Institute of High Energy Physics Chinese Academy of Sciences

University of Chinese Academy of Sciences

Dispersive analysis of $\eta(1405/1475)$ on the recent BESIII decay $J/\psi \to \gamma K_0^S K_0^S \pi^0$

Lin Qiu (邱林)

In collaboration with: Dr. Yin Cheng (程茵), Dr. MengChuan Du (杜蒙川) and Prof. Qiang Zhao (赵强)

Mini-workshop on Light QCD Exotic States

IHEP, Beijing, China

October 18, 2024

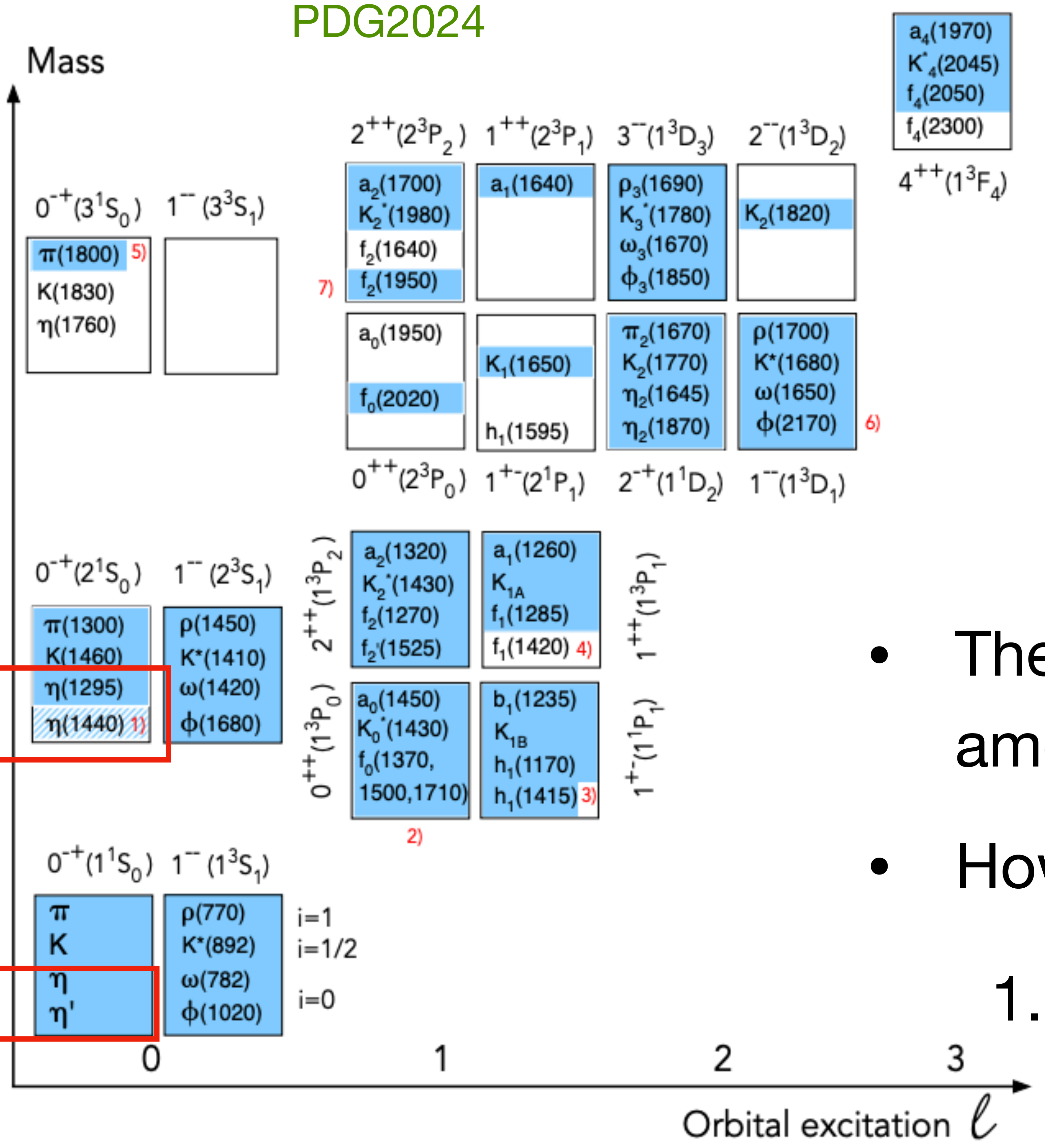
Outline

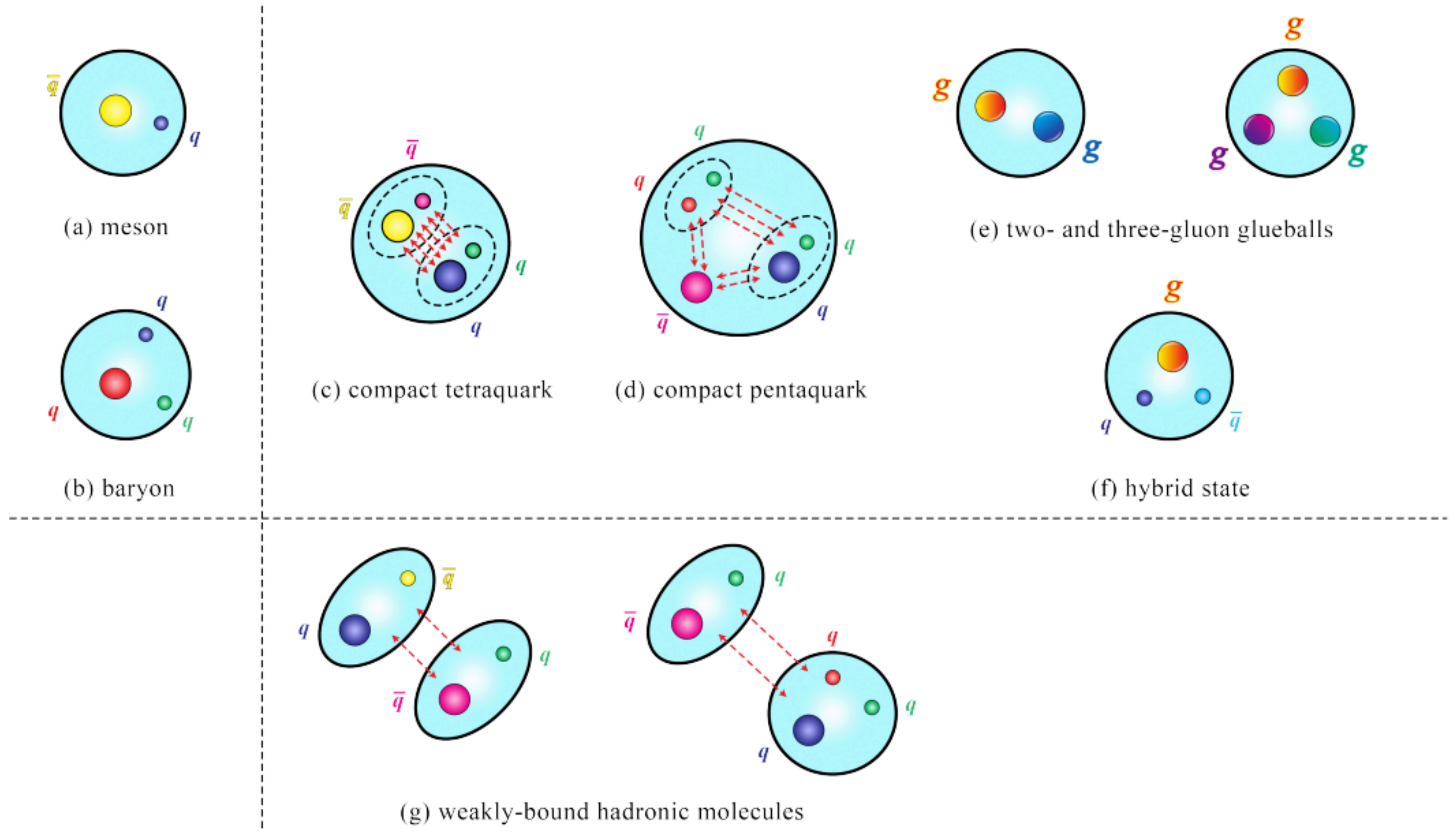
- *Introduction & Motivation*
  - *Light-meson spectrum, status of* $\eta(1405/1475)$
  - *Triangle singularity in dispersive perspective*
- *Dispersive Framework of* $J/\psi \to \gamma K_0^S K_0^S \pi^0$
  - *Muskhelishvili-Omnès Framework: two-body unitary interactions*
  - *Khrui-Trieman Framework: three-body unitary interactions*
- *Discussions*
  - *Monte-Carlo outputs, BESIII spectra*

Introduction & Motivation

Light-meson spectrum

H.X.Chen,et al.,Rept.Prog.Phys.86(2023)2,026201





- The first radially-excited states of $\eta-\eta'$ are assigned to $\eta(1295)-\eta(1405/1475)$, among which one of the states is regarded as 0^{-+} glueball candidate.
- However, there are many puzzles:

1. Controversial observations: one state observed in $K\bar{K}\pi, \gamma V, \eta\pi\pi$ but two states observed (only) in other $K\bar{K}\pi$ final states;

   X(2370)?BESIII, PRL132.181901

2. LQCD simulation: the mass of 0^{-+} glueballs are calculated to be above 2GeV;
3. Supernumerary problem: one or two states? their natures?

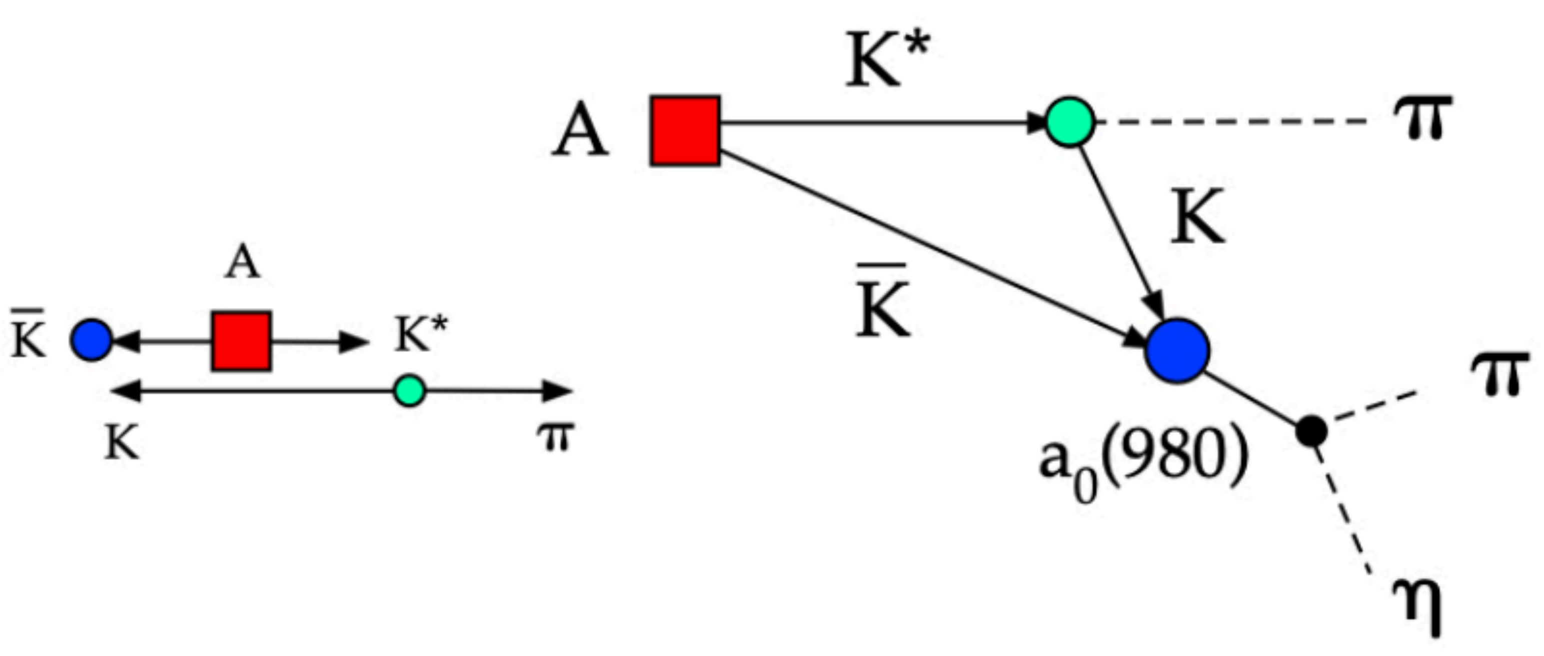
High-statistics BESIII data $J/\psi \to \gamma K_S^0 K_S^0 \pi^0$

BESIII, JHEP03(2023)121

Recently, BESIII collaboration reports the high-statistics $J/\psi \to \gamma K_S^0 K_S^0 \pi^0$ data and the data seem to favor the two-state scenario.

