Off-lightcone Wilson-line operators in gradient flow

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Outlines

- Motivation and Introduction of Gradient Flow
- Renormalization of Off-lightcone Wilson-line Operator and Matching relations
- One-loop Matching Calculations and Results
- Summary and Outlook

Motivation 1: Quasi-PDFs/LCDAs

- Quasi-PDFs/LCDAs: Hadronic matrix elements of following operator,

$$\mathcal{O}_\Gamma(zv) = \bar{\psi}(zv)\Gamma W(zv,0)\psi(0), \tag{1}$$

$$\mathcal{O}^{\mu\nu\alpha\beta}(zv) = g^2 F^{\mu\nu}(zv) W(zv,0) F^{\alpha\beta}(0), \tag{2}$$

where Γ is a Dirac matrix and the space-like Wilson-line

$$W(zv,0) \equiv \mathrm{P}\exp\left(ig\int_0^z ds\, v\cdot A(sv)\right). \tag{3}$$

- Quasi-PDFs/LCDAs (Lattice) $\xrightarrow{\text{LaMET}}$ Lightcone PDFs/LCDAs.

See X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and Y. Zhao, Rev. Mod. Phys. 93 (2021) 035005 for a review.

Motivation 2: $\mathcal{E}_3$ & $G_{\mathbf{E}}, G_{\mathbf{B}}$

- Vacuum expectation value of $g^2 F^{\mu\nu}(\tau)W(\tau,0)F^{\alpha\beta}(0)$ is related to P-wave quarkonium decay in the framework of potential NRQCD (time-like, $\mathbf{E}$ is the chromoelectric field)

  See N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo, PRL 88 (2002) 012003, hep-ph/0109130

$$\mathcal{E}_3 = \frac{T_F}{N_c}\int_0^\infty d\tau\, \tau^3 \langle 0|g\mathbf{E}(\tau,\mathbf{0})W(\tau,0)g\mathbf{E}(0,\mathbf{0})|0\rangle, \tag{4}$$

- and heavy quarkonium (quark) diffusion coefficient ($\mathbf{B}$ is the chromomagnetic field),

  See PRL. 132 (2024) 5, 051902, 2401.06733, 2402.09337 ...

$$G_{\mathbf{E}} \sim \langle 0|g\mathbf{E}(\tau,\mathbf{0})W(\tau,0)g\mathbf{E}(0,\mathbf{0})|0\rangle, \tag{5}$$

$$G_{\mathbf{B}} \sim \langle 0|g\mathbf{B}(\tau,\mathbf{0})W(\tau,0)g\mathbf{B}(0,\mathbf{0})|0\rangle. \tag{6}$$

Motivation 3: Spin-dependent potentials

- Relativistic corrections of QCD static potentials can be defined in terms of chromomagnetic ($\mathbf{B}$) and chromoelectric field ($\mathbf{E}$) insertions into the Wilson loops.

  See N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. 77 (2005) 1423 for a review.

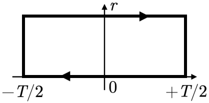
- Spin-dependent potentials, such as

$$V^{(1,1)}_{L_2S_1}(r) \quad = -i\frac{c_F(m,\mu)}{r^2}\mathbf{r}\cdot \lim_{T\to\infty}\int_0^T dt\, t\langle\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\rangle, \tag{7}$$

  where $\langle\langle \dots \rangle\rangle \equiv \langle \dots W_\square\rangle / \langle W_\square\rangle$,

- $W_\square$ is the Wilson-loop (time-like) defined as

$$W_\square \equiv \mathrm{P}\exp\left\{-ig\oint_{r\times T_W} dz^\mu A_\mu(z)\right\}. \tag{8}$$



Lattice-perturbative matching

- Lattice-perturbative matching: $G_{\mathbf{E}}^{\text{lattice}} = c_{\mathbf{E}}\, G_{\mathbf{E}}^{\overline{\text{MS}}}$, $G_{\mathbf{B}}^{\text{lattice}} = c_{\mathbf{B}}\, G_{\mathbf{B}}^{\overline{\text{MS}}}$....

- Higher-order perturbative calculation of lattice-perturbative matching coefficient ($c_{\mathbf{E}}, c_{\mathbf{B}}...$) is very important for precise lattice computations.

- Difficult using lattice regularization (a), but relatively simple in gradient flow scheme. There are many two-loop matching calculations in gradient flow for local operators.

- For off-lightcone Wilson-line operators, only several one-loop matching calculations in gradient flow at matrix element level, with very complicated approach or giving wrong results.

