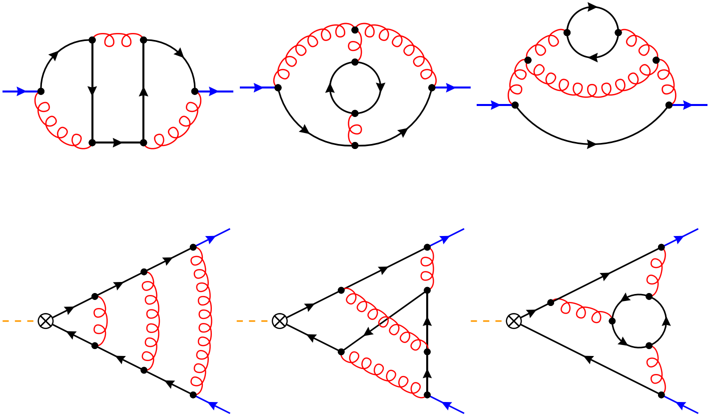
$$Z_S(=Z_P?): \quad \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R}\right] + \frac{1}{6q^2}\mathrm{Tr}\left[2m_R\Lambda_{\mathrm{P},R}\gamma_5\not{q}\right] = 1$$

- $S_R(p)$ is the renormalized massive quark propagator
- $\Lambda_{\Gamma,R}$ is the renormalized amputated operator matrix element ($\Gamma = S, P, V, A$)

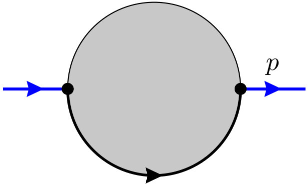
Generate Feynman Diagram Representations for Form Factors

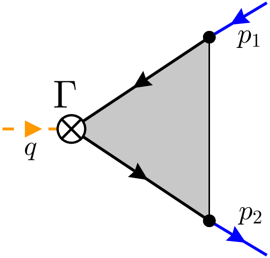
- Feynman diagrams generated using DiaGen
- Lorentz and Dirac algebra done using FORM
- **Full *ζ*-dependence in form factors kept at off-shell kinematics**

  (non-singlet type only)



IBP Reduction and Evaluation of Master Integrals



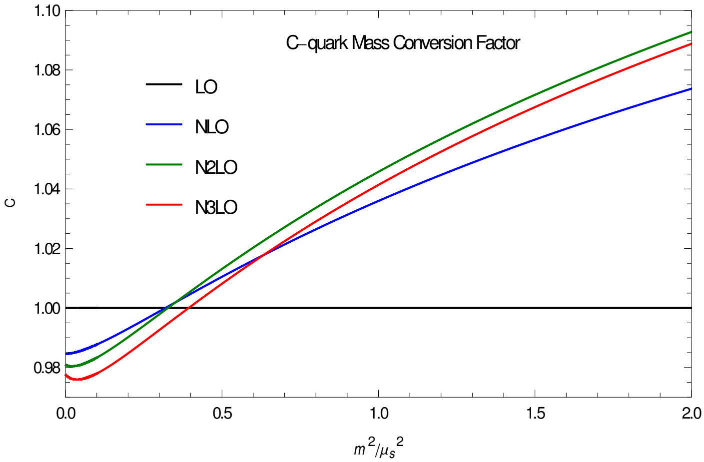


- Due to **massive quark off-shell**, number of masters 1115 **more than doubled** compared to the cutting-edge 3-loop on-shell heavy-quark form factors [Fael et al. 2022]
- Optimize the master basis simply by minimizing the size of DE ("trial-and-error", $\rightarrow$ 280 MB)
- Full reduction to this basis with symbolic ϵ and $m_s \equiv m^2/\mu_s^2$ takes (Kira+FireFly) "several weeks" on a machine with $\sim$200 CPUs and 2T RAM

- Off-shell momenta + Massive propagator $\Rightarrow$ **(Diagrammatic) Large-Mass Expansion**
- Piecewise (generalized) power-series expansion using DE (DESolver utility from AMFlow [Liu Ma 22])
- Boundary conditions obtained using AMFlow **modified** to take **directly** external ready-to-use DE (avoiding the **most time-consuming** step of setting up DE with symbolic m_s, **critical @ 3-loop**)

Results for $C_{\overline{\text{MS}}/\text{RImSMOM}}(\mu_s, m_R, \mu, \xi)$ @ 3-Loop in QCD

- Extending the previous result by **two more loop order**!
- The size of $\mathcal{O}(\alpha_s^3)$ is **comparable** to $\mathcal{O}(\alpha_s^2)$ in the typical scenario of m_c application
- A window where $C_{\overline{\text{MS}}/\text{RImSMOM}}$ is **smaller** than $C_{\overline{\text{MS}}/\text{RISMOM}}$, good for reducing systematic uncertainties
- $C_{\overline{\text{MS}}/\text{RImSMOM}}$ depends on ξ, and ξ renormalization is needed for $\xi \neq 0$
- $C_{\overline{\text{MS}}/\text{RImSMOM}}(\xi = 0)$'s dependence on μ satisfies the same RGE as $\overline{\text{MS}}$-mass



