QCD meets Inverse Problem

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第七届重味物理与量子色动力学研讨会@南京

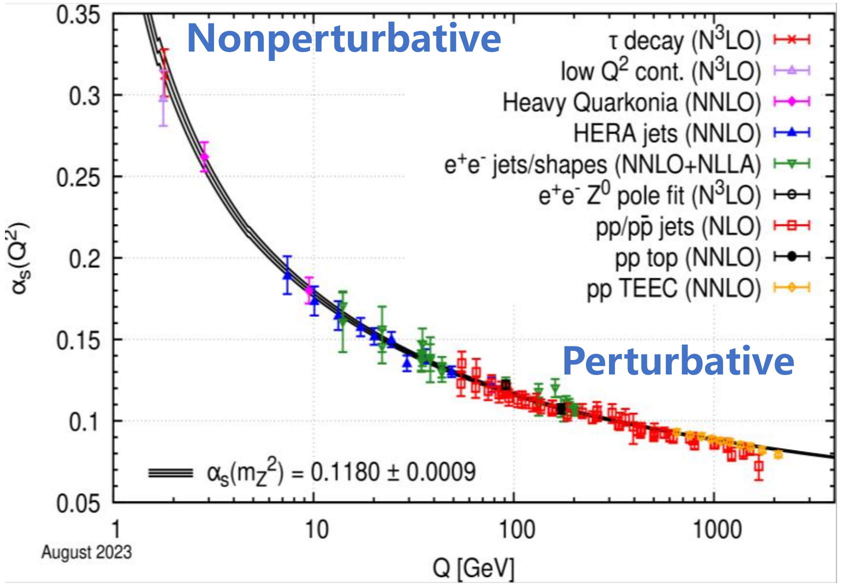
Outline

- Introduction
- Inverse problem approach for pion decay constant: ***dispersion relation***
- Inverse problem approach for pion LCDA: ***moments problem***
- Conclusion

Introduction

- QCD: nonperturbative at low energy scale

- Nonperturbative problem limits us:
  - Understand Quark confinement
  - Indirectly search for new physics



- Nonperturbative method:
  - First principle Lattice QCD: requires huge computing resources
  - QCDSR, Dyson-Schwinger equation …: Model dependence

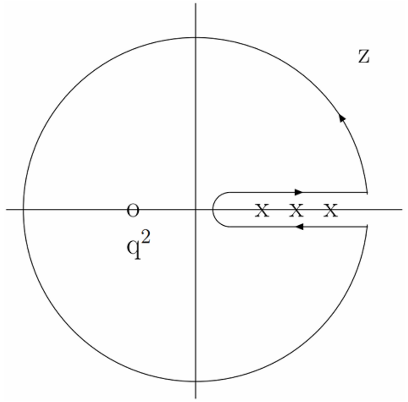
- New nonperturbative method: Inverse problem approach
  - Combined with the methods to explore the nonperturbative QCD

Introduction: Inverse problem approach

- Proposed by profs. Li and Yu. etl.
- Based on the analyticity in QFT: dispersion relation an inverse problem

Calculable; To be solved

$$I_n(q^2) - I_n(\sigma_0) = \frac{(q^2-\sigma_0)}{\pi}\int_0^\infty \frac{\mathrm{Im} I_n(s)\; ds}{(s-\sigma_0)(s-q^2)},$$



$$Kf = g\,, \qquad K^{-1} \to \infty$$

- Ill-posedness: solution has existence, uniqueness, ***but instability***

A.S.Xiong, T.Wei, F.S.Yu, arXiv:2211.13753

- Tikhonov Regularization:

$$f_\alpha = \left(K^\dagger K + \alpha I\right)^{-1} K^\dagger g\,,$$

- The regularization constant α: **determined by *L*-curve method**
- **No artificial assumptions and parameters.**
- **Regularization solution converge to true solution with the input error approaches zero.**

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Inverse problem approach for pion decay constant

- Correlation function:

$$\Pi_{(n)}(z,q) = i\int d^4x e^{iq\cdot x} \langle 0|\, T\left\{O_n(x)O_0^\dagger(0)\right\}|0\rangle \;=(z\cdot q)^{n+2} I_n(q^2)\,,$$

$$\langle 0|\, O_n(x) \equiv \bar{d}(x)\not{z}\gamma_5(iz\cdot \overleftrightarrow{D})^n u(x)\,|\pi(q)\rangle \equiv i(z\cdot q)^{n+1} f_\pi \langle \xi^n\rangle\;,$$

- Dispersion relation:

Calculable

To be solved

$$\boxed{I_n(q^2) - I_n(\sigma_0)} = \frac{(q^2-\sigma_0)}{\pi}\int_0^\infty \frac{\boxed{\mathrm{Im} I_n(s)}\; ds}{(s-\sigma_0)(s-q^2)}\,,$$