See C. J. Monahan, PRD 97 (2018) 5, 054507, A. M. Eller, PhD Thesis (2021), PRD 110 (2024) 9, 094057.

Two international workshops on gradient flow

1st international workshop on gradient flow in 03/2023, Trento, Italy

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Abstract

The gradient flow field transformation is a continuous smoothing transformation that removes ultraviolet fluctuations. It can serve as a tool to renormalize quantum field theories, allowing numerical studies of strongly coupled systems. The flow has been used extensively in lattice gauge theory calculations, both in QCD and in beyond the standard model settings for applications including scale

2nd international workshop on gradient flow in 02/2025, Zurich, Switzerland

Zürich Gradient Flow Workshop 2025

12 Feb 2025, 08:30 → 14 Feb 2025, 17:50 Europe/Zurich

Y36-K08 (University of Zürich)

Description The **Zürich Gradient Flow Workshop 2025** will take place from Feb. 12 - 14, 2025, at the University of Zürich, Switzerland, with the goal to bring together scientists working actively on non-perturbative and perturbative aspects of the gradient flow.

The gradient flow has been instrumental in a large variety of recent developments in non-perturbative QCD and it plays a crucial role in reaching a better understanding of strong dynamics and to provide input in precision phenomenology studies, searching for physics beyond the Standard Model.

A workshop with a similar theme was last held at the ECT* Center in Trento, Italy, in March 2023: https://indico.ectstar.eu/event/164/overview.

Gradient flow and its advantages

Gradient Flow: 4+1 dimension (x,t) QCD in Euclidean spacetime , where x is the spacetime position, t is the flow time with mass dimension -2:

Martin Lüscher, JHEP 08 (2010) 071, 1150+ citations

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi. \tag{9}$$

$\mathcal{L}_B, \mathcal{L}_\chi$ depend on flowed gauge and quark field $B(x,t), \chi(x,t)$, with the boundary condition $B(x,t=0) = A(x)$ and $\chi(x,t=0) = \psi(x)$.

- Improve signal to noise ratio and efficiency in lattice computations.
- As a UV regulator ($t > 0, a \to 0$), no composite operator renormalization besides QCD parameters (m, g_s) renormalization.
- Dimensional regularization can be used in perturbative gradient flow, matching from gradient flow to $\overline{\text{MS}}$ is much easier.

  See recent progress on lattice calculation of heavy quark diffusion coefficient using gradient flow: L. Altenkort *et al.* [HotQCD], PRL 132 (2024) 051902; PRL 130 (2023) 231902; Application on renormalon subtraction: M. Beneke and H. Takaura, *PoS* RADCOR 2023 (2024) 062
- To simplify the matching calculation, we need to understand its renormalization.

Off-lightcone Wilson-line operators in Generalized HQET

- The renormalized generalized version of HQET ($v^2 = \pm 1$):

$$\mathcal{L}_{h_v} = Z_{h_v} \bar{h}_v (iv \cdot \partial - i\delta m) h_v + g Z_g Z_A^{\frac{1}{2}} Z_{h_v} \bar{h}_v v \cdot A^a T^a h_v, \tag{10}$$

- Using the generalized HQET, the off-lightcone Wilson-line operators can be substituted with products of local current operators, for instance,

$$\begin{aligned} \mathcal{O}_\Gamma^{\mathrm{B}}(zv) &= \bar{\psi}_0(zv) \Gamma W(zv, 0) \psi_0(0) = \int d^d x \, \delta\left(\frac{v \cdot x}{v^2} - z\right) \\ &\times \langle \bar{\psi}_0(x) h_{v,0}(x) \Gamma \bar{h}_{v,0}(0) \psi_0(0) \rangle_{h_v}, \end{aligned} \tag{11}$$

$$\begin{aligned} \mathcal{O}^{\mu\nu\alpha\beta,\mathrm{B}}(zv) &= g^2 F_0^{\mu\nu}(zv) W(zv, 0) F_0^{\alpha\beta}(0) = \int d^d x \, \delta\left(\frac{v \cdot x}{v^2} - z\right) \\ &\times \langle g^2 F_0^{\mu\nu}(x) h_{v,0}(x) \bar{h}_{v,0}(0) F_0^{\alpha\beta}(0) \rangle_{h_v}, \end{aligned} \tag{12}$$

where the superscript “B” indicates bare composite operators.