S.X.Nakamura et al., PRD.109.014021;PRD.107.L091505

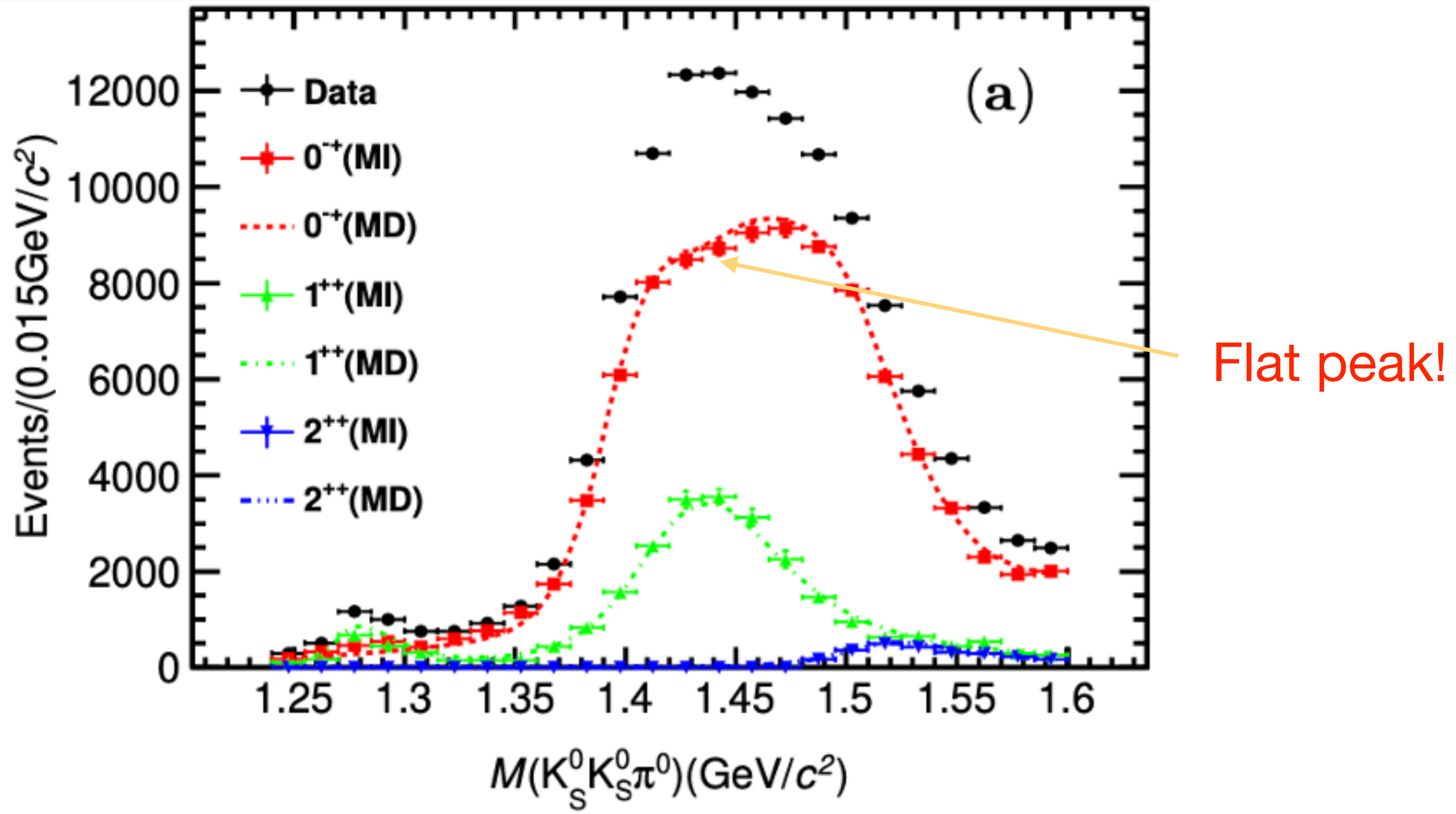
Wu et al.,PRL.108.081803 PDG2024



Triangle singularity window:

$$K\bar{K}\pi \in [1.388,1.437]GeV$$

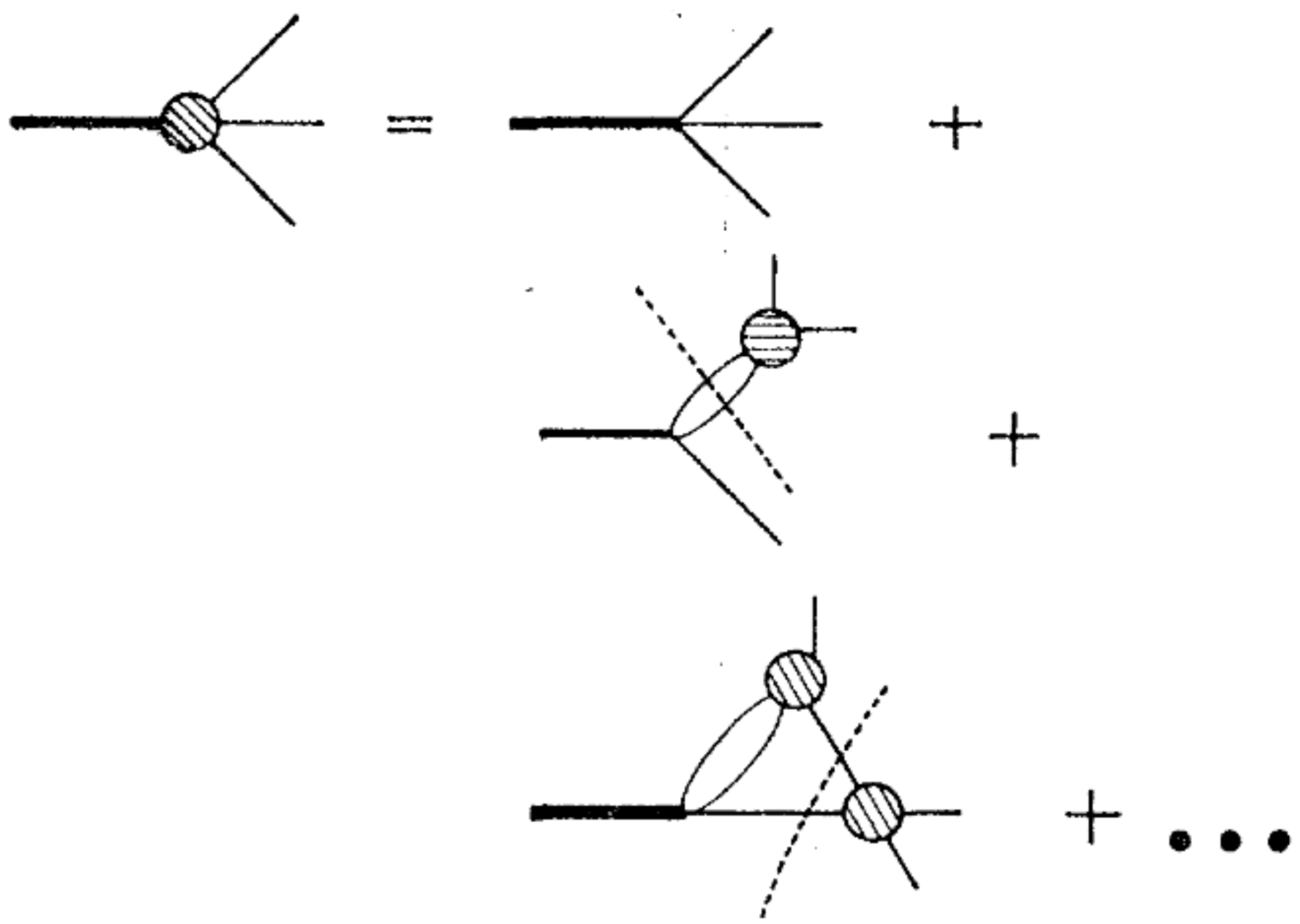
$$K\bar{K} \in [0.991,1.029]GeV$$



A better understanding of 0^{-+} spectrum in $1.2 \sim 1.5$GeV is strongly desired!

Motivation: Dispersive approach

C.Kacser,PhysRev.132.2712

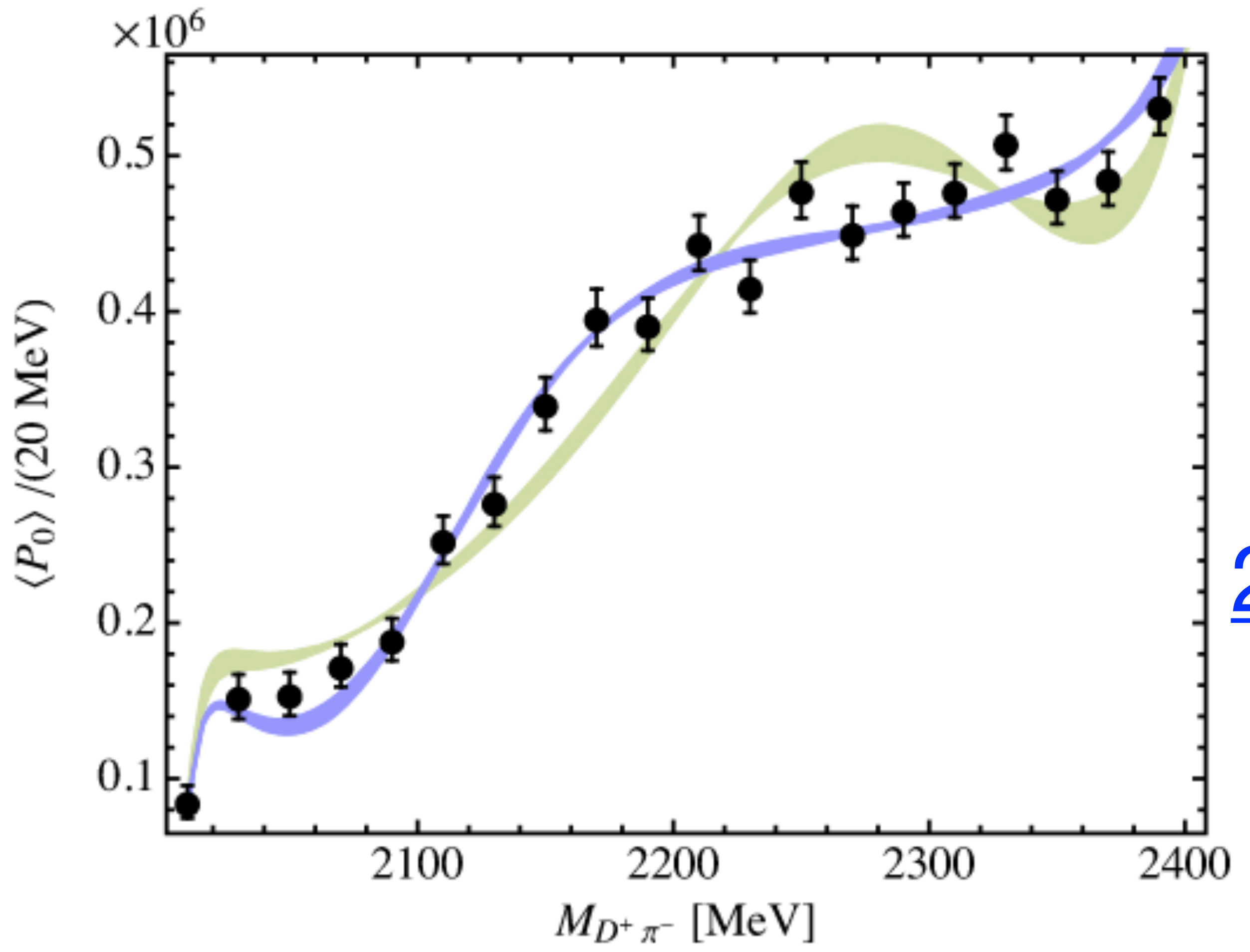


In the light sector, the dispersion theory benefits to respect

- ✓ Unitarity
  - 2-body: sub-channel interactions determined from scattering data
  - 3-body: normal discontinuity (RHC) + partial-wave projection (LHC)
- ✓ (Maximal) Analyticity: $\mathbf{q} > 1GeV$
- ✓ Crossing symmetry

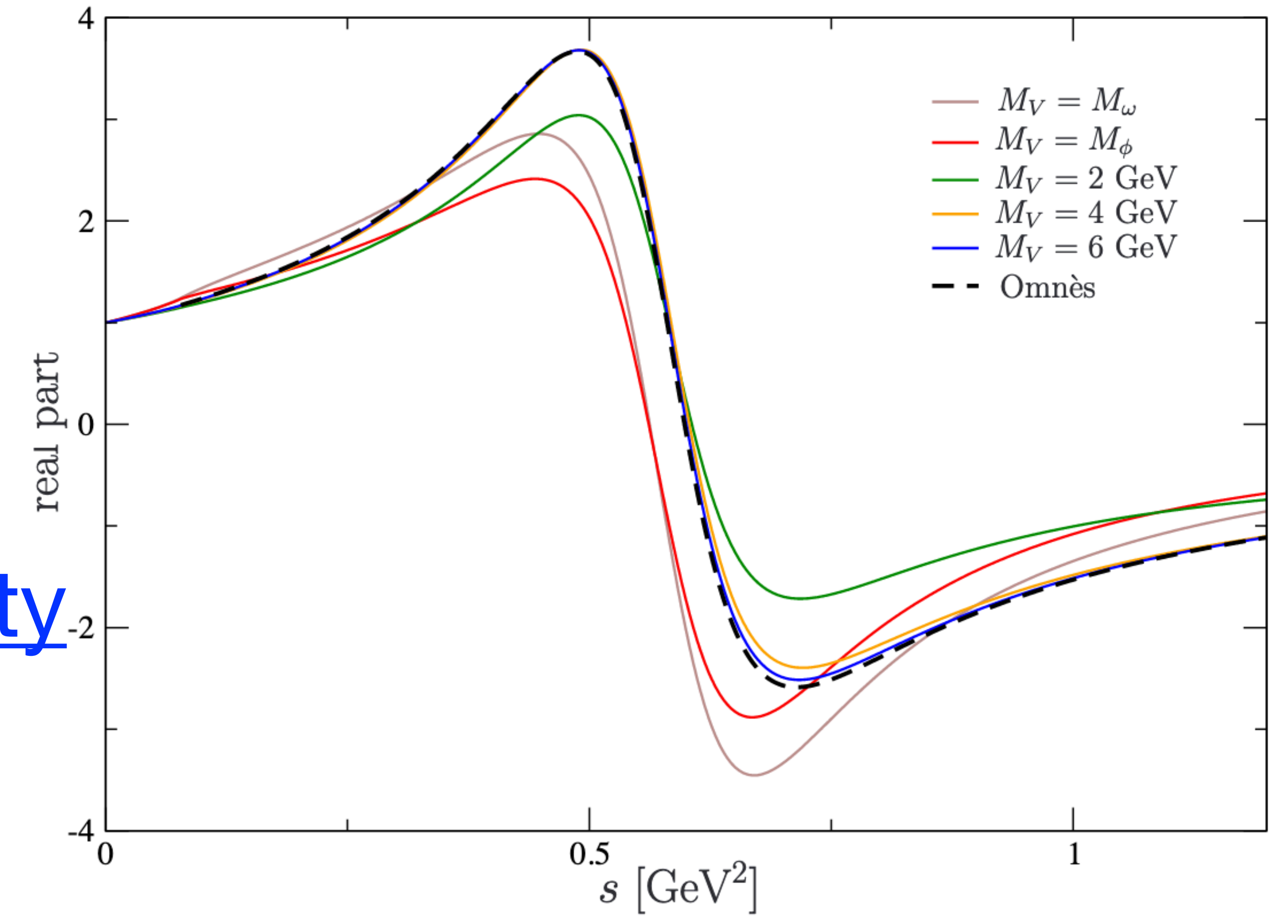
The inefficiency of isobar-model is manifesting in many aspects such as:

$B^- \to D^+\pi^-\pi^-$ Du et al., PRL126192001(2021)