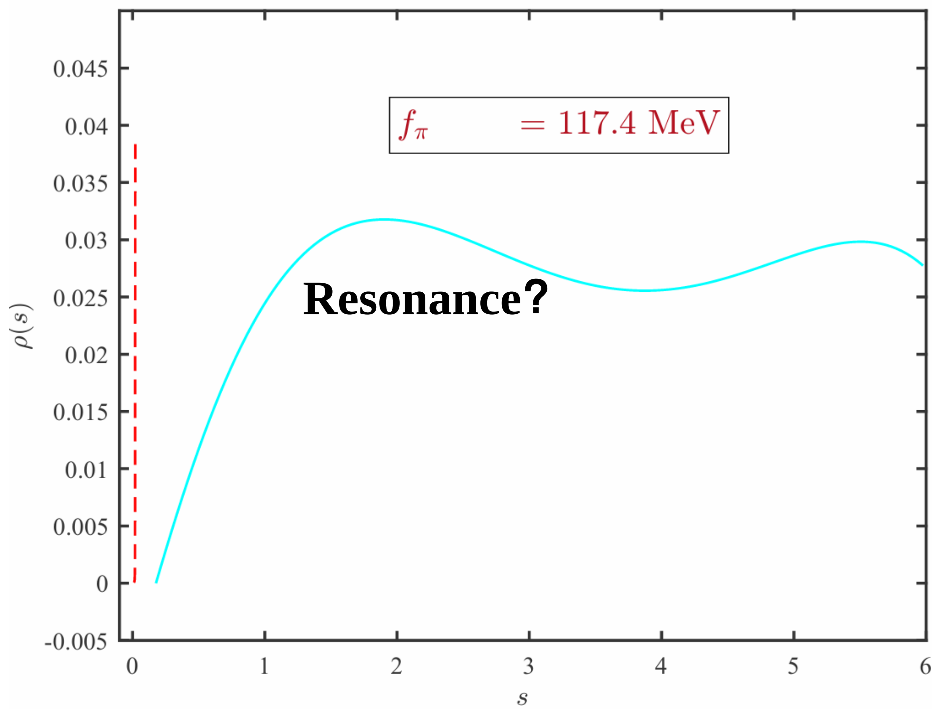
$$\frac{1}{\pi}\mathbf{Im} I_0(s) \equiv f_\pi^2\delta(s-m_\pi^2) + \rho^0(s)\theta(\Lambda - s) + \frac{1}{\pi}\mathbf{Im} I_0^{(pert)}(s)\theta(s-\Lambda)\,,$$

- Decay constant and continuum spectrum can be calculated simultaneously.

Inverse problem approach for pion decay constant

$$\frac{1}{\pi}\mathrm{Im}I_0(s) \equiv f_\pi^2\delta(s-m_\pi^2) + \rho^0(s)\theta(\Lambda - s) + \frac{1}{\pi}\mathrm{Im}I_0^{(pert)}(s)\theta(s-\Lambda)\,,$$

- The pion state and spectrum **are solved**

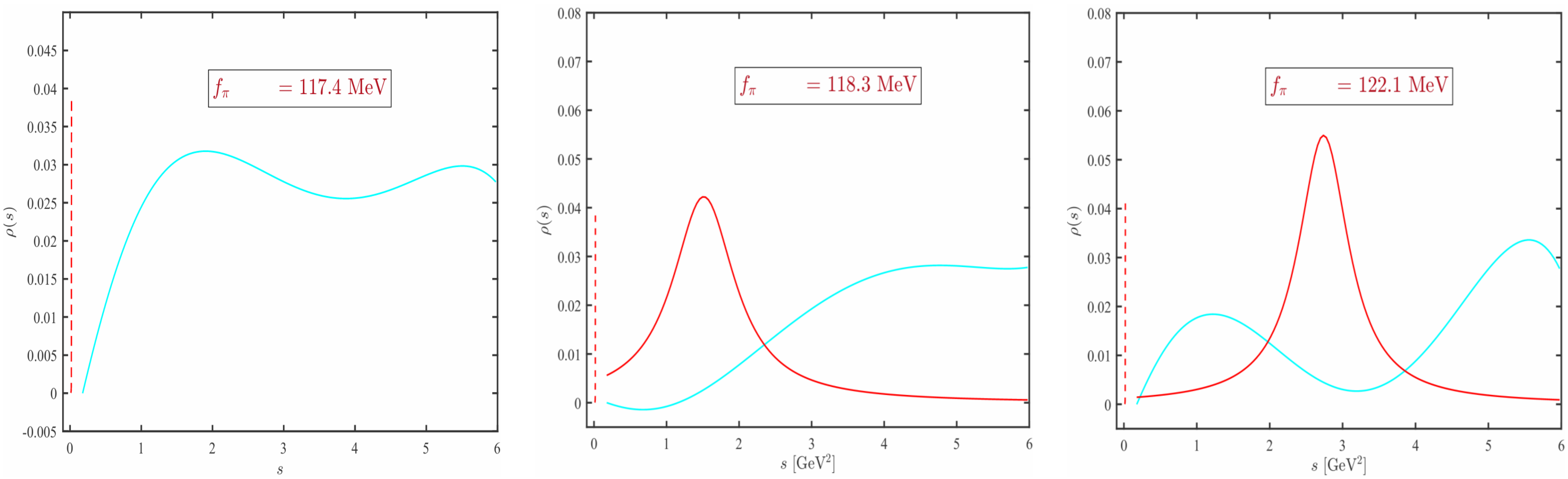


- The non-trivial structure of solutions will tell that there must be some resonances at 1-2 GeV.
- Form PDG, there are $a_1(1260)$ and $a_1(1640)$.