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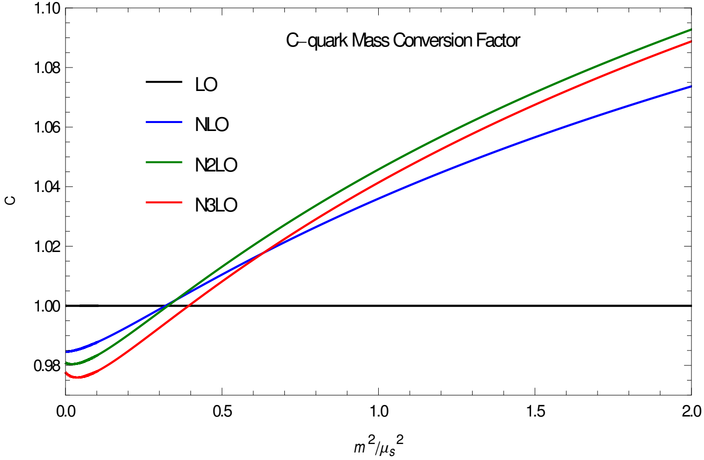

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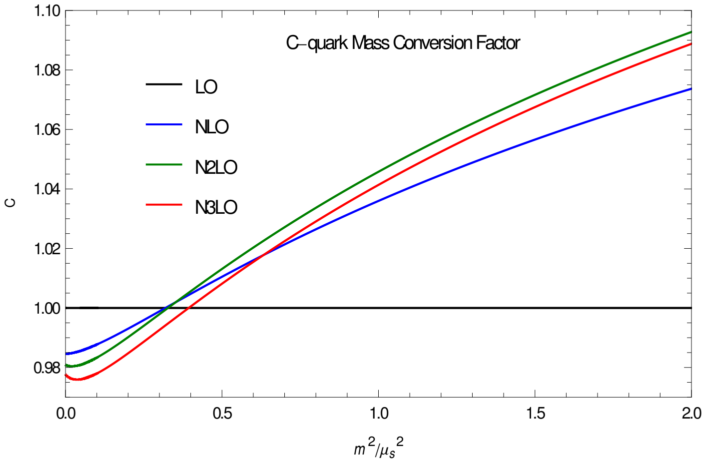

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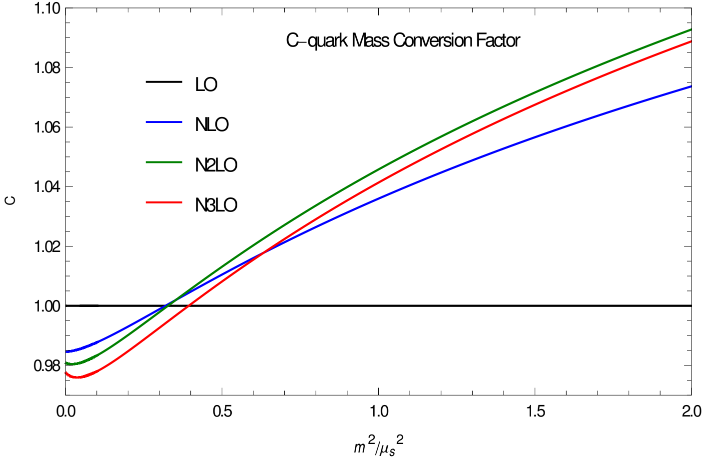

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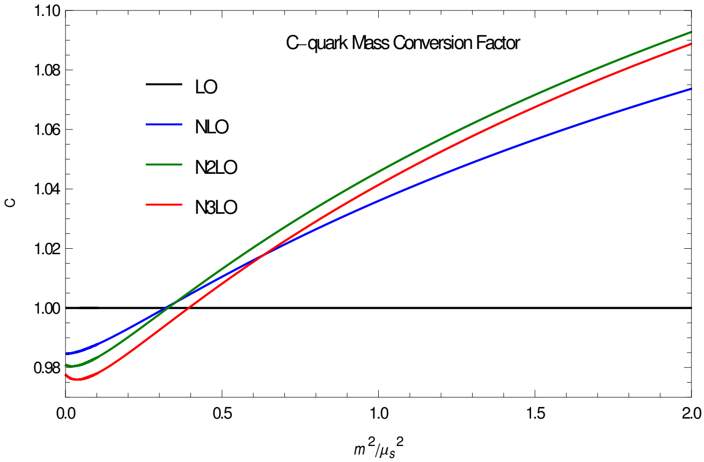

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RI(m)SMOM Conditions in DR Re-interpreted in a Weaker Sense

The original RI(m)SMOM conditions interpreted as **exact equations to all orders in ϵ** in Dimensional Regularization:

$$\frac{1}{12p^2}\,\mathrm{Tr}\left[-iS_R(p)^{-1}\not{p}\right] = 1\,, \quad Z_q$$
$$\frac{1}{12i}\,\mathrm{Tr}\left[\Lambda_{P,R}\gamma_5\right] = 1\,, \quad Z_P = Z_m$$

solved for $Z \equiv 1 + \sum_{i=1}^{3}\sum_{j=-i}^{3-i} Z_{ij}(\mu_s, m_R, \mu, \xi)\,\alpha_s^i\,\epsilon^j + \mathcal{O}(\alpha_s^4)$