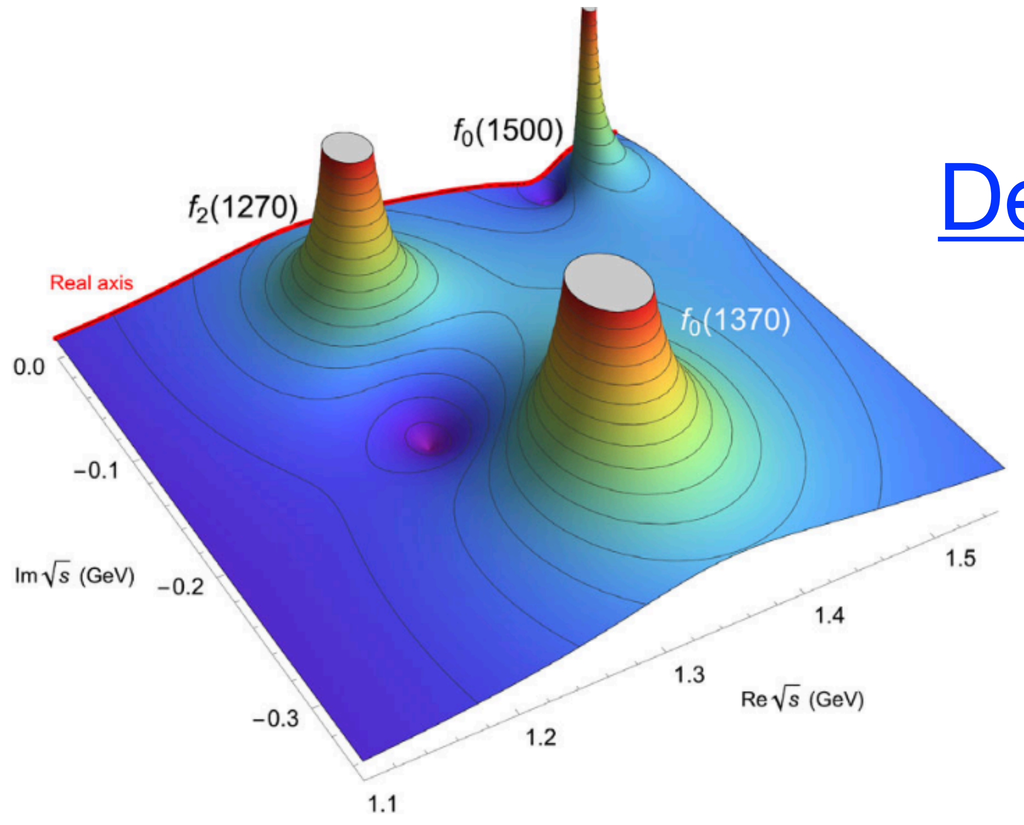


2&3-b unitarity

$V \to 3\pi$ F.Niecknig et al., EPJC722014(2012)



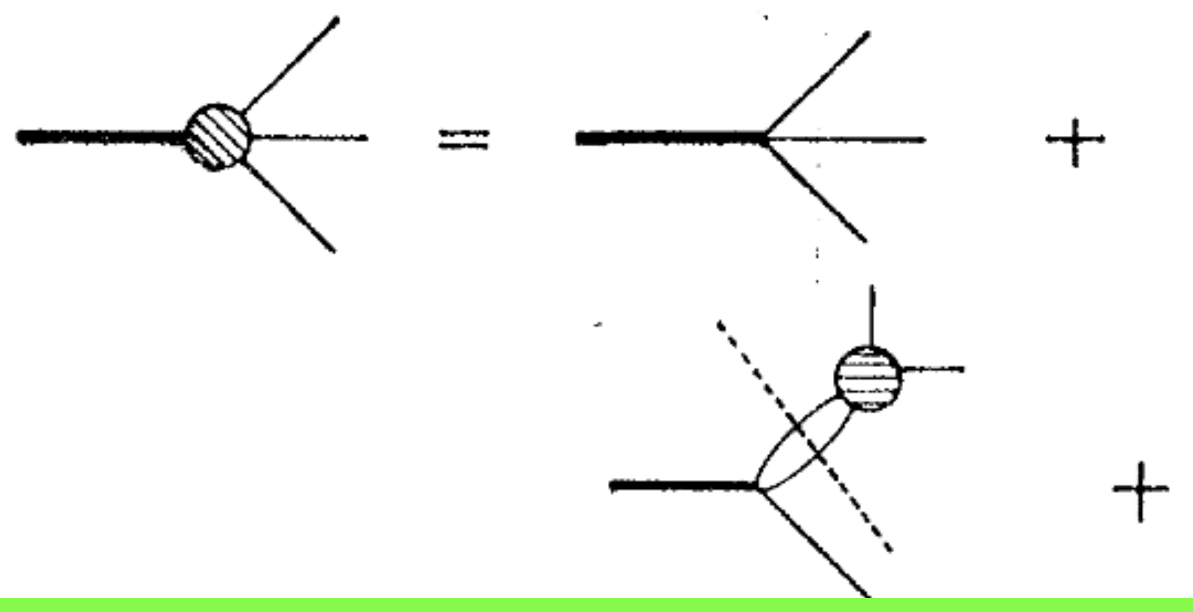
$f_0(1370)$ J.R.Peláez, PRL130051902(2023)



Deep pole

Motivation: Dispersive approach

C.Kacser,PhysRev.132.2712



In the light sector, the dispersion theory benefits to respect

✓ Unitarity

- 2-body: sub-channel interactions determined from scattering data
- 3-body: normal discontinuity (RHC) + partial-wave projection (LHC)

✓

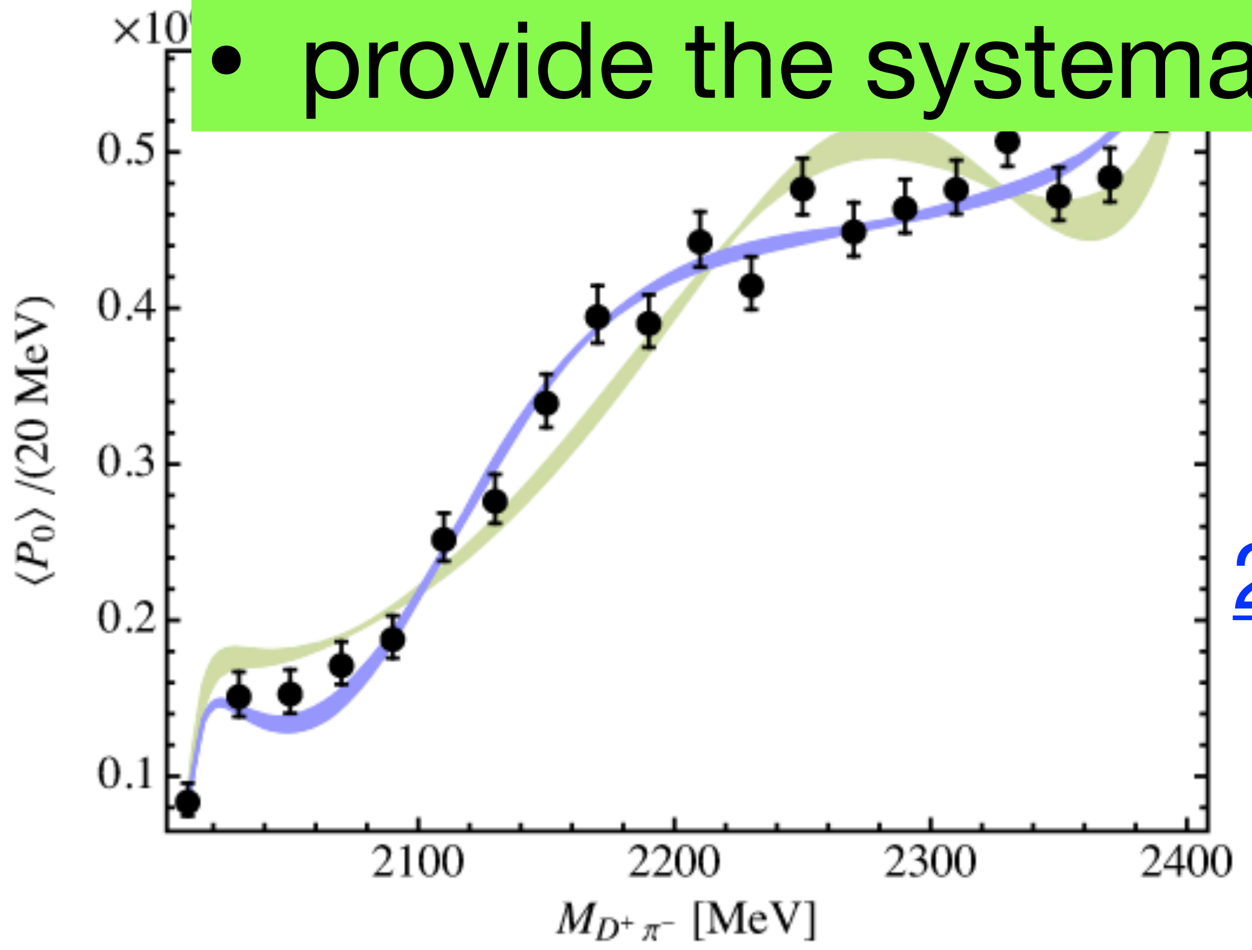
✓ …

The

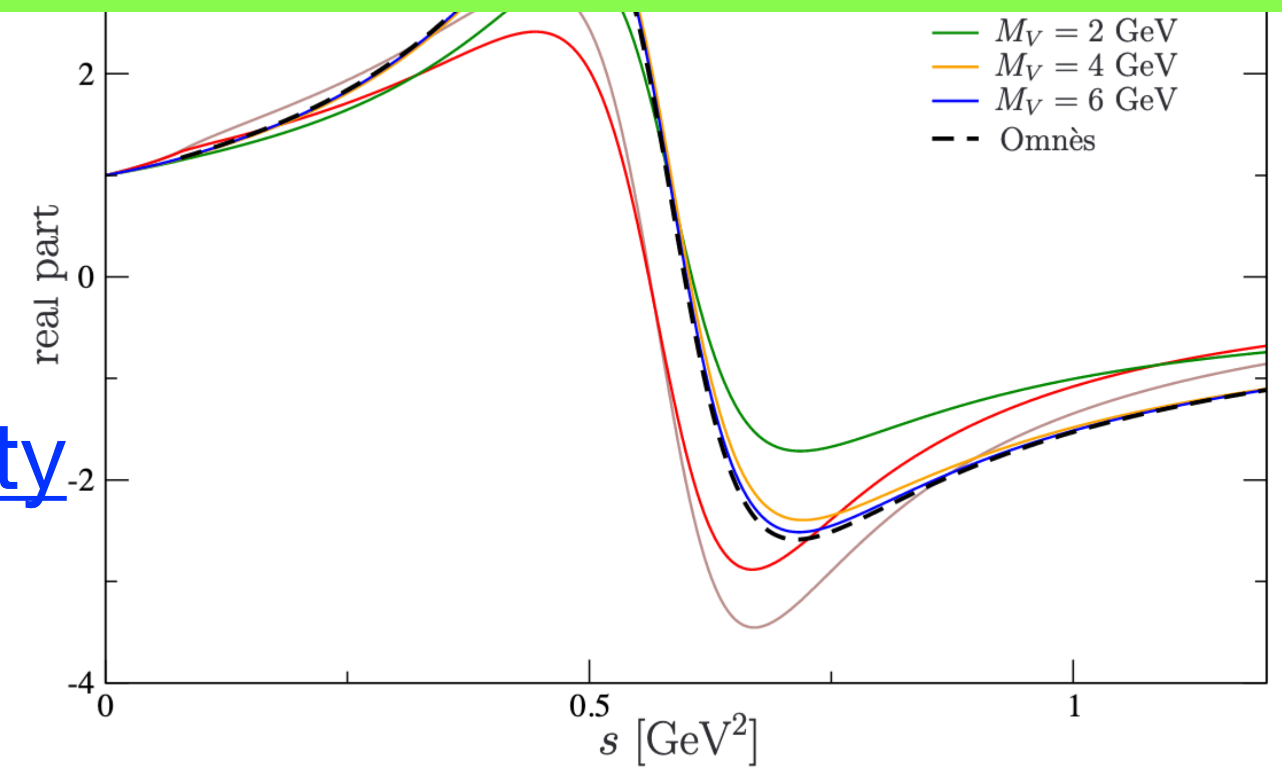
B^- … 023)

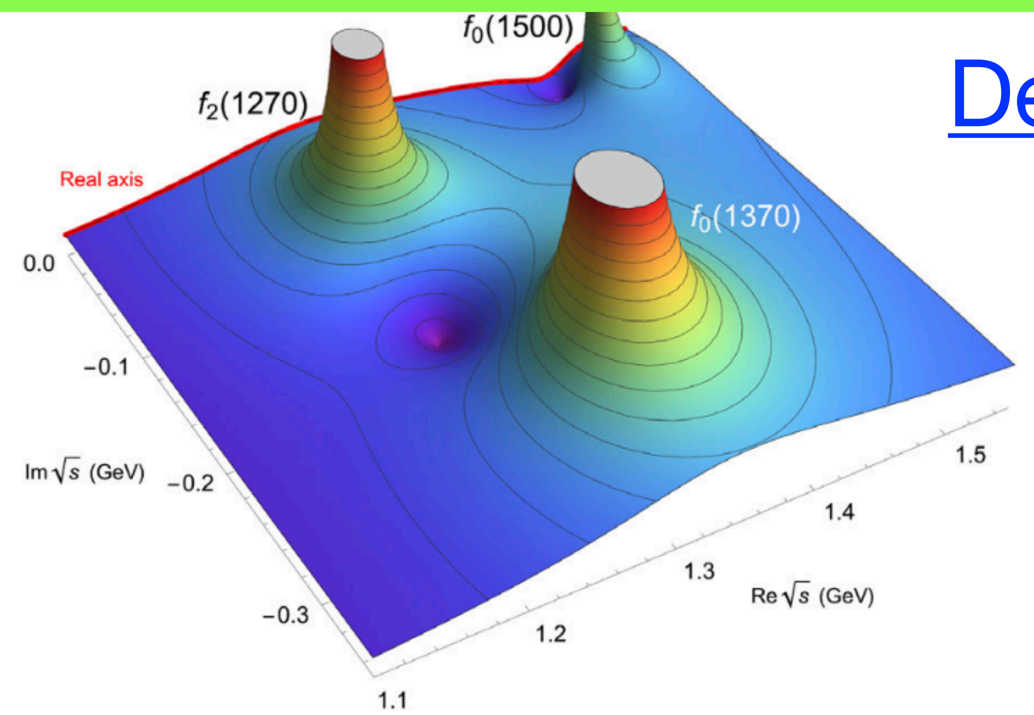
We hope to:

- understand the "distortion" of the three-body unitarity over the two-body one and the triangle singularity mechanism;
- provide the generic $K\bar{K}\pi, \eta\pi\pi, 3\pi$ FSIs below 1.6GeV;
- provide the systematic prescriptions for iso-scalar pseudo-scalar spectra.