Inverse problem approach for pion decay constant

- **Resonances emergences**

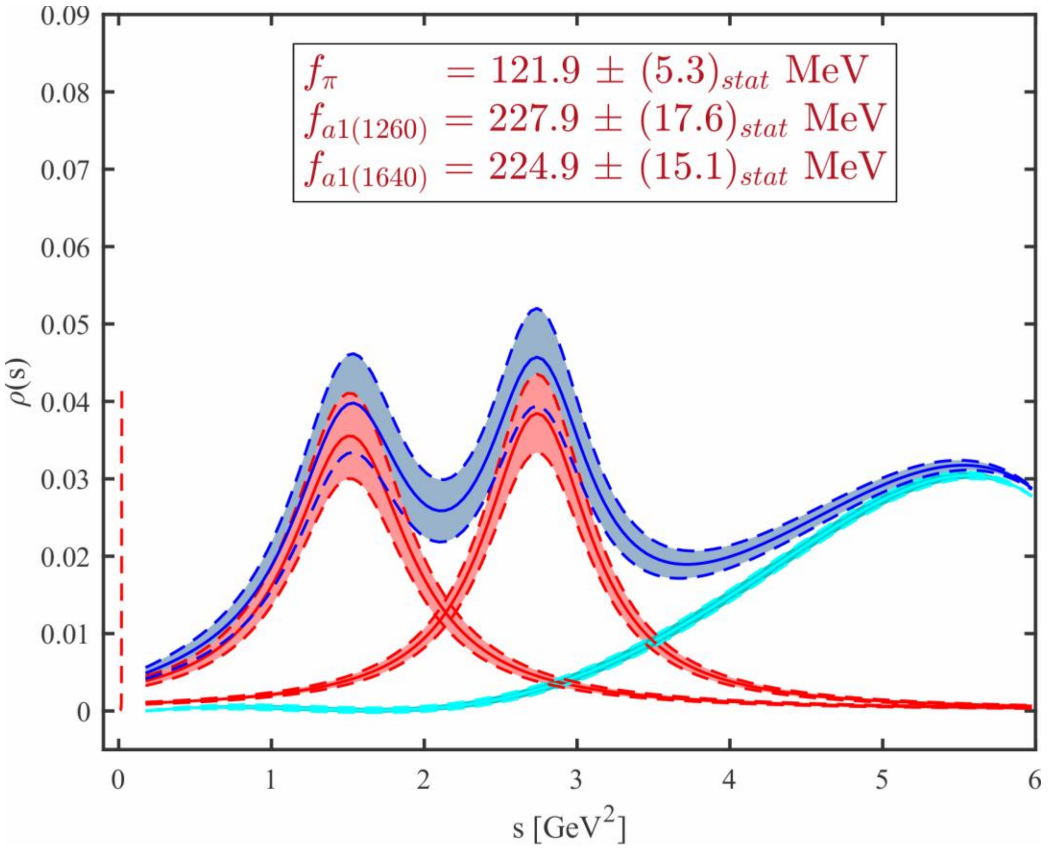
$a_1(1260)$ *as a priori* $\quad$ $a_1(1640)$ *as a priori*



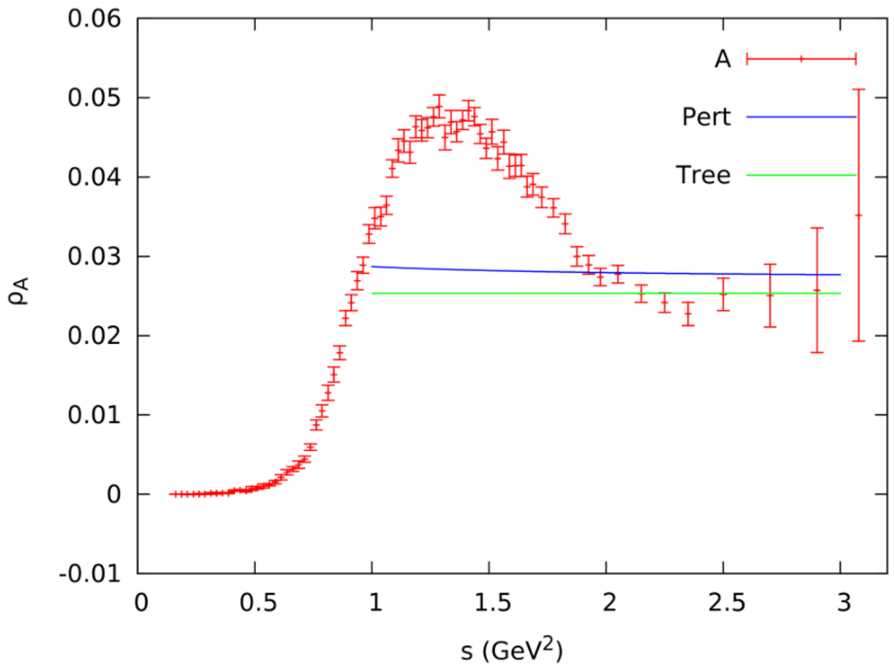
- The non-trivial structure of solutions will tell that there must be some resonances.
- **Our method is powerful.**

Inverse problem approach for pion decay constant

- Decay constants of excited states can be extracted well.



$(f_\pi)_{exp} = 130.4 \pm 0.2$ MeV



Eur. Phys. J., C74(3):2803, 2014.
arXiv:hep-ex/0506072
Precision physics in Hadronic Tau Decays

- The result for decay constant and spectrum are consistent with experiment!

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Inverse problem approach for LCDA

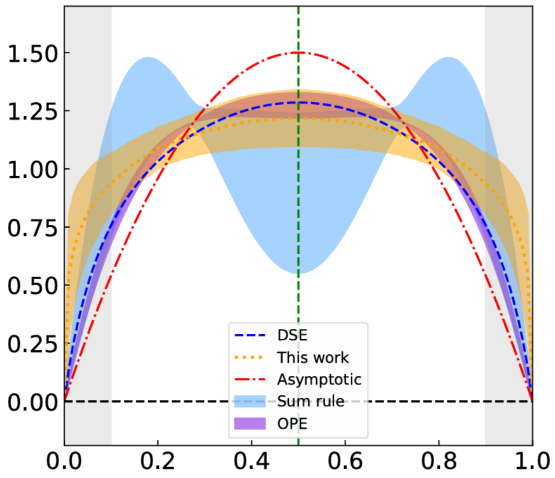
- LCDA describes the inner structure of hadron, which is important nonperturbative input for factorization mothed
- For pion LCDA:
    - The results given by different methods very greatly vary.
    - The endpoint region is difficult for Lattice.
- LCDA is trivial in inverse problem approach.