Decomposition of $F^{\mu\nu}$

- For convenience, we introduce the following projectors,

$$g_{\parallel}^{\mu\nu} = \frac{v_\mu v_\nu}{v^2}, \quad g_{\perp}^{\mu\nu} = g^{\mu\nu} - \frac{v_\mu v_\nu}{v^2}, \tag{13}$$

to project out the parallel and the transverse components of $F^{\mu\nu}$ by defining,

$$F_{\parallel\perp}^{\mu\nu} = g_{\parallel}^{\mu\alpha} g_{\perp}^{\nu\beta} F^{\alpha\beta}, \tag{14}$$

$$F_{\perp\perp}^{\mu\nu} = g_{\perp}^{\mu\alpha} g_{\perp}^{\nu\beta} F^{\alpha\beta}. \tag{15}$$

- When $v = (1,0,0,0)$ and $\mu \neq \nu$, $F_{\parallel\perp}^{\mu\nu}, F_{\perp\perp}^{\mu\nu}$ are proportional to the chromoelectric field **E** and the chromomagnetic field **B**.
- Define the following gluonic Wilson-line operators,

$$\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta}(zv) = g^2 F_{\parallel\perp}^{\mu\nu}(zv) W(zv,0) F_{\parallel\perp}^{\alpha\beta}(0), \quad (E-E) \tag{16}$$

$$\mathcal{O}_{\perp\perp}^{\mu\nu\alpha\beta}(zv) = g^2 F_{\perp\perp}^{\mu\nu}(zv) W(zv,0) F_{\perp\perp}^{\alpha\beta}(0). \quad (B-B) \tag{17}$$

Renormalization of local current operators

- Off-lightcone Wilson-lone operators can be expressed in terms of product of local current operators.
- We define the local “heavy-to-light” and “heavy-to-gluon” current operators

$$J_q(x) = \bar{\psi}(x) h_v(x), \tag{18}$$

$$J_{\parallel\perp}^{\mu\nu}(x) = g F_{\parallel\perp}^{\mu\nu}(x) h_v(x), \tag{19}$$

$$J_{\perp\perp}^{\mu\nu}(x) = g F_{\perp\perp}^{\mu\nu}(x) h_v(x). \tag{20}$$

- The renormalization formulas for the above local current operators are given by,

$$J_q^{\mathrm{B}}(x,\Lambda) = Z_{J_q}(\Lambda,\mu) J_q^{\mathrm{R}}(x,\mu), \tag{21}$$

$$J_{\parallel\perp}^{\mu\nu,\mathrm{B}}(x,\Lambda) = Z_{J,\parallel\perp}(\Lambda,\mu) J_{\parallel\perp}^{\mu\nu,\mathrm{R}}(x,\mu), \tag{22}$$

$$J_{\perp\perp}^{\mu\nu,\mathrm{B}}(x,\Lambda) = Z_{J,\perp\perp}(\Lambda,\mu) J_{\perp\perp}^{\mu\nu,\mathrm{R}}(x,\mu), \tag{23}$$

which are well known in HQET.

Multiplicative renormalizability

- The renormalizations of the off-lightcone Wilson-line operators:

$$\mathcal{O}_{\Gamma}^{\mathrm{B}}(zv,\Lambda) = Z_q(\Lambda,\mu)e^{\delta m(\Lambda)z}\mathcal{O}_{\Gamma}^{\mathrm{R}}(zv,\mu), \quad (24)$$

$$\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta,\,\mathrm{B}}(zv,\Lambda) = Z_{\parallel\perp}(\Lambda,\mu)e^{\delta m(\Lambda)z}\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta,\,\mathrm{R}}(zv,\mu), \quad (25)$$

$$\mathcal{O}_{\perp\perp}^{\mu\nu\alpha\beta,\,\mathrm{B}}(zv,\Lambda) = Z_{\perp\perp}(\Lambda,\mu)e^{\delta m(\Lambda)z}\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta,\,\mathrm{R}}(zv,\mu), \quad (26)$$

in which,

$$Z_q(\Lambda,\mu) = Z_{J_q}^2(\Lambda,\mu), \quad (27)$$

$$Z_{\parallel\perp}(\Lambda,\mu) = Z_{J,\,\parallel\perp}^2(\Lambda,\mu), \quad (28)$$

$$Z_{\perp\perp}(\Lambda,\mu) = Z_{J,\,\perp\perp}^2(\Lambda,\mu). \quad (29)$$

Multiplicative renormalizability

- For the spin dependent potential, such as

$$V^{(1,1)}_{L_2S_1}(r) = -i\frac{c_F}{r^2}\mathbf{r}\cdot \lim_{T\to\infty}\int_0^T dt\, t\langle\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\rangle, \tag{30}$$