2&3-b unitarity





Deep pole

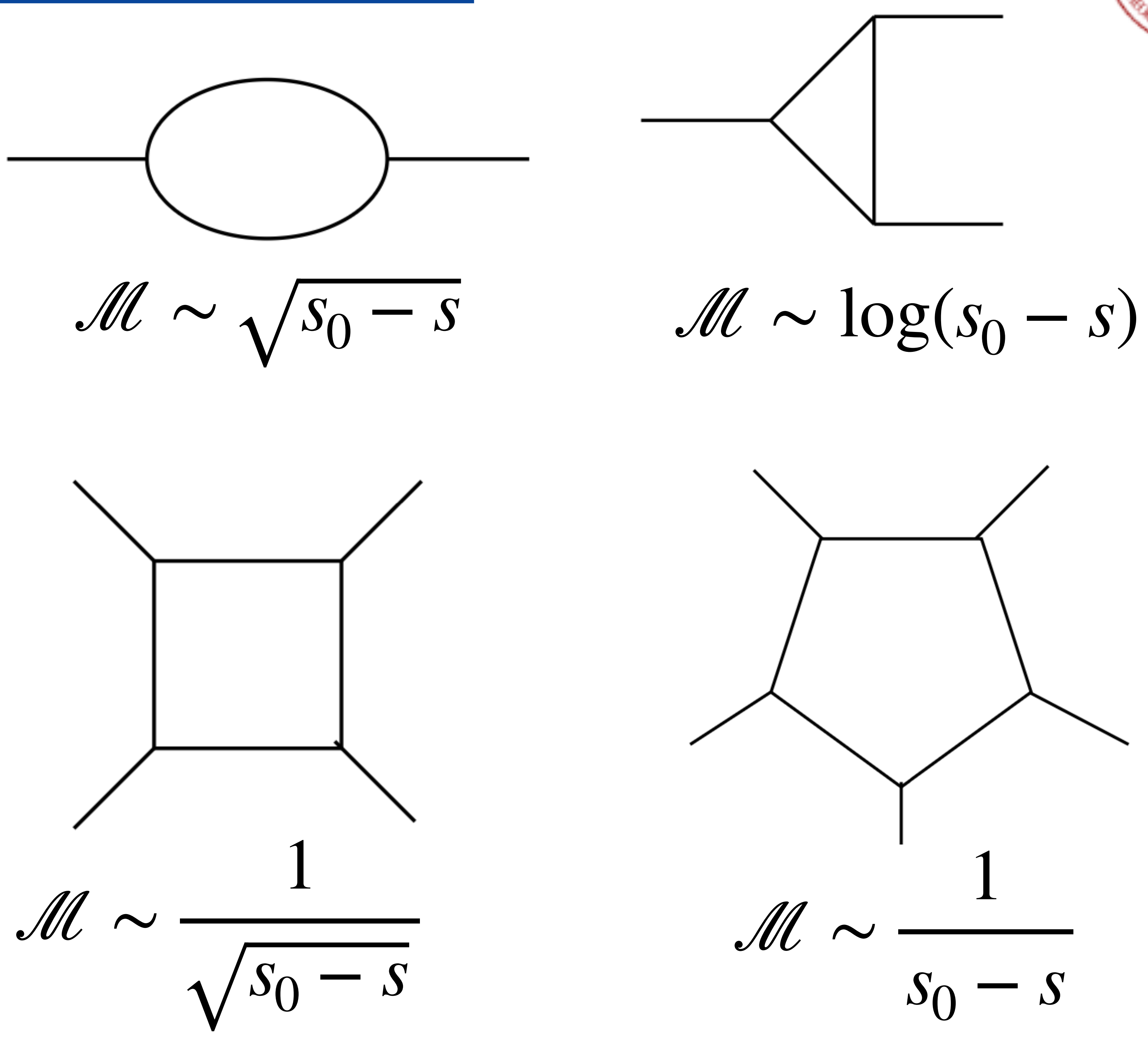
Loop integral and Landau singularities

L.D.Landau, NP13(1959)181; F.K.Guo et al.,PPNPhys.112(2020)103757

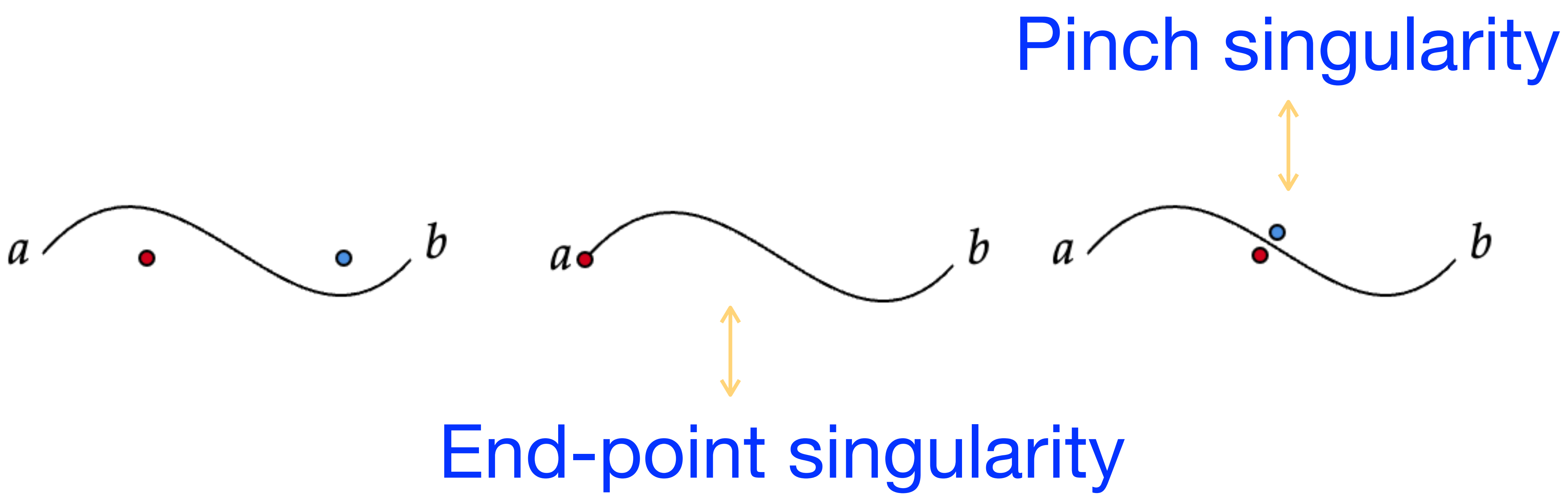
l loops integral with n propagators

$$I(p_1,\ldots,p_m) = \int \frac{d^4q_1\ldots d^4q_l}{[(2\pi)^4 i]^l}\frac{1}{(k_1^2-m_1^2+i\epsilon)\ldots(k_n^2-m_n^2+i\epsilon)},$$
$$= \int_0^1 \prod_{i=1}^{n} d\alpha_i\,\delta\left(\sum_i \alpha_i - 1\right)\int \prod_{j=1}^{l}\frac{d^4q_j}{(2\pi)^4 i}\frac{1}{[J(\alpha,q,p)+i\epsilon]^n},$$

$$\begin{cases} \sum_i \pm\alpha_i k_i^{\mu} = 0 & \text{for each loop,} \\ \alpha_i = 0 \ \text{ or } \ k_i^2 - m_i^2 = 0 & \text{for each } i, \end{cases}$$

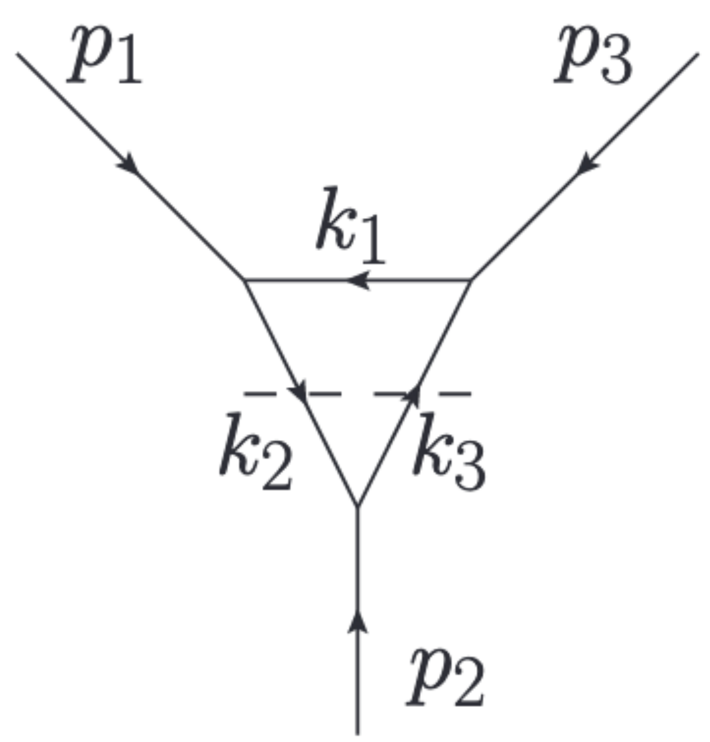


The Landau singularities manifest only when they get close to the physical axis!



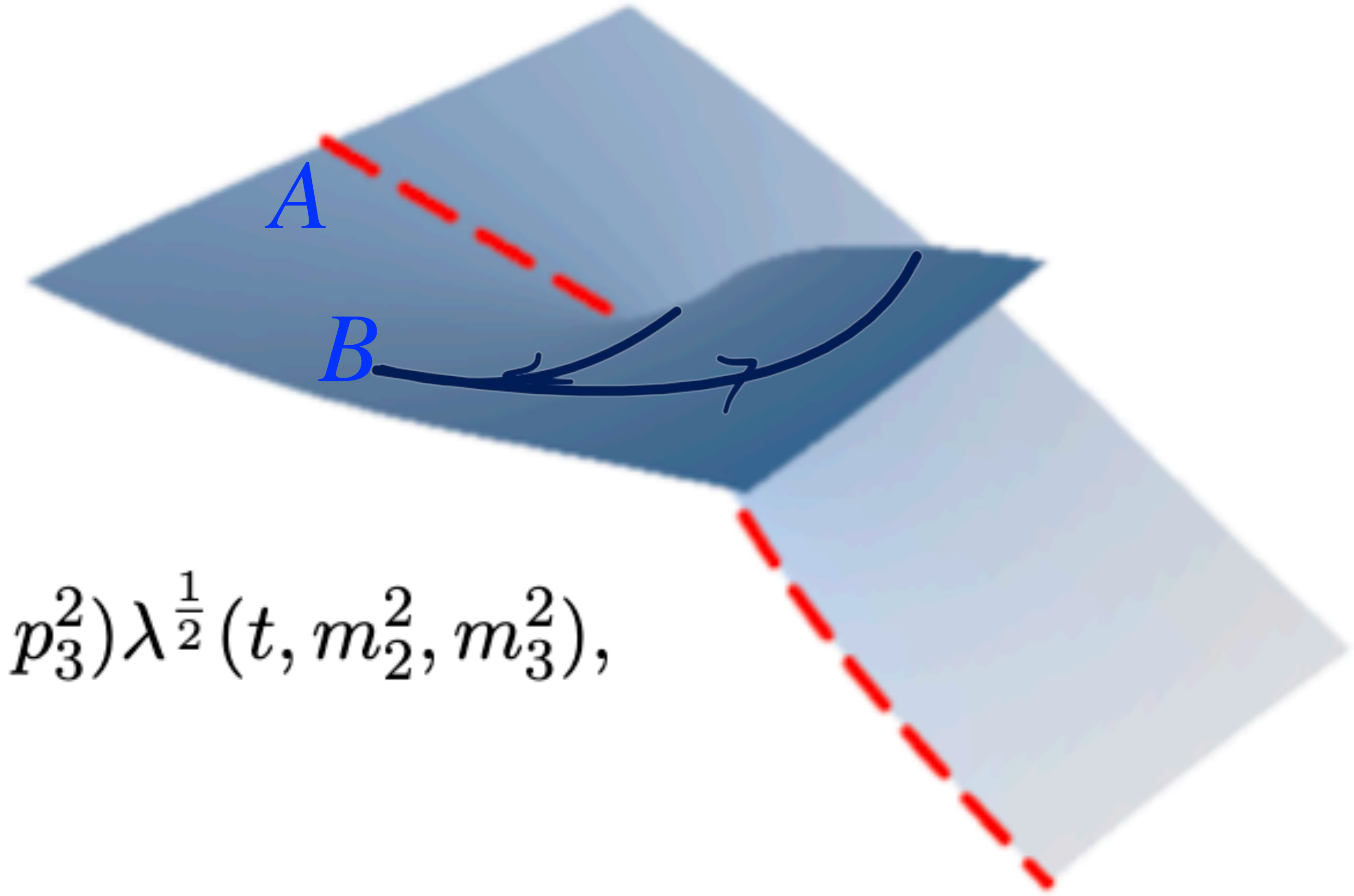
TS in dispersive perspective

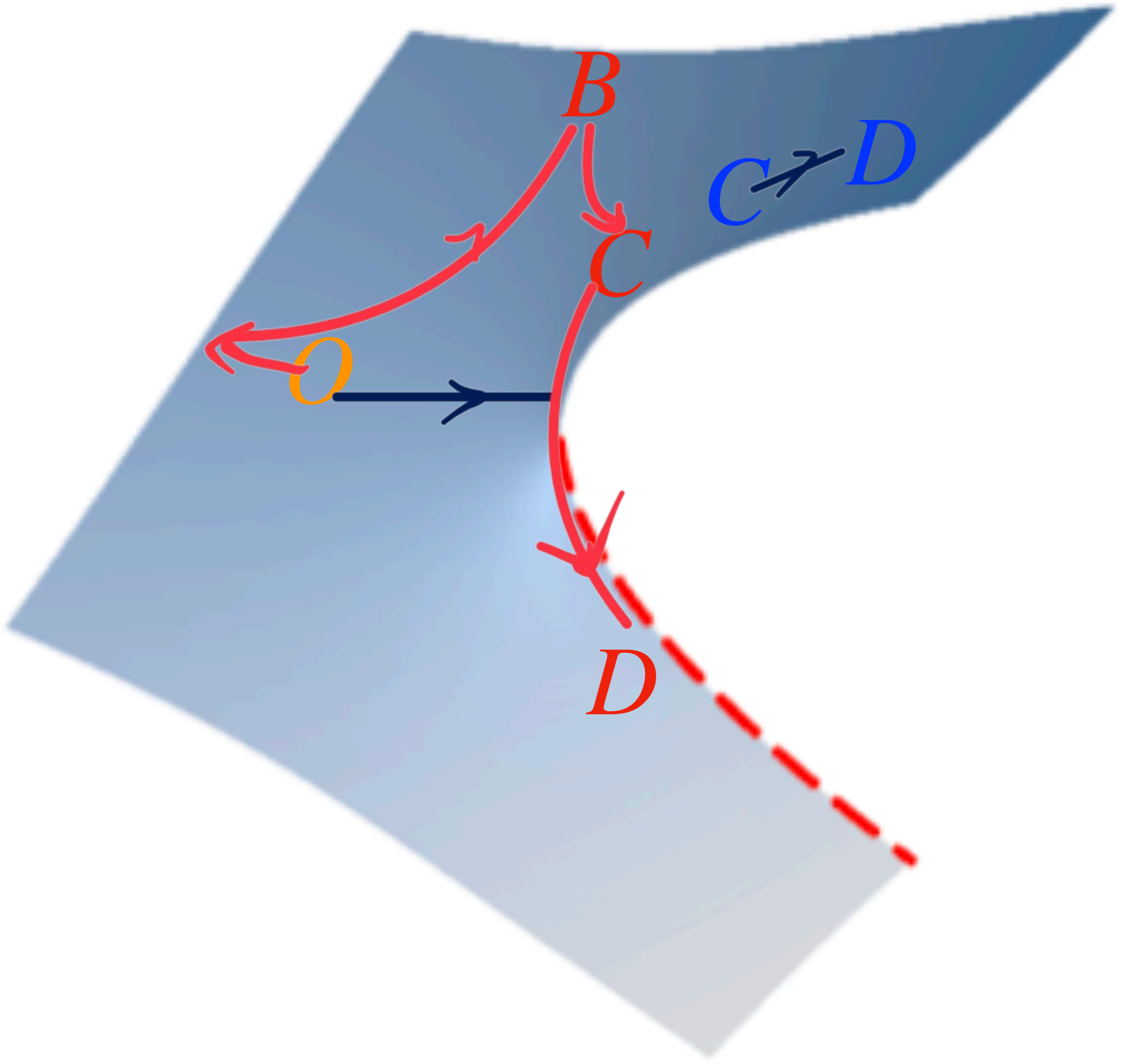
S.Mutke et al., JHEP07(2024)276.