Calculable — To be solved

$$\underbrace{I_n(q^2) - I_n(\sigma_0)}_{\text{Calculable}} = \frac{(q^2-\sigma_0)}{\pi}\int_0^\infty \frac{\overbrace{\mathrm{Im} I_n(s)}^{\text{To be solved}}\ ds}{(s-\sigma_0)(s-q^2)},$$

$$\frac{1}{\pi}\mathrm{Im} I_n(s) \equiv f_\pi^2 \langle\xi^n\rangle\, \delta(s-m_\pi^2) + \rho^n(s)\theta(\Lambda - s) + \frac{1}{\pi}\mathrm{Im} I_n^{(pert)}(s)\theta(s-\Lambda)\,,$$

$$\langle\xi^n\rangle = \int_0^1 dx(2x-1)^n\phi(x)$$



(Lattice Parton Collaboration) PRL **129**, 132001

$\langle\xi^2\rangle_\pi$	$\langle\xi^4\rangle_\pi$	$\langle\xi^6\rangle_\pi$	$\langle\xi^8\rangle_\pi$	$\langle\xi^{10}\rangle_\pi$	$\langle\xi^{12}\rangle_\pi$
0.289	0.055	0.093	0.060	0.035	0.057
	$\langle\xi^{14}\rangle_\pi$	$\langle\xi^{16}\rangle_\pi$	$\langle\xi^{18}\rangle_\pi$	$\langle\xi^{20}\rangle_\pi$	
	0.042	0.028	0.017	0.0066	

- As long as known moments ξ^n is large, we can capture enough information about LCDA.

Inverse problem approach for LCDA: Moment problem

- Moment problem: $\xi^n \rightarrow \phi_\pi(x)$

$$\langle \xi^n \rangle = \int_0^1 dx (2x-1)^n \phi(x)$$

- **Moment problem encountered in physics:**
  - QCDSR for LCDA, Dyson-Schwinger for LCDA (PDF) ……
    arXiv:2102.03989v2 arXiv:2403.01937 [hep-ph], 2308.14871 [hep-ph]

- **Moment problem in mathematic: *an inverse problem***
  - **Ill-posedness:** solution has existence, uniqueness, ***but instability***
  - Standard scheme: *Tikhonov regularization*
  - Regulation constant α: selected by mathematical method

$$\phi = (K^\dagger K + \alpha I)^{-1} K^\dagger \langle \xi^n \rangle$$

Inverse problem approach for LCDA: Moment problem

- Real pion LCDA can be reconstructed.
  - Whole complete form is obtained.
  - Broader than the asymptotic and relatively flat in the central region.
  - The central region is consistent with Lattice.
  - Smooth in the endpoint: Can be complementary to Lattice.

$$\langle \xi^n \rangle = \int_0^1 dx (2x-1)^n \phi(x)$$

$\langle \xi^2 \rangle_\pi$	$\langle \xi^4 \rangle_\pi$	$\langle \xi^6 \rangle_\pi$	$\langle \xi^8 \rangle_\pi$	$\langle \xi^{10} \rangle_\pi$	$\langle \xi^{12} \rangle_\pi$
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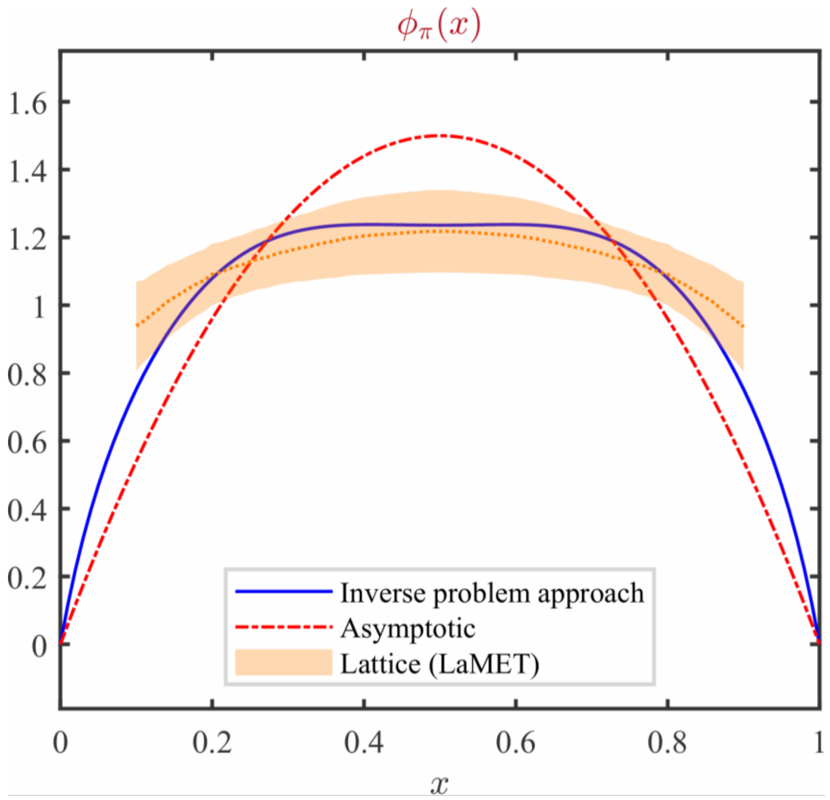


Difficult for the endpoint region in Lattice

(Lattice Parton Collaboration) PRL **129**, 132001

- Pion LCDA can be expressed by gegenbauer polynomial.