We find that the following **weak variant**

$$\frac{1}{12p^2}\,\mathrm{Tr}\left[-iS_R(p)^{-1}\not{p}\right]\Big|_{\epsilon\to 0} = 1\,, \quad \tilde{Z}_q$$
$$\frac{1}{12i}\,\mathrm{Tr}\left[\Lambda_{P,R}\gamma_5\right]\Big|_{\epsilon\to 0} = 1\,, \quad \tilde{Z}_P = \tilde{Z}_m$$

solved for $\tilde{Z} \equiv 1 + \sum_{i=1}^{3}\sum_{j=-i}^{0} Z_{ij}(\mu_s, m_R, \mu, \xi)\,\alpha_s^i\,\epsilon^j + \mathcal{O}(\alpha_s^4)$

lead to different, albeit **simpler**, Z_m, but the **same** $C_{\overline{\mathrm{MS}}/\mathrm{RImSMOM}}(\mu_s, m_R, \mu, \xi) = Z_m^{\mathrm{RImSMOM}}/\tilde{Z}_m$!

$C_{\overline{\text{MS}}/\text{RImSMOM}}(\mu_s, m_R, \mu, \xi)$ Re-obtained in an Acrobatic Way

$$\frac{1}{12p^2}\,\mathrm{Tr}\left[-iS_R(p)^{-1}\not{p}\right]\Big|_{\epsilon\to 0} = 1\,, \quad \tilde{Z}_q$$

$$\frac{1}{12i}\,\mathrm{Tr}\left[\Lambda_{\mathrm{P},R}\gamma_5\right]\Big|_{\epsilon\to 0} = 1\,, \quad \tilde{Z}_P = \tilde{Z}_m$$

solved for $\tilde{Z} \equiv 1 + \sum_{i=1}^{3}\sum_{j=-i}^{0} Z_{ij}(\mu_s, m_R, \mu, \xi)\,\alpha_s^i\,\epsilon^j + \mathcal{O}(\alpha_s^4)$

- **The normal way**:
  insert *Laurent ϵ-expansions* of $S(p)$ and Λ_{P} and extract **exact ϵ-free algebraic equations** for Z_{ij} (truncated precisely to ϵ^0), solved exactly.

- **An *acrobatic* way**:
  insert $S(p)$ and Λ_{P} evaluated at numerical samples of ϵ [Liu, Ma 19, 22] , **every single algebraic equation so-extracted for Z_{ij} is incorrect(!)**, miraculously, **the same** $C_{\overline{\text{MS}}/\text{RImSMOM}}(\mu_s, m_R, \mu, \xi)$ **is restored by extrapolation in $\epsilon \to 0$** (conceptually similar as extrapolation in a in Lattice)

RImSMOM's statement on $Z_S = Z_P(Z_m)$ Revised

- **RISMOM**

$$\frac{1}{12}\mathrm{Tr}\left[\Lambda_{S,R}\right]\Big|_{m\to 0} = 1$$

$Z_S = Z_P$ holds in this *chiral* limit $m \to 0$

- **RImSMOM**

$$\frac{1}{12}\mathrm{Tr}\left[\Lambda_{S,R}\right] + \frac{1}{6q^2}\mathrm{Tr}\left[2m_R\Lambda_{P,R}\gamma_5\not{q}\right] = 1$$

- **We observe, however,**

$$Z_S \neq Z_P$$

in general (accidentally equal in Feynman-gauge @ 1-loop), but approach each other again in the *chiral* limit $m \to 0$

- Alternatively, we **suggest** to simply take

$$Z_S = Z_P$$

in the Scalar-Operator renormalization away from chiral limit (completely detached from the others), as part of the **definition of the revised RImSMOM prescription**.

Summary and Outlook

- ☑ We present the **first 3-loop result for mass conversion factor** $C_{\overline{\mathrm{MS}}/\mathrm{RImSMOM}}(\mu/\mu_s, m_R/\mu_s)$ away from chiral limit (extending the previous result by two more loop orders)

- ☑ The $\mathcal{O}(\alpha_s^3)$ correction is quite **sizable** in the typical scenario of m_c-determination, but there exists a window where $C_{\overline{\mathrm{MS}}/\mathrm{RImSMOM}}$ is **smaller** than RISMOM counterpart, good for reducing systematic uncertainties

- ☑ We have provided an alternative interpretation of the original RI(m)SMOM conditions in DR in a weaker sense (holding just in 4-dimensional limit rather than exactly in D dimensions)

- ☑ Furthermore, when solving the weaker variant of the RImSMOM conditions, an exact explicit truncation to ϵ^0 is shown to be not necessary.

- ☑ The original RImSMOM's claim on scalar-operator renormalization shall be **revised**.

- ○ Extend to tensor operators, the effect of a second mass via the singlet-type diagrams...

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