$$C_0(t) = \frac{1}{2\pi i}\int_{t_{\rm thr}}^{\infty} \mathrm{d}t' \, \frac{\mathrm{disc}\, C_0(t')}{t' - t - i\epsilon}. \qquad \kappa(t) \equiv \lambda^{\frac{1}{2}}(t, p_1^2, p_3^2)\lambda^{\frac{1}{2}}(t, m_2^2, m_3^2),$$

$$\mathrm{disc}\, C_0(t) = -\frac{2\pi i \theta(t - t_1)}{\lambda^{\frac{1}{2}}(t, p_1^2, p_3^2)} \log \boxed{\frac{Y(t) + \kappa(t)}{Y(t) - \kappa(t)}},$$



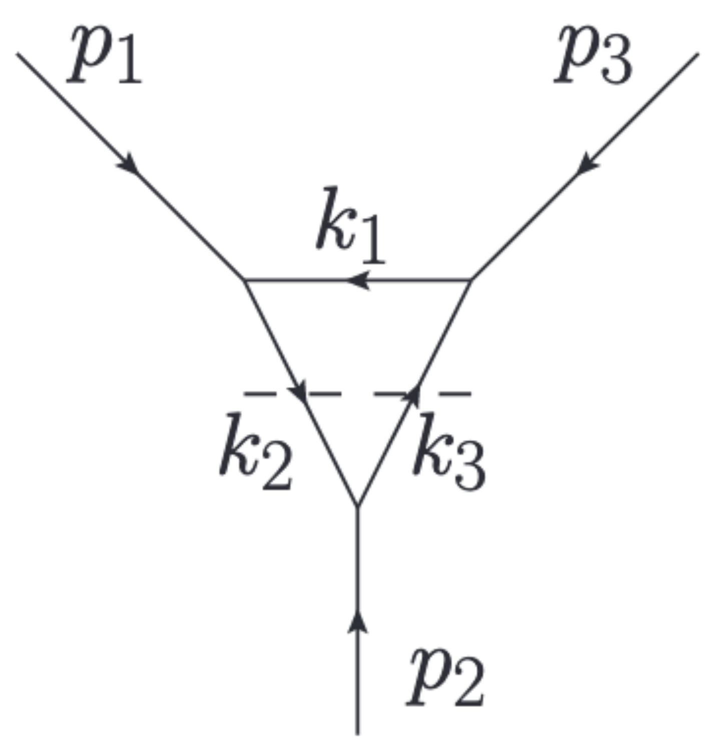


Logarithm cut between t^- $\leftrightarrow$ t^+

O	A	B	C	D
$(m_1 - m_2)^2 \le p_1^2 \le (m_1 + m_2)^2$	$p_1^2 > (m_1 + m_2)^2$			
$(m_1 - m_3)^2 \le p_3^2 \le (m_1 + m_3)^2$	$p_3^2 > (m_1 + m_3)^2$	$(m_1 - m_3)^2 \le p_3^2 \le (m_1 + m_3)^2$	$p_3^2 < (m_1 + m_3)^2$	
Stable	Anomalous integral t^+		2 logarithms	Triangle singularity t^-

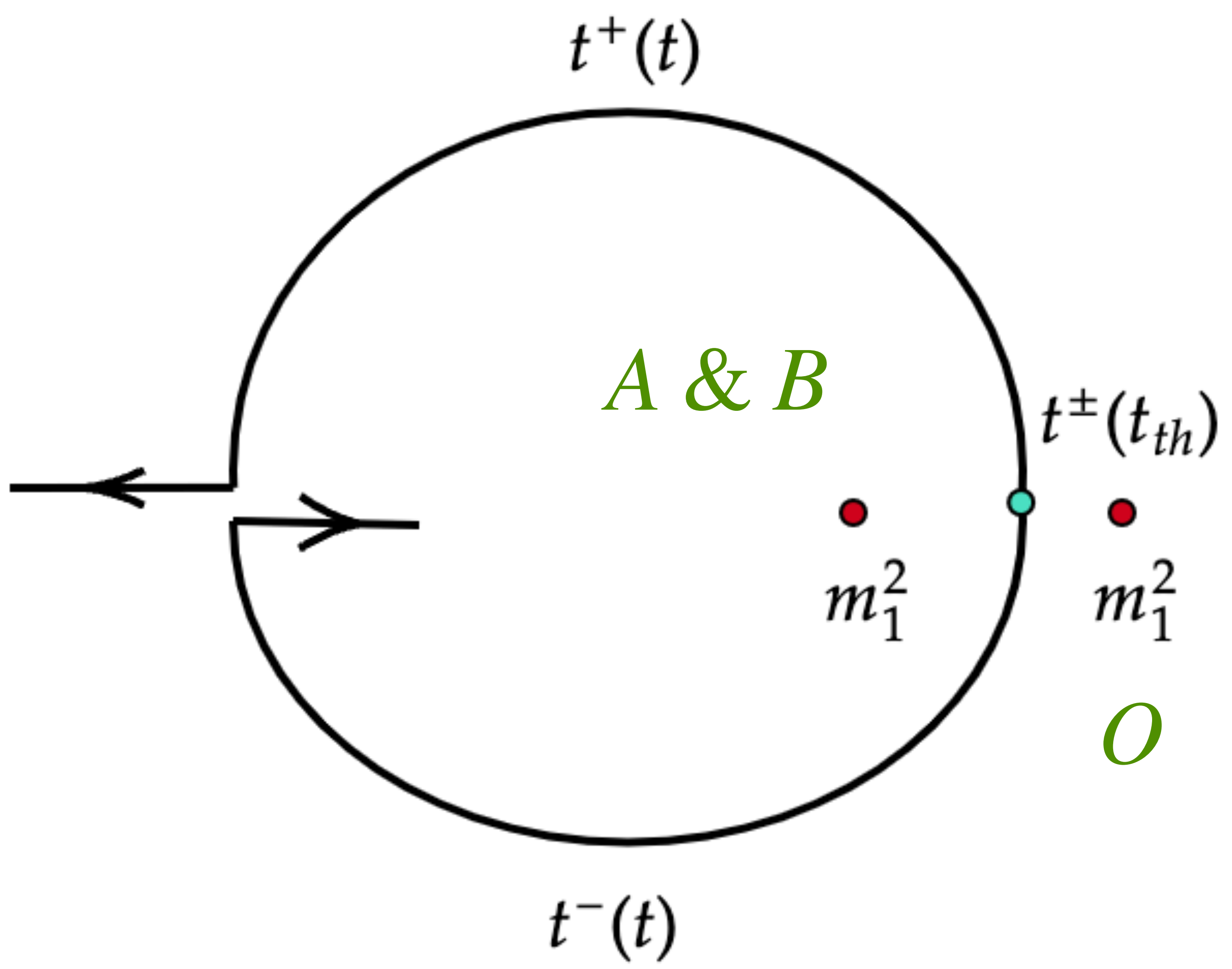
TS in dispersive perspective

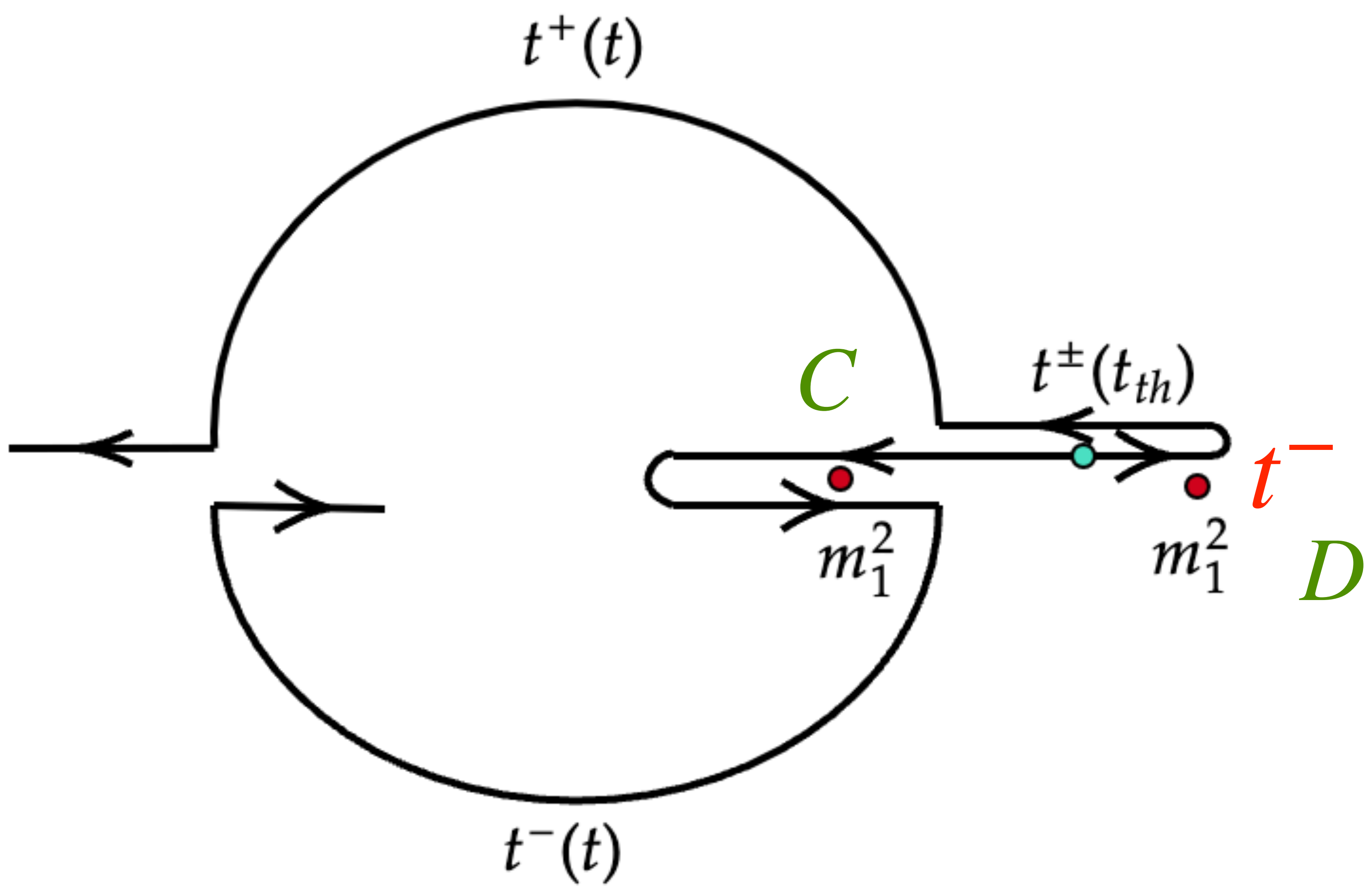
S.Mutke et al., JHEP07(2024)276.



$$C_0(t) = \frac{1}{2\pi i}\int_{t_{\rm thr}}^{\infty} \mathrm{d}t' \, \frac{\mathrm{disc}\, C_0(t')}{t' - t - i\epsilon}.$$

$$\text{disc } C_0(t) \propto \frac{1}{\kappa(t)} \int_{\mathcal{C}_t} \mathrm{d}s \, \frac{1}{s - m_1^2},$$





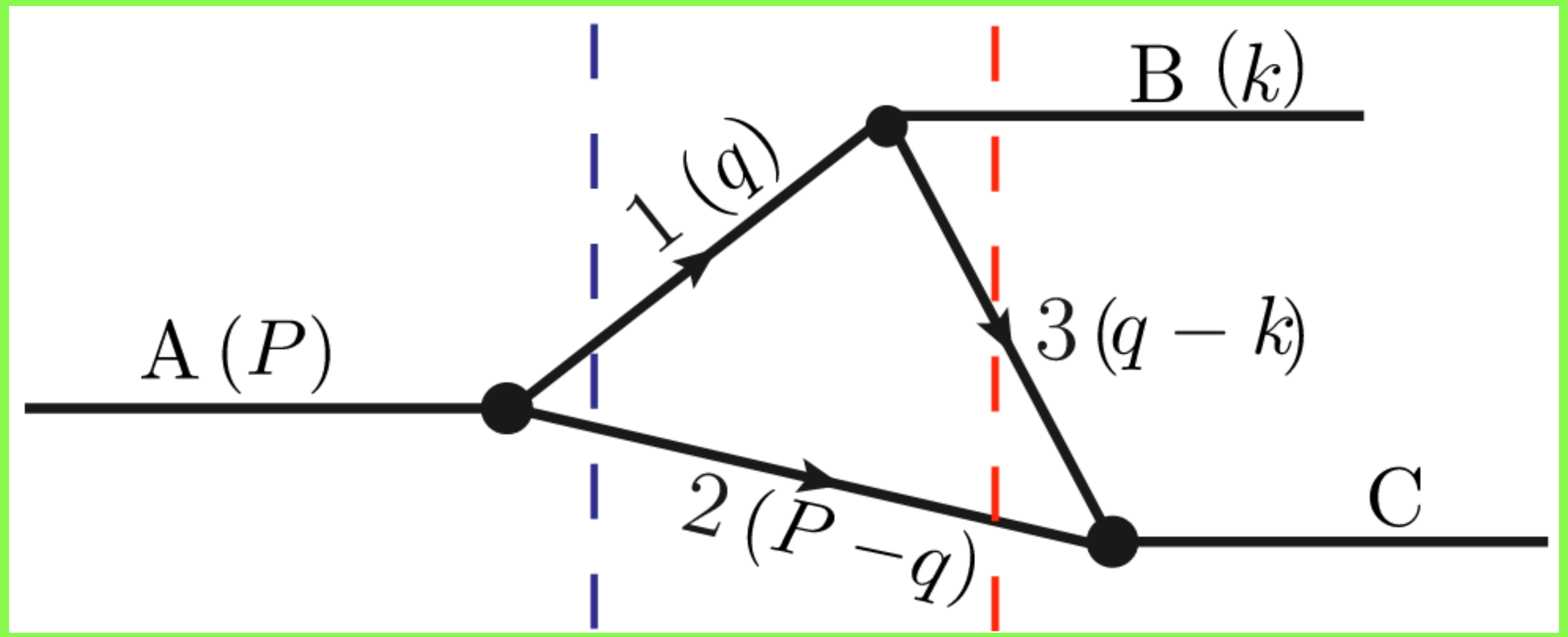
O	A	B	C	D
$(m_1 - m_2)^2 \leq p_1^2 \leq (m_1 + m_2)^2$	$p_1^2 > (m_1 + m_2)^2$			
$(m_1 - m_3)^2 \leq p_3^2 \leq (m_1 + m_3)^2$	$p_3^2 > (m_1 + m_3)^2$	$(m_1 - m_3)^2 \leq p_3^2 \leq (m_1 + m_3)^2$	$p_3^2 < (m_1 + m_3)^2$	
Stable	Anomalous integral t^+		2 logarithms	Triangle singularity t^-

TS in dispersive perspective

S.Mutke et al., JHEP07(2024)276.