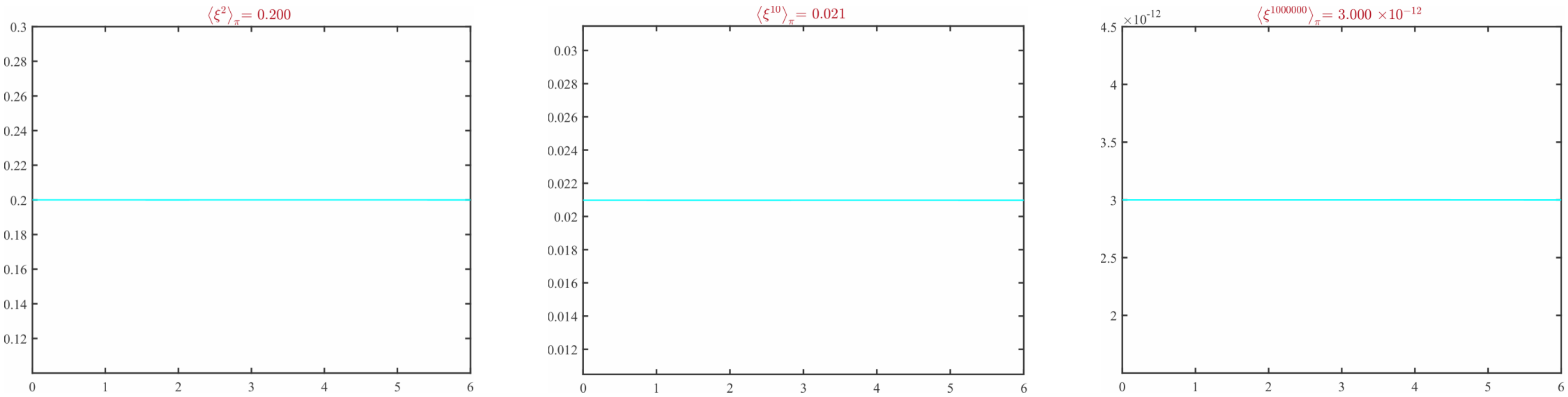
$$\varphi_P(x,\mu) = 6x(1-x)\left[1+\sum_{n=1}^{\infty} a_n^P(\mu) C_n^{(3/2)}(2x-1)\right], \qquad a_n = \frac{4(2n+3)}{6(n+1)(n+2)}\int_0^1 dx\phi_\pi(x) C_n^{3/2}(2x-1).$$

Methods	$a_2^\pi(2\text{GeV})$	$a_4^\pi(2\text{GeV})$	
Our result	0.123	0.010	
Lattice QCD (OPE) [76]	$0.116^{+0.019}_{-0.020}$	$--$	JHEP (2019) [RQCD]
Lattice QCD (OPE) [77]	0.136 ± 0.021	$--$	PRD (2015) [V. M. Braun…]
Lattice QCD (OPE) [78]	0.233 ± 0.065	$--$	PRD (2011) [R. Arthur …]
Lattice QCD (LaMET) [75]	0.258 ± 0.087	0.122 ± 0.055	PRL **129**, 132001
Inverse Matrix Method [70]	$0.1775^{+0.0036}_{-0.0040}$	$0.0957^{+0.0011}_{-0.0012}$	arXiv:2205.06746
QCD sum rules [65]	0.157 ± 0.029	0.032 ± 0.007	arXiv:2102.03989v2
QCD sum rules [79]	$0.057^{+0.024}_{-0.019}$	$-0.013^{+0.022}_{-0.019}$	arXiv:1405.0959 [hep-ph]
QCD sum rules [80]	$0.149^{+0.052}_{-0.043}$	$-0.096^{+0.063}_{-0.058}$	arXiv:hep-ph/0103119

- Our result is consistent with Lattice QCD calculated by OPE.

Inverse problem approach for asymptotic LCDA: Moments

- Asymptotic LCDA's moments: be reproduced perfectly, $n \to \infty$



- It is due to the input is exact completely.