- Each insertion of a **B** field into a Wilson loop can be related to the following local current operator as,

$$J_F^{\mu\nu}(x) = \bar{h}_v(x) g F_{\perp\perp}^{\mu\nu}(x) h_v(x), \tag{31}$$

where the renormalization formula for this operator is ,

$$J_F^{\mu\nu,\mathrm{B}}(x) = Z_F J_F^{\mu\nu,\mathrm{R}}(x) = Z_A^{1/2} Z_{h_v} Z_F^V J_F^{\mu\nu,\mathrm{R}}(x). \tag{32}$$

Small flow time expansion for the flowed current operators

- Matching relation for the local current operators,

$$\mathcal{O}^{\mathrm{R}}(t) = c_{\mathcal{O}}(t,\mu)\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) + O(t), \tag{33}$$

which is small-distance (t) operator-product-expansion (OPE).

- For $\bar{\psi}h_v$, $gF^{\mu\nu}_{\parallel\perp}h_v$, $gF^{\mu\nu}_{\perp\perp}h_v$, and $g\bar{h}_vF^{\mu\nu}_{\perp\perp}h_v$, we can split $c_{\mathcal{O}}(t,\mu)$ into

$$\begin{aligned}
c_{\psi h_v}(t,\mu) &= \mathring{\zeta}_{\psi}(t,\mu)\zeta^{F}_{h_v}(t,\mu)\zeta_{\psi h_v}(t,\mu), && (34a)\\
c_{\parallel\perp}(t,\mu) &= \zeta_{A}(t,\mu)\zeta^{A}_{h_v}(t,\mu)\zeta_{\parallel\perp}(t,\mu), && (34b)\\
c_{\perp\perp}(t,\mu) &= \zeta_{A}(t,\mu)\zeta^{A}_{h_v}(t,\mu)\zeta_{\perp\perp}(t,\mu), && (34c)\\
c_{F}(t,\mu) &= \zeta_{A}(t,\mu)(\zeta^{F}_{h_v}(t,\mu))^2\zeta_{F}(t,\mu). && (34d)
\end{aligned}$$

Matching for the Wilson-line operators

- Key insights from the multiplicative renormalization of the off-lightcone Wilson-line operators:

$$\mathcal{O}^{\mathrm{R}}_{\Gamma}(zv,t) = \mathcal{C}_{\psi}(t,\mu)e^{\delta m z}\mathcal{O}^{\overline{\mathrm{MS}}}_{\Gamma}(zv) + O(t), \quad (35)$$

$$\mathcal{O}^{\mathrm{R}}_{\parallel\perp}(zv,t) = \mathcal{C}_{\parallel\perp}(t,\mu)e^{\delta m z}\mathcal{O}^{\overline{\mathrm{MS}}}_{\parallel\perp}(zv) + O(t), \quad (36)$$

$$\mathcal{O}^{\mathrm{R}}_{\parallel\perp}(zv,t) = \mathcal{C}_{\perp\perp}(t,\mu)e^{\delta m z}\mathcal{O}^{\overline{\mathrm{MS}}}_{\perp\perp}(zv) + O(t), \quad (37)$$

in which,

$$\mathcal{C}_{\psi}(t,\mu) = c^2_{\psi h_v}(t,\mu), \quad (38)$$

$$\mathcal{C}_{\parallel\perp}(t,\mu) = c^2_{\parallel\perp}(t,\mu), \quad (39)$$

$$\mathcal{C}_{\perp\perp}(t,\mu) = c^2_{\perp\perp}(t,\mu). \quad (40)$$

which are independent from the distance z.

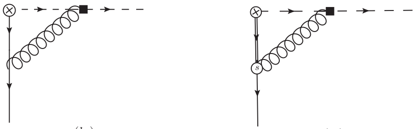
One-loop matching calculations

We need to calculate

- δm: linear divergence coming from Wilson-line self-energy
- $\zeta_{h_v}^{F/A}, \mathring{\zeta}_{\psi}, \zeta_A$: matching coefficient of the heavy field h_v, quark field ψ, gluon field A
- $\zeta_{\psi h_v}, \zeta_{\parallel\perp}, \zeta_{\perp\perp}, \zeta_F$:matching coefficient of the vertices $\bar{\psi}h_v$, $gF^{\mu\nu}_{\parallel\perp}h_v$, $gF^{\mu\nu}_{\perp\perp}$, $\bar{h}_v gF^{\mu\nu}_{\perp\perp}h_v$

The loop integrals in $\overline{\mathrm{MS}}$ scheme ($t=0$) are all scaleless, which is proportional to $\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}}$.
The matching is very much similar to HQET.