$C_0(t) =$

disc $C_0(t)$



- When the intermediators are resonances or composite

  ↪ Muskhelishvili-Omnès Framework

- When the three-body unitarity is important

  ↪ Khrui-Trieman Framework

The dispersion theory is a powerful tool!

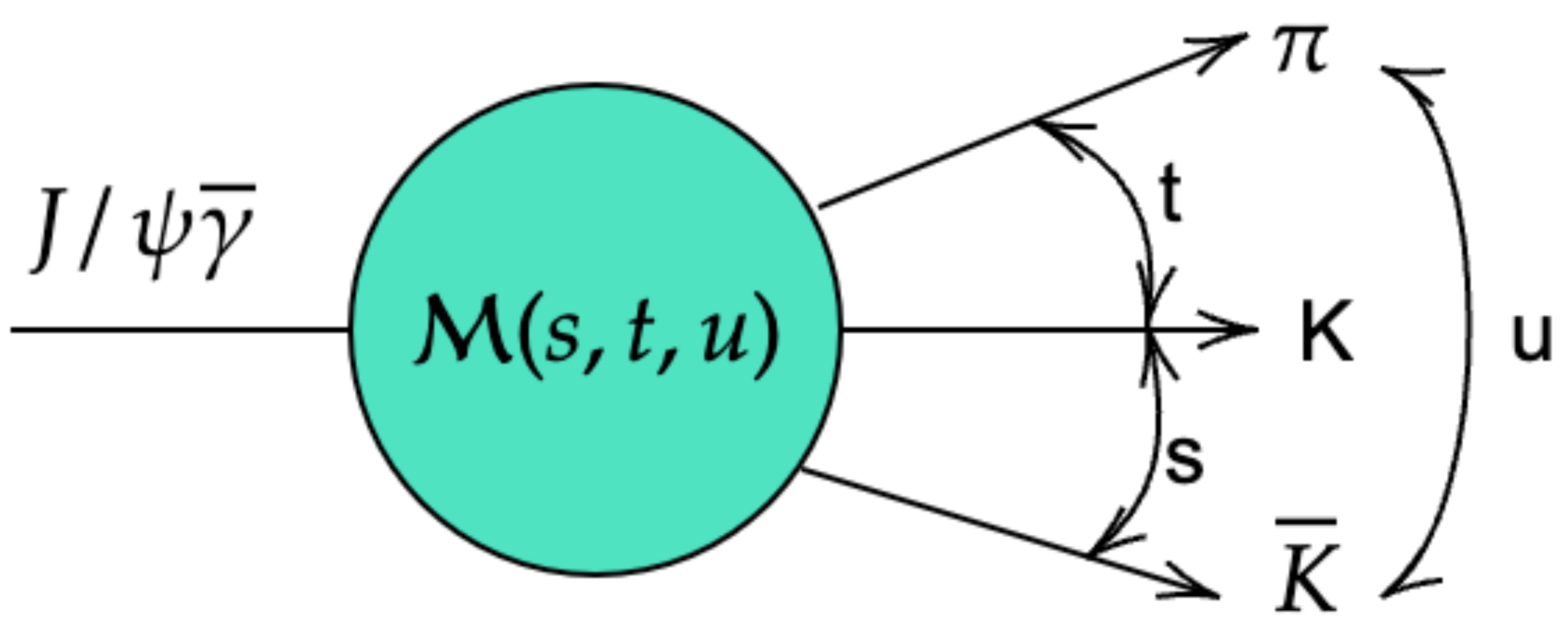
$(m_1$			
$(m_1$			
Stable	Anomalous integral t^+	2 logarithms	Triangle singularity t^-

Dispersive Framework of $J/\psi \to \gamma K_0^S K_0^S \pi^0$

Amplitudes on the Mandelstam plane

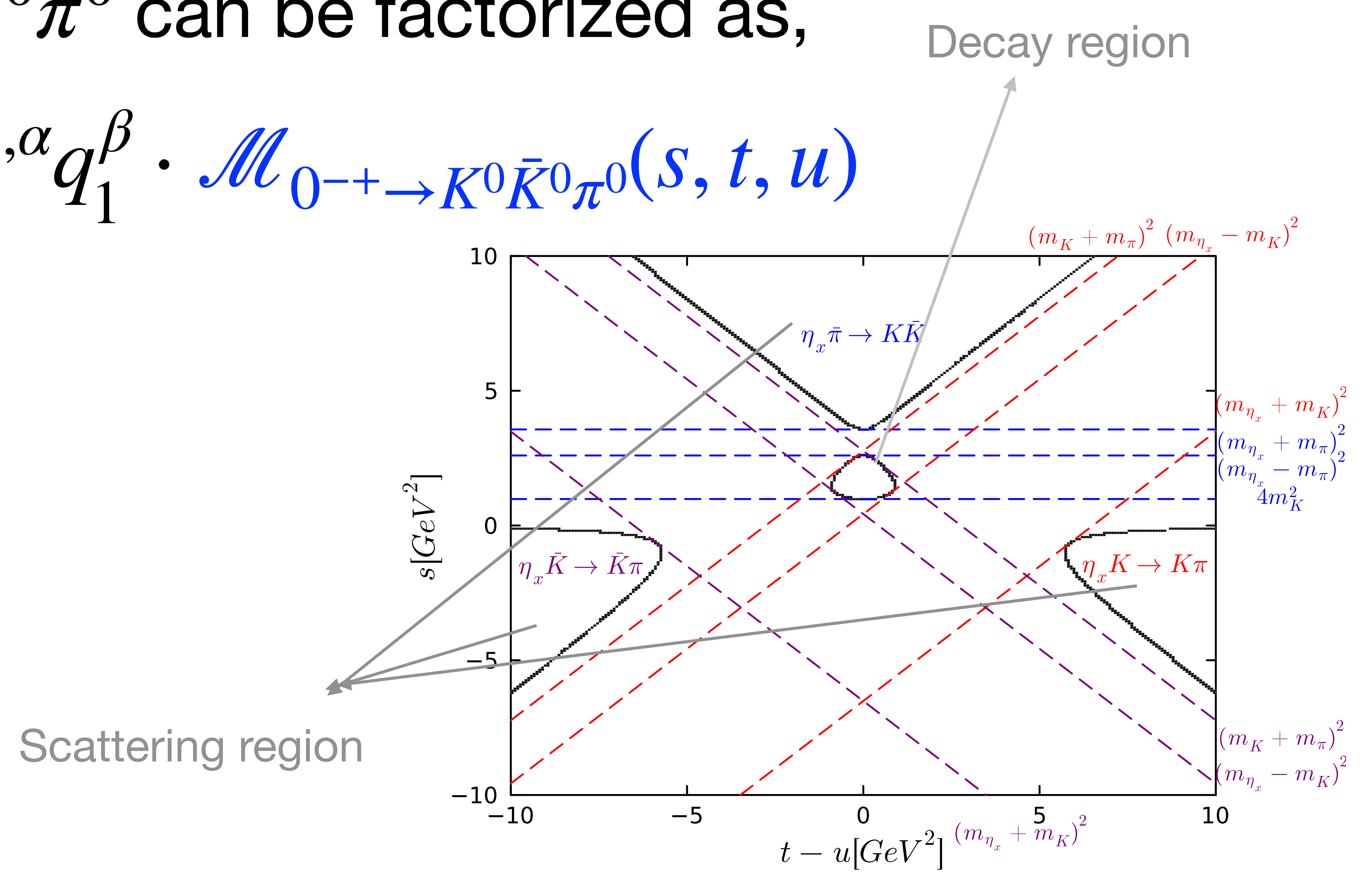
The LO amplitude of $J/\psi \to \gamma 0^{-+} \to \gamma K^0 \bar{K}^0 \pi^0$ can be factorized as,

$$\mathcal{M}_{J/\psi\to\gamma K^0\bar{K}^0\pi^0} \propto \epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}_{J/\psi}P^{\nu}\epsilon_{\gamma}^{*,\alpha}q_1^{\beta} \cdot \mathcal{M}_{0^{-+}\to K^0\bar{K}^0\pi^0}(s,t,u)$$



κ & $K^*(892)$

$a_0(980)$

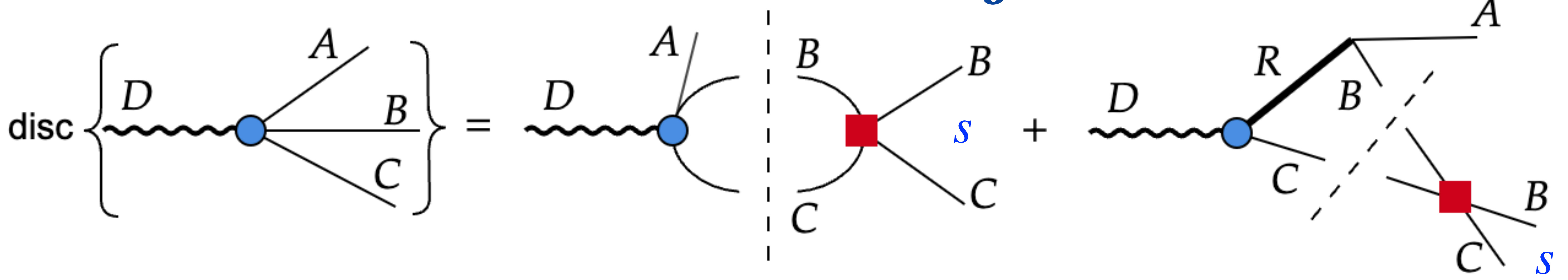


By virtue of crossing symmetry and reconstruction theorem,

$$\mathcal{M}(s,t,u) = \frac{1}{\sqrt{2}}\mathcal{F}_0^1(s) + [(-\frac{1}{\sqrt{3}})\mathcal{F}_0^{1/2}(t) + (-\frac{1}{\sqrt{3}})(t(s-u)-\Delta)\mathcal{F}_1^{1/2}(t)] + [t \leftrightarrow u]$$

The single-variable amplitudes $\mathcal{F}_J^I(x)$ are then all what we desire!

Single-variable amplitudes $\mathcal{F}^I_J(x)$



Dispersion relation for two-body scattering,

$$\mathrm{disc}\mathcal{F}^I_J(s) = 2iT^{I*}_J(s+i\epsilon)\Sigma(s)(\mathcal{F}^I_J(s+i\epsilon)+\hat{\mathcal{F}}^I_J(s+i\epsilon))$$

The most general solution (inhomogeneous Omnès problem),

$$x = s, t$$

$$\mathcal{F}^I_J(x) = \Omega^I_J(x)(a_0 + a_1 x + \cdots + a_n x^n + \frac{x^{n+1}}{\pi}\int_{x_{th}}^{\infty}\frac{dx'}{x'^{n+1}}\frac{\Omega^{-1}(x')T^{I*}_J(x')\Sigma(x')\hat{\mathcal{F}}^I_J(x')}{x'-x-i\epsilon})$$

FSIs — Sub-channel interaction — Crossed-channel projection

With $\Omega^I_J(x) = \exp(\frac{x}{\pi}\int_{x_{th}}\frac{\delta^I_J(x')dx'}{x'(x'-x-i\epsilon)}) \to x^{-\delta^I_J(\infty)/\pi}$.

Froissart-Martin bound: $\mathcal{F}(x) \lesssim x\log^2(x)$

$$\Rightarrow \begin{cases} \delta^1_0(\infty) = \pi \to P_2(s) \\ \delta^{\frac{1}{2}}_0(\infty) = 2\pi \to P_3(t) \\ \delta^{\frac{1}{2}}_1(\infty) = \pi \to P_0(t) \end{cases}$$

$[t(s-u)-\Delta] \asymp \mathcal{O}(t^2)$

Inhomogeneities $\hat{\mathcal{F}}^I_J(x)$: p.w.a of $\mathcal{F}^I_J(x)$

$$\hat{\mathcal{F}}^1_0(s) = (-\sqrt{\frac{2}{3}})2\langle\mathcal{F}^{1/2}_0\rangle_{t_s} + (-\sqrt{\frac{2}{3}})[\frac{1}{2}(\Sigma_0 - s)(3s-\Sigma_0) - 2\Delta]\langle\mathcal{F}^{1/2}_1\rangle_{t_s} + (-\sqrt{\frac{2}{3}})s\cdot\kappa_{K\bar{K}}\langle z_s\mathcal{F}^{1/2}_1\rangle_{t_s} + (-\sqrt{\frac{2}{3}})\frac{\kappa^2_{K\bar{K}}}{2}\langle z_s^2\mathcal{F}^{1/2}_1\rangle_{t_s}$$

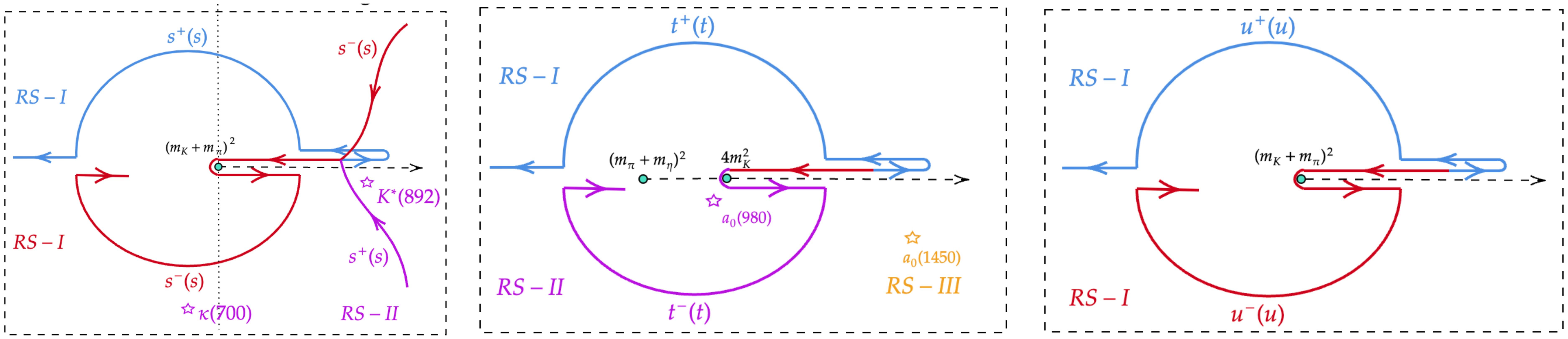
$$\langle f\rangle_{x_y} = \int_{-1}^{1} dz_y f(x(y,z_y)) \longrightarrow \int_{s^-(s)}^{s^+} ds', \quad \int_{t^-(s)}^{t^+} dt', \quad \int_{u^-(s)}^{u^+} du'$$