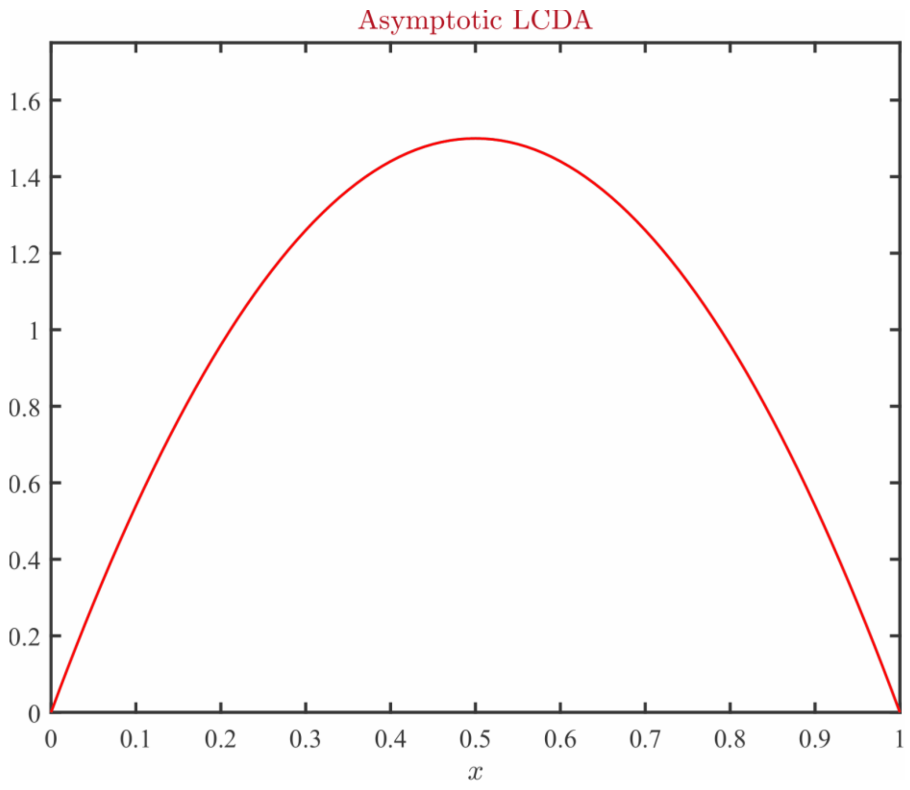
$$\underbrace{I_n(q^2) - I_n(\sigma_0)}_{\text{Calculable}} = \frac{(q^2 - \sigma_0)}{\pi} \int_0^\infty \frac{\overbrace{\mathrm{Im} I_n(s)}^{\text{To be solved}} \; ds}{(s - \sigma_0)(s - q^2)}, \qquad I_n(q^2) = -\frac{3 \ln(-q^2/\mu^2)}{4\pi^2 (n+1)(n+3)},$$

- **Solution converge to true solution as long as the input error approaches zero.**
- Inverse problem approach can be improved systematically by considering higher precision in OPE calculation.

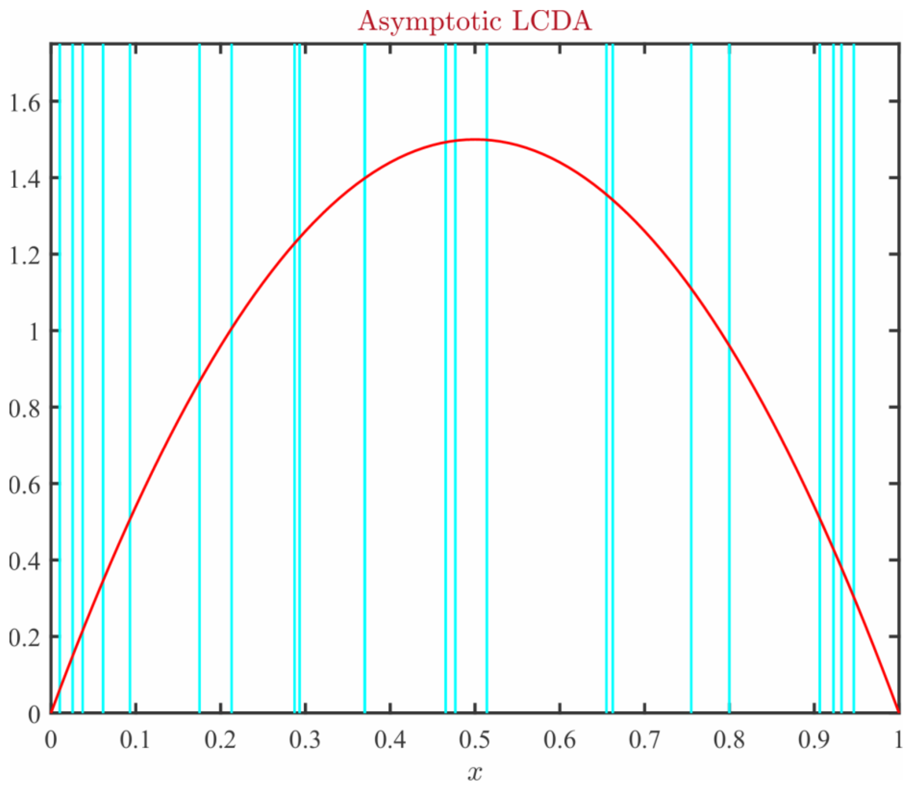
Inverse problem approach for asymptotic LCDA: Moment problem

$$\langle \xi^n \rangle = \int_0^1 dx (2x-1)^n \phi(x)$$

- Asymptotic LCDA can be reproduced perfectly, indicates our approach is self-consistency



Inverse problem approach



Direct brute-force solving

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Conclusion

- First, we solved the complete spectral function of the pion system using a rigorous inverse problem approach.
  - The spectral function consistent with the experiments.
  - The errors therein can be systematically estimated.

- We consider the ‘moments problem’ as an inverse problem, then the pion LCDA can be obtained well.

- Asymptotic LCDA and its moments are reproduced very well.
  - Indicates that the inverse problem approach can be systematically improved by enhancing the accuracy of the input.

- Inverse problem provides a new perspective to explore nonperturbative QCD.

Thank you for your attention!