Example: one-loop calculation of $\zeta_{\psi h_v}$



$$\mathcal{M}^{\text{flowed}} = -g^2\tilde{\mu}^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{(\gamma\cdot v)\,\not{k}}{(-k\cdot v - i0)\,(k^2)^2}e^{-2tk^2}$$
$$= g^2\tilde{\mu}^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}e^{-2tk^2}$$
$$= \frac{\alpha_s C_F}{4\pi}\left[-\frac{1}{\epsilon_{\text{IR}}} - \log\left(2\mu^2 t e^{\gamma_E}\right) - 1\right].$$

$$\mathcal{M}^{\text{un-flowed}} = -g^2\tilde{\mu}^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{(\gamma\cdot v)\,\not{k}}{(-k\cdot v - i0)\,(k^2)^2}$$
$$= g^2\tilde{\mu}^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}$$
$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}\right].$$

- e^{-2tk^2} suppresses UV modes and converts UV pole $(1/\epsilon_{\text{UV}})$ into $\log(t\ldots)$
- Matching: compare the flowed and un-flowed vertex $\zeta_{\psi h_v}$ gives ($1/\epsilon_{\text{UV}}$ term is removed by renormalization, diagram (b_2) does not contribute),

$$\zeta_{\psi h_v} = 1 - \frac{\alpha_s C_F}{4\pi}\left[\log\left(2\mu^2 t e^{\gamma_E}\right) + 1\right] + O(\alpha_s^2). \quad (41)$$

One-loop results of matching coefficients

- At one-loop, we obtain the matching coefficients for the local current operators

$$
\begin{aligned}
\delta m &= -\frac{\alpha_s}{4\pi} C_R \frac{\sqrt{2\pi}}{\sqrt{t}} + O(\alpha_s^2), && (42)\\
c_{\psi h_v}(t,\mu) &= 1 - \frac{1}{2}\frac{\alpha_s}{4\pi} C_F \left[3\log\left(2\mu^2 t e^{\gamma_E}\right) + \log(432) + 2\right] + O(\alpha_s^2), && (43)\\
c_{\parallel\perp}(t,\mu) &= 1 + O(\alpha_s^2), && (44)\\
c_{\perp\perp}(t,\mu) &= 1 + \frac{\alpha_s}{4\pi} C_A \times \log\left(2\mu^2 t e^{\gamma_E}\right) + O(\alpha_s^2), && (45)\\
c_F(t,\mu) &= 1 + \frac{\alpha_s}{4\pi} C_A \times \log\left(2\mu^2 t e^{\gamma_E}\right) + O(\alpha_s^2). && (46)
\end{aligned}
$$

- Relations between off-lightcone Wilson-line operators and local current operators

$$
\begin{aligned}
\mathcal{C}_\psi(t,\mu) &= c_{\psi h_v}^2(t,\mu), && (47)\\
\mathcal{C}_{\parallel\perp}(t,\mu) &= c_{\parallel\perp}^2(t,\mu), && (48)\\
\mathcal{C}_{\perp\perp}(t,\mu) &= c_{\perp\perp}^2(t,\mu). && (49)
\end{aligned}
$$

Applications of our results

- $c_F(t,\mu)$: spin-dependent relativistic corrections of QCD static potential, heavy quark diffusion coefficient
- $c_{\psi h_v}(t,\mu)$: quark quasi-PDFs/LCDAs...
- $c_{\parallel\perp}(t,\mu)$: gluon quasi-PDFs/LCDAs, heavy quarkonium diffusion coefficient, P-wave quarkonium decay long distance matrix element in the framework of pNRQCD
- $c_{\perp\perp}(t,\mu)$: gluon quasi-PDFs/LCDAs, heavy quarkonium diffusion coefficient...

Summary and outlook

- We have developed a systematic approach for matching from the gradient-flow scheme to the $\overline{\mathrm{MS}}$ scheme for off-lightcone Wilson-line operators in the small flow-time limit.
- Matching of off-lightcone Wilson-line operators is reduced into the matching of local current operators, which greatly simplifies the matching calculations.
- Applications: lattice calculations of quasi-PDFs/LCDAs, spin-dependent potentials, heavy quark (quarkonium) diffusion coefficient, P-wave quarkonium decay in pNRQCD and so on.
- We are looking forward to extend the matching calculations at two-loop level, which will be more helpful for precise lattice computations.