- elastic $K\pi \curvearrowright K\pi$ $\longrightarrow$ Riemann sheet 1 only
- inelastic $K\bar{K} \curvearrowright K\pi,\ K\pi \curvearrowright K\bar{K}$ $\longrightarrow$ Riemann sheet 1&2
- $\mathcal{F}$ has the right-hand-cut as Ω

When $s^+(s)$ crosses K^* pole,

$m_{K\bar{K}\pi} \sim 1.433$ GeV,

$m_{K\bar{K}} \sim 0.973$GeV

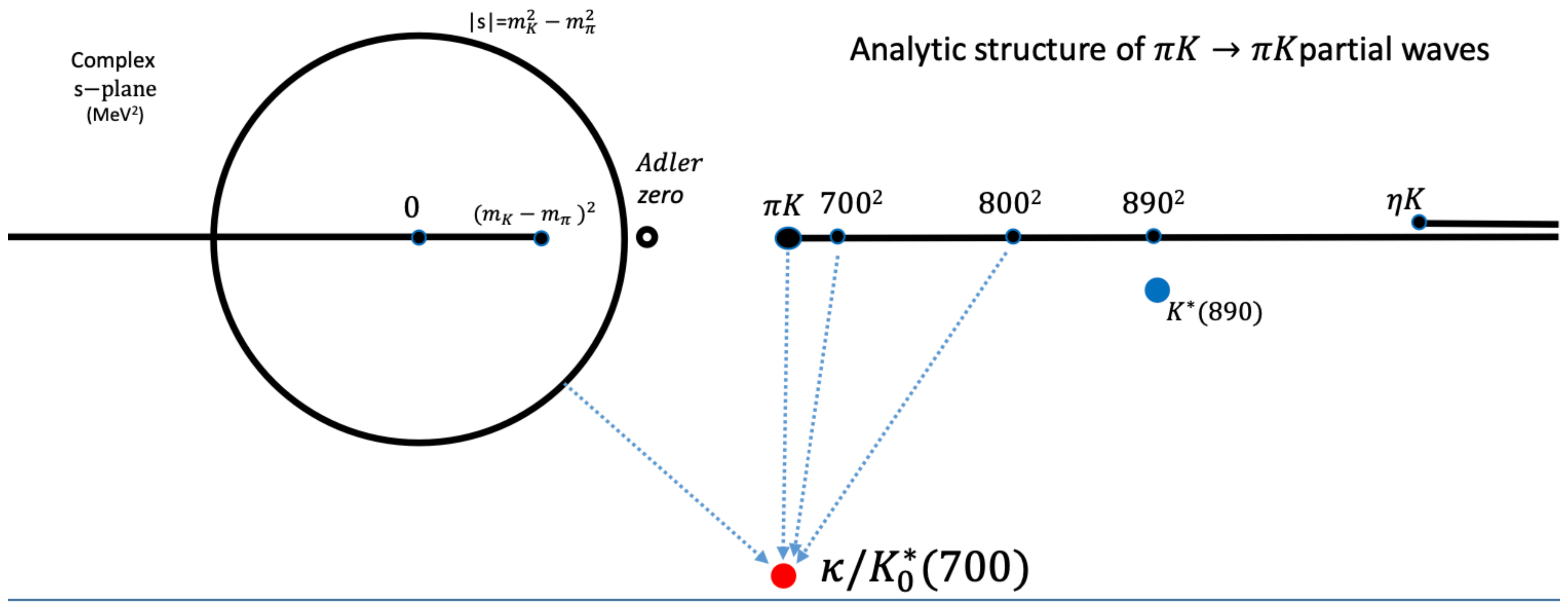


The integration needs to be continued on unphysical sheet!

Analytical structure and continuation

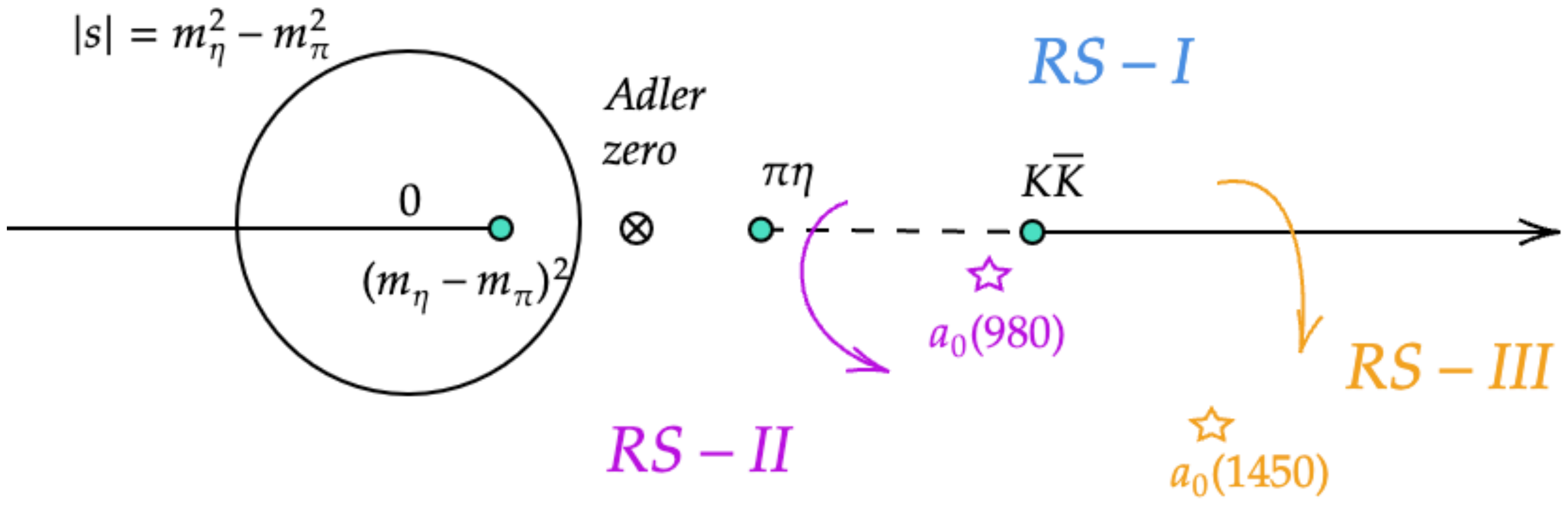
$K\pi$ scattering

J.R.Peláez, j.physrep.2022.03.004



$\pi\eta - K\bar{K}$ scattering

M.Albaladejo et al., EPJC(2015)75:488



- Single-channel continuation: $\Omega^{II}(s) = \Omega^{I}(s)/\hat{S}(s)$.
- Model-independent accesses to $\hat{S}(s)$:

A. Conformal expansion: $T_J^I(s) = \dfrac{1}{\sigma(s)}\dfrac{1}{\cot\delta_J^I(s) - i}, \quad \cot\delta_J^I(s) = \dfrac{\sqrt{s}}{2q^{2J+1}}F(s)\sum_n B_n\omega(s)^n;$

B. Schlesinger fraction method: $C_N(s) = F_1(s)/(1 + \dfrac{a_1(s-s_1)}{1+\dfrac{a_2(s-s_2)}{\cdots a_{N-1}(s-s_{N-1})}});$

C. Padé series: $P_M^N(s,s_0) = \dfrac{Q_N(s,s_0)}{R_M(s,s_0)}$

These methods give consistent results!

$(I, J) = (1/2, 0 \ \& \ 1)\ K\pi$ *scattering*

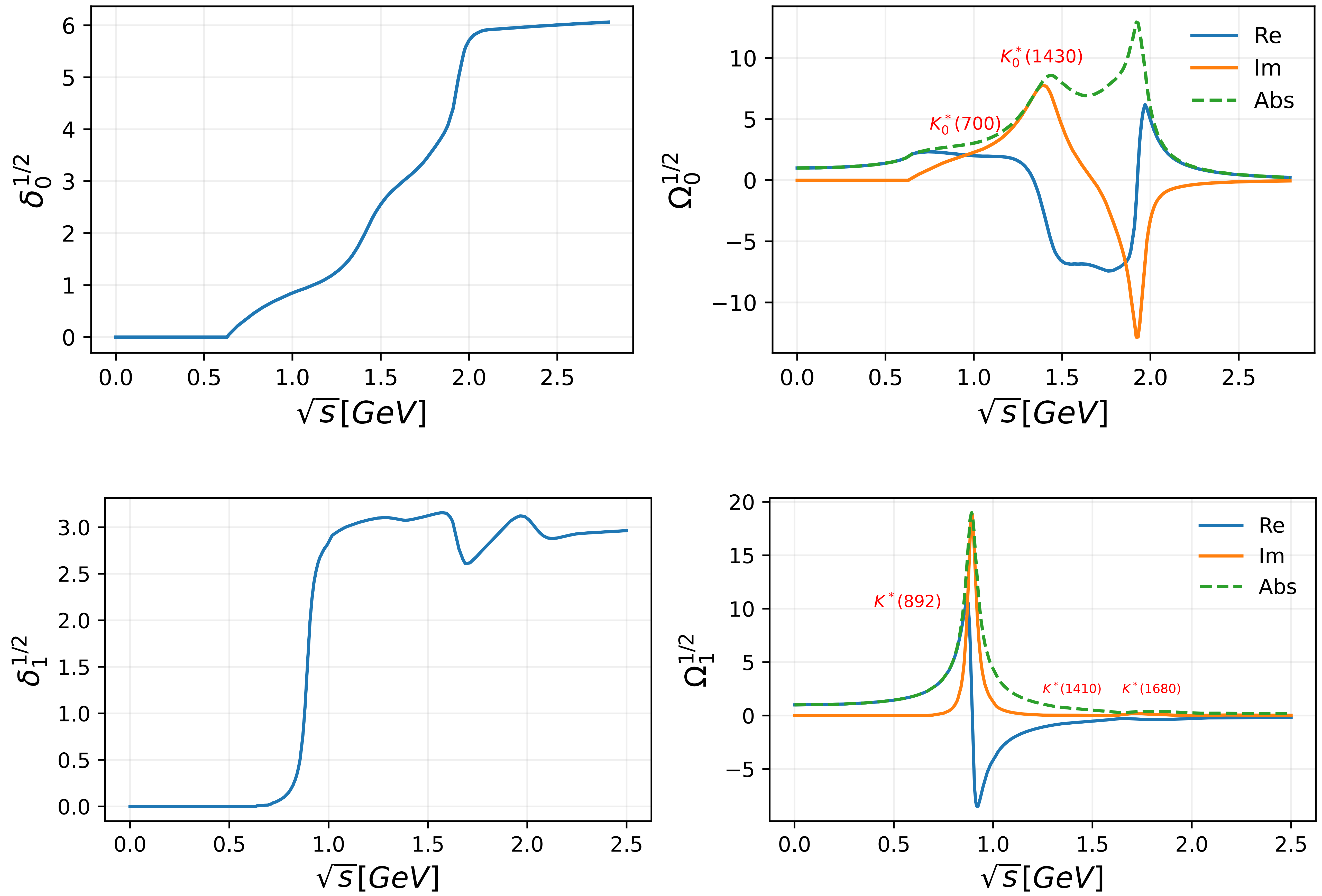
- Elastic up til $K\eta'$ threshold; L. von Detten et al., EPJC(2021) 81:420
- Treatment: conformal expansion ($\sim K\eta$ threshold)

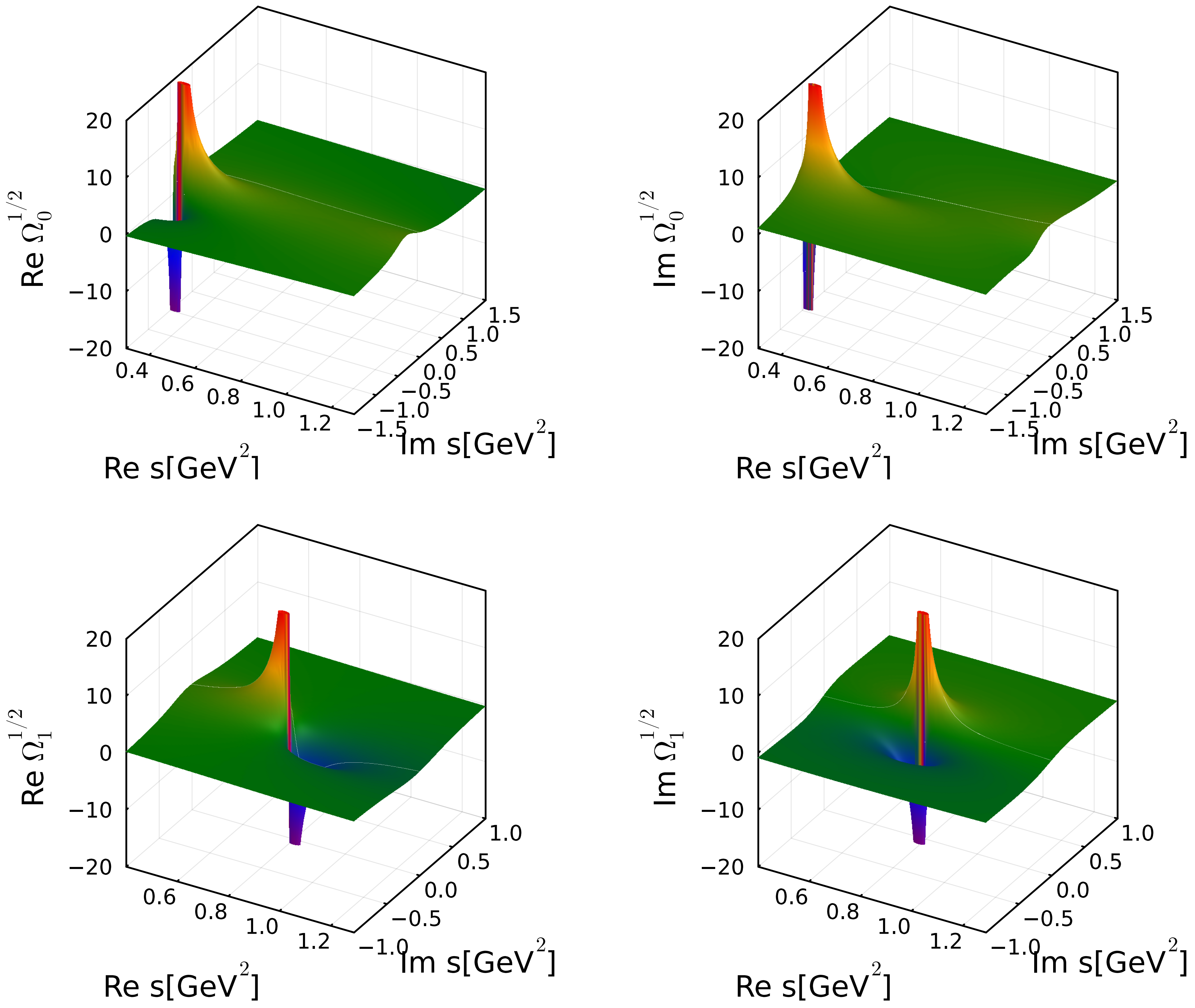
$\Rightarrow$ Schlesinger fraction (~ 1.3GeV).