Back up

OPE calculation

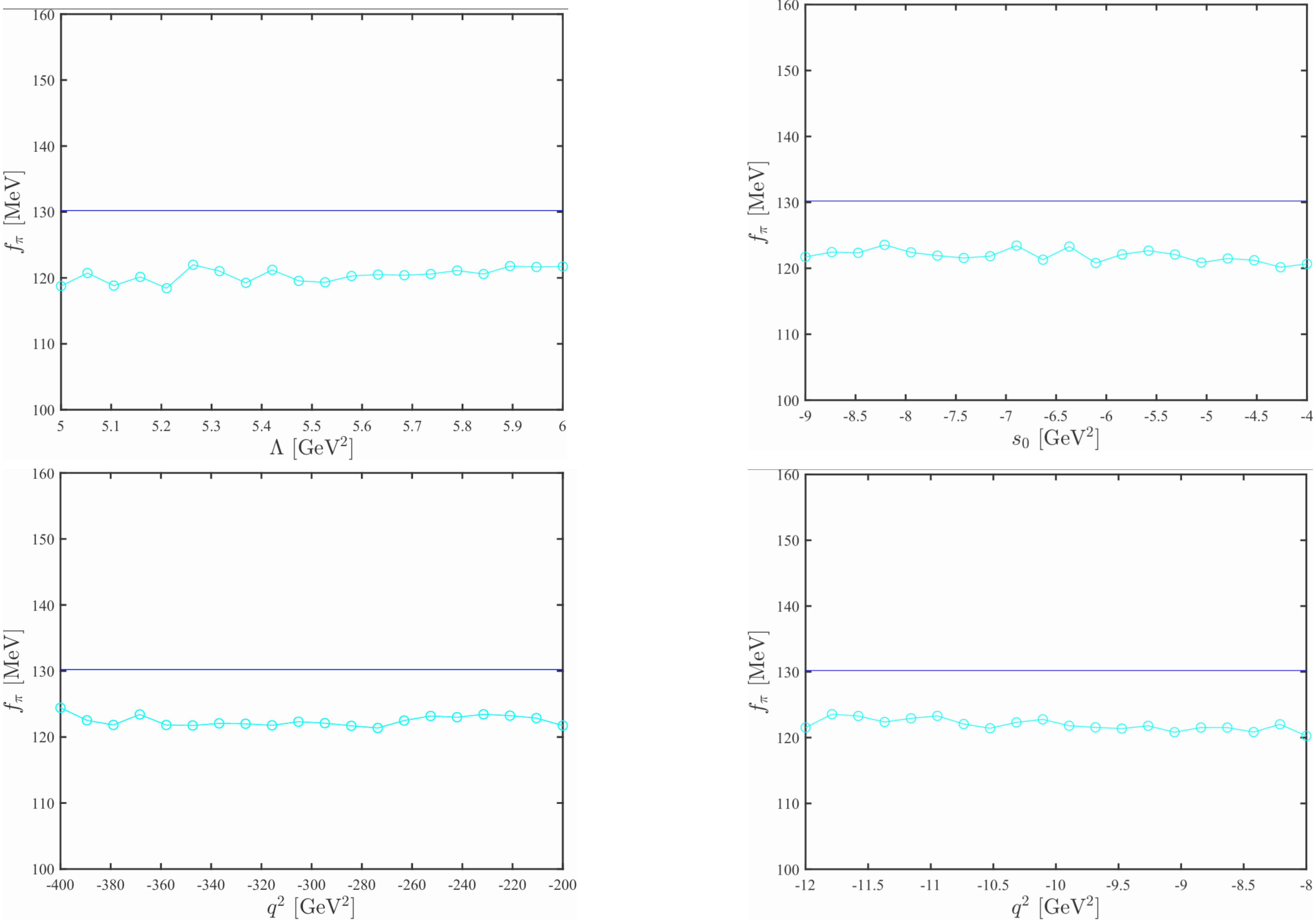
$$I_{n0}^{cond}(q^2) = \frac{m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle}{(q^2)^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + \frac{8n+1}{9}\frac{m_u\langle g_s\bar{u}\sigma TGu\rangle + m_d\langle g_s\bar{d}\sigma TGd\rangle}{(q^2)^3} + \frac{n\theta(n-2)}{24\pi^2}\frac{\langle g_s^3 fG^3\rangle}{(q^2)^3} - \frac{4(2n+1)}{81}\frac{\langle g_s\bar{u}u\rangle^2 + \langle g_s\bar{d}d\rangle^2}{(q^2)^3} - \frac{[C_n+P_n]}{243\pi^2}\frac{\sum_{\psi=u,d,s}\langle g_s^2\bar{\psi}\psi\rangle^2}{(q^2)^3}, \quad (13.13)$$

where $C_, P_n$ are respectively,

$$C_n = 3(17n+35) + \theta(n-2)\left\{\frac{49n^2+100n+56}{n} - 25(2n+1)\left[\psi(\frac{n+1}{2}) - \psi(\frac{n}{2}) + ln4\right]\right\},$$

$$P_n = [2(51n+25) - 2n\theta(n-2)]\, ln\left(\frac{-q^2}{\mu^2}\right),$$

2205.06746 [Hsiang-nan Li]   2102.03989v2 [Tao Zhong ...]



f_π [MeV]
Λ [GeV^2]
s_0 [GeV^2]
q^2 [GeV^2]
100
110
120
130
140
150
160
5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
6
-9
-8.5
-8
-7.5
-7
-6.5
-6
-5.5
-5
-4.5
-4
-400
-380
-360
-340
-320
-300
-280
-260
-240
-220
-200
-12
-11.5
-11
-10.5
-10
-9.5
-9
-8.5
-8

2. ill-posedness of the inverse problem

- Most of inverse problems are ill-posed

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5 \end{cases} \quad \Rightarrow \quad x_1 = 1, \;\; x_2 = 1$$

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5.001 \end{cases} \quad \Rightarrow \quad x_1 = -5, \;\; x_2 = 5$$

- A very small noise might cause a large change of solutions

2. ill-posedness of the inverse problem

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5 \end{cases} \quad \Rightarrow \quad x_1 = 1, \quad x_2 = 1$$

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5.001 \end{cases} \quad \Rightarrow \quad x_1 = -5, \quad x_2 = 5$$

- A very small noise might cause a large change of solutions

$$Ax = y, \qquad x = A^{-1}y$$

$$A = \begin{pmatrix} 2 & 3 \\ 1.9999 & 3.0001 \end{pmatrix}, \qquad |A| = 0.0005, \qquad A^{-1} = \frac{A^*}{|A|} = \begin{pmatrix} 6000.2 & -6000 \\ -3999.8 & 4000 \end{pmatrix}$$

A^{-1} enhances the errors

Leading Twist LCDA of Meson $\phi(x)$:
can be expanded by Gegenbauer polynomial

$$Q\frac{\partial}{\partial Q}\phi(x,Q) = \frac{\alpha_s(Q^2)}{4\pi}\left\{\int_0^1 dy\,\frac{V(x,y)}{y(1-y)}\phi(y,Q) - 2\phi(x,Q)\right\} \tag{90}$$

where the evolution potential is

$$V(x,y) = 4C_F\left\{x(1-y)\theta(y-x)\left(\delta_{-h,\bar{h}} + \frac{\Delta}{y-x}\right) + \begin{pmatrix} x \leftrightarrow 1-x \\ y \leftrightarrow 1-y \end{pmatrix}\right\} = V(y,x). \tag{91}$$

Operator Δ in the potential is defined by

$$\Delta\frac{\phi(y,Q)}{y(1-y)} \equiv \frac{\phi(y,Q)}{y(1-y)} - \frac{\phi(x,Q)}{x(1-x)}. \tag{92}$$

$$Q\frac{\partial}{\partial Q}\phi(x,Q) = \frac{\alpha_s(Q^2)}{4\pi}\left\{\int_0^1 dy\, \frac{V(x,y)}{y(1-y)}\,\phi(y,Q) - 2\phi(x,Q)\right\} \qquad (90)$$

#8 The evolution potential $V(x,y)$ can be treated as an integral operator. Being symmetric it has real eigenvalues $\tilde{\gamma}_n$ and eigensolutions $\phi_n(y)$ that satisfy $\int dy\, V(x,y)\, w(y)\, \phi_n(y) = \tilde{\gamma}_n\, \phi_n(x)$ where integration weight $w(y) \equiv 1/(y(1-y))$. The eigensolutions must be orthogonal with respect to weight $w(x)$, from which it immediately follows that $\phi_n(x) \propto x(1-x)\, C_n^{3/2}(2x-1)$ where $C_n^{3/2}$ is a Gegenbauer polynomial. It is a straightforward exercise to now extract analytic expressions for the eigenvalues. Given the eigenvalues a general solution of the evolution equation can be written down as an expansion on the complete set of eigensolutions, as we do here.

$$\phi(x, Q) = x(1-x)\sum_{n=0}^{\infty} a_n C_n^{3/2}(2x-1)\left(\log\frac{Q^2}{\Lambda^2_{QCD}}\right)^{-\gamma_n/2\beta_0}. \qquad (93)$$

where[#9]

$$\gamma_n = 2C_F\left\{1 + 4\sum_{k=2}^{n+1}\frac{1}{k} - \frac{2\delta_{-h,\bar{h}}}{(n+1)(n+2)}\right\} \geq 0. \qquad (94)$$

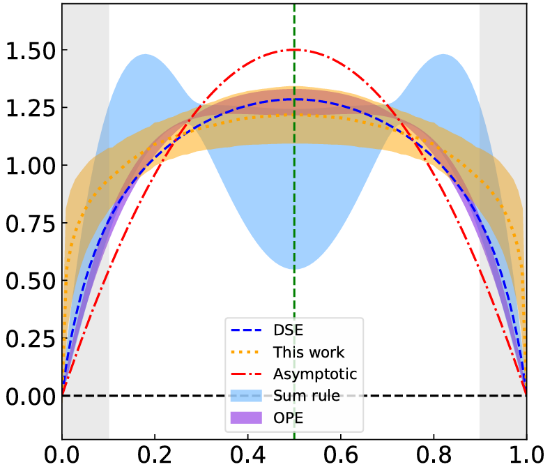
- ***The expansion is generic and is valid for all leading twist quark antiquark meson distribution amplitudes, albeit with different coefficients and anomalous dimensions.***

The Uses of conformal symmetry in QCD. [Prog. Part. Nucl. Phys., 51:311–398, 2003.]

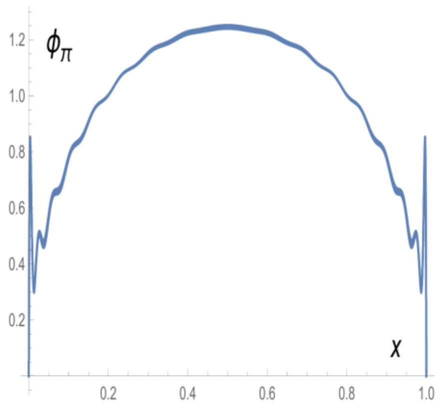
- ***The Gegenbauer polynomials also naturally in this context, as a consequence of the residual conformal symmetry of QCD at short distances.***

$$\varphi_P(x,\mu) = 6x(1-x)\left[1+\sum_{n=1}^{\infty} a_n^P(\mu)\, C_n^{(3/2)}(2x-1)\right],$$

$$a_n = \frac{4(2n+3)}{6(n+1)(n+2)}\int_0^1 dx\varphi_\pi(x)C_n^{3/2}(2x-1)\,.$$



(Lattice Parton Collaboration) PRL **129**, 132001



Hsiang-nan Li [arXiv: 2205.06746]