Continued to lower-half plane on RS-II

Pole positions (MeV):

- κ: $667 - i335$;
- $K^*(892)$: $892 - i28$





$(I, J) = (1,0)\ \pi\eta - K\bar{K}$ scattering

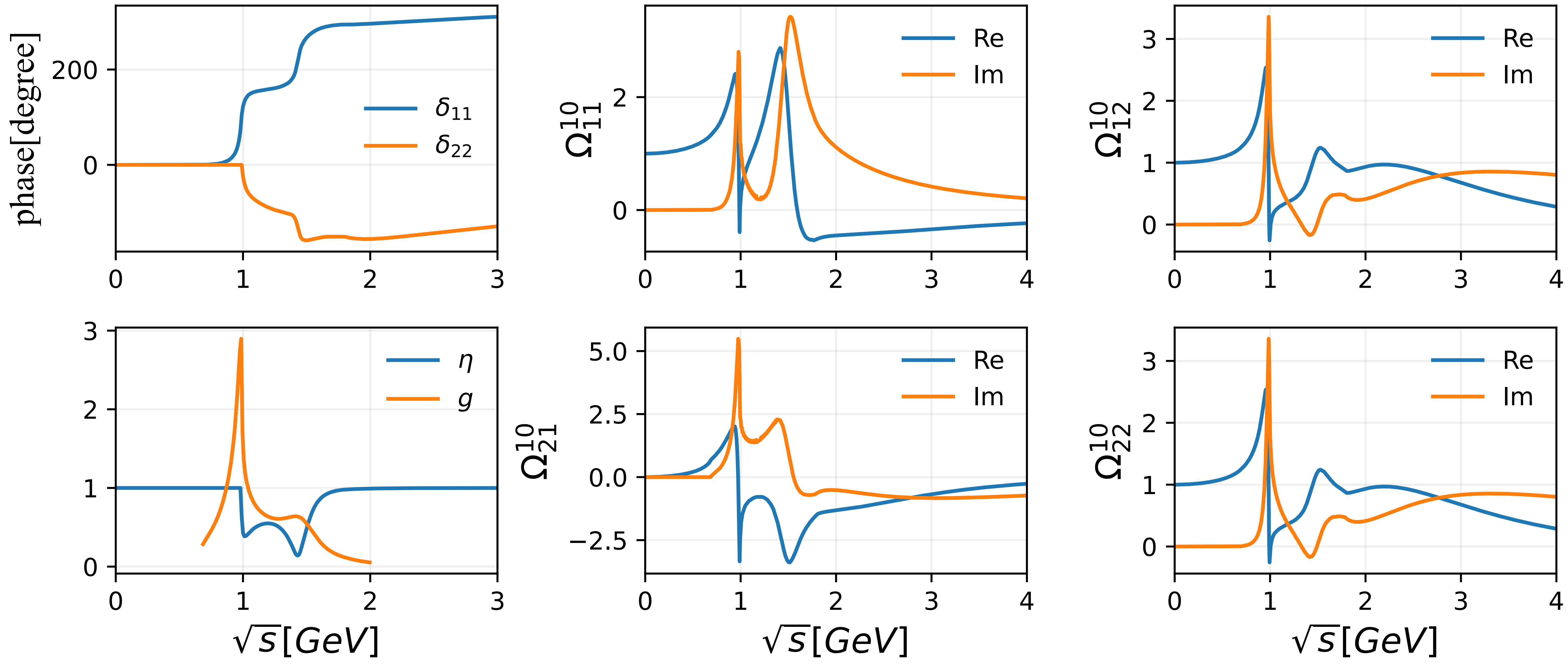
The isovector $\pi\eta - K\bar{K}$ coupling has a significant inelastic effect due to the onset of $a_0(980)$ and $a_0(1450)$. We adopt the following δ, η, g which satisfies the most ($5 \sim 6$) chiral constraints,

$$\Omega(s) = \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{T^*(s')\Sigma(s')\Omega(s')}{s' - s - i\epsilon}$$

B.Moussallam, EPJC14,111–122(2000)

J.F.Donoghue, NPB343(1990)

M.Doring,JHEP10(2013)011



$$T(s) = \begin{pmatrix} \frac{\eta e^{2i\delta_{11}} - 1}{2i\sigma_1} & g e^{i\phi_{12}} \\ g^{i\phi_{12}} & \frac{\eta e^{2i\delta_{22}} - 1}{2i\sigma_2} \end{pmatrix}$$

M.Albaladejo et al.,EPJC(2015)75:488

M.Albaladejo et al., EPJC(2017)77:508

“Effective” elastic $K\bar{K}$ scattering

The 2×2 Omnès matrix describes the coupling between the production amplitudes $J/\psi\bar{\gamma}\bar{\pi} \to \pi\eta$ and $J/\psi\bar{\gamma}\bar{\pi} \to K\bar{K}$,

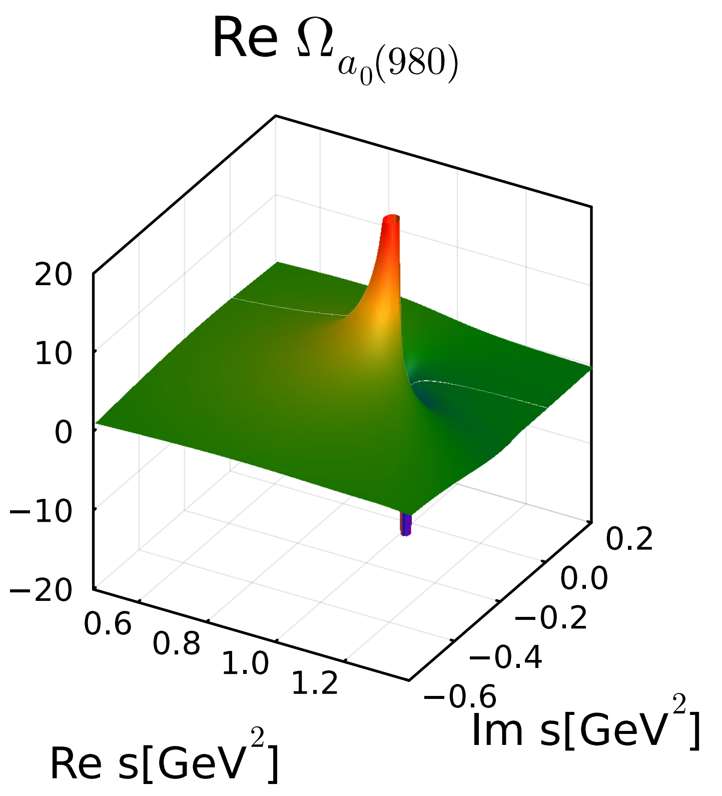
$$\begin{pmatrix} \mathscr{M}^{\pi\eta} \\ \mathscr{M}^{K\bar{K}} \end{pmatrix} = \begin{pmatrix} \Omega_0^1(s)_{\pi\eta\to\pi\eta} & \Omega_0^1(s)_{K\bar{K}\to\pi\eta} \\ \Omega_0^1(s)_{\pi\eta\to K\bar{K}} & \Omega_0^1(s)_{K\bar{K}\to K\bar{K}} \end{pmatrix} \begin{pmatrix} \mathscr{M}^{\chi,\pi\eta} \\ \mathscr{M}^{\chi,K\bar{K}} \end{pmatrix}$$

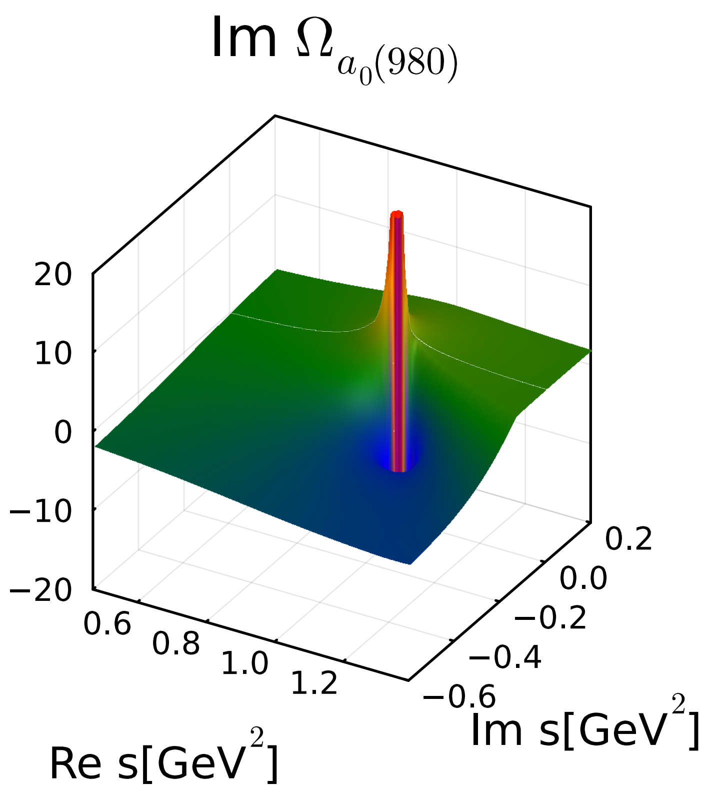
T.Isken et al., EPJC(2017)77:489;E.Kou et al. ,JHEP12(2023)17

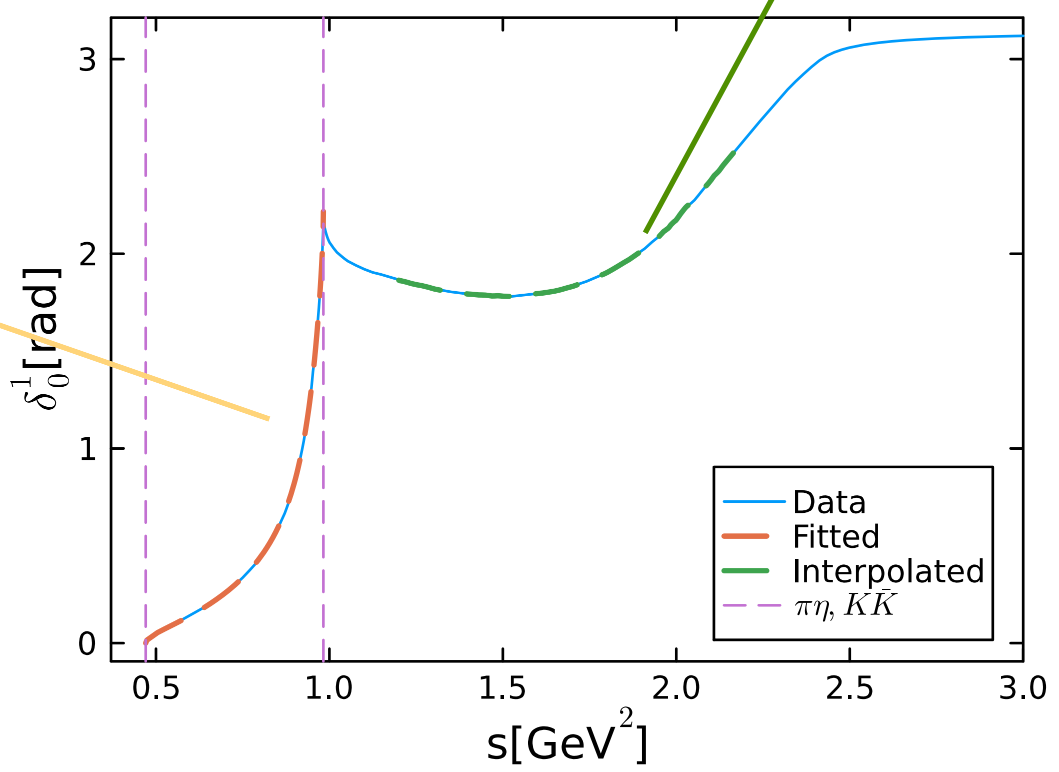
To simplify the problem, we adopt the idea of “effective phase shift” (LO approximation of production form factors),

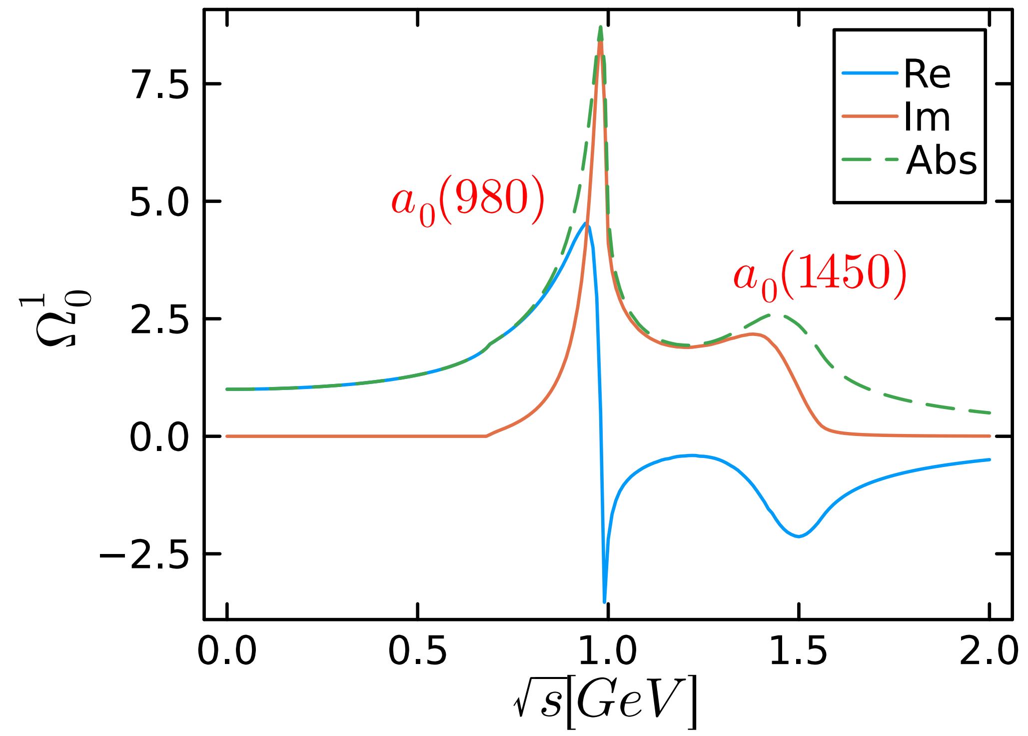
$$\mathscr{M}^{K\bar{K}} = \Omega(s)_{\pi\eta\to K\bar{K}}P_1(s) + \Omega(s)_{K\bar{K}\to K\bar{K}}P_2(s) = (\xi \cdot \Omega_{\pi\eta\to K\bar{K}}(s) + \Omega_{K\bar{K}\to K\bar{K}}(s))P_{eff}(s) =: \Omega_{eff}(s)P_{eff}(s).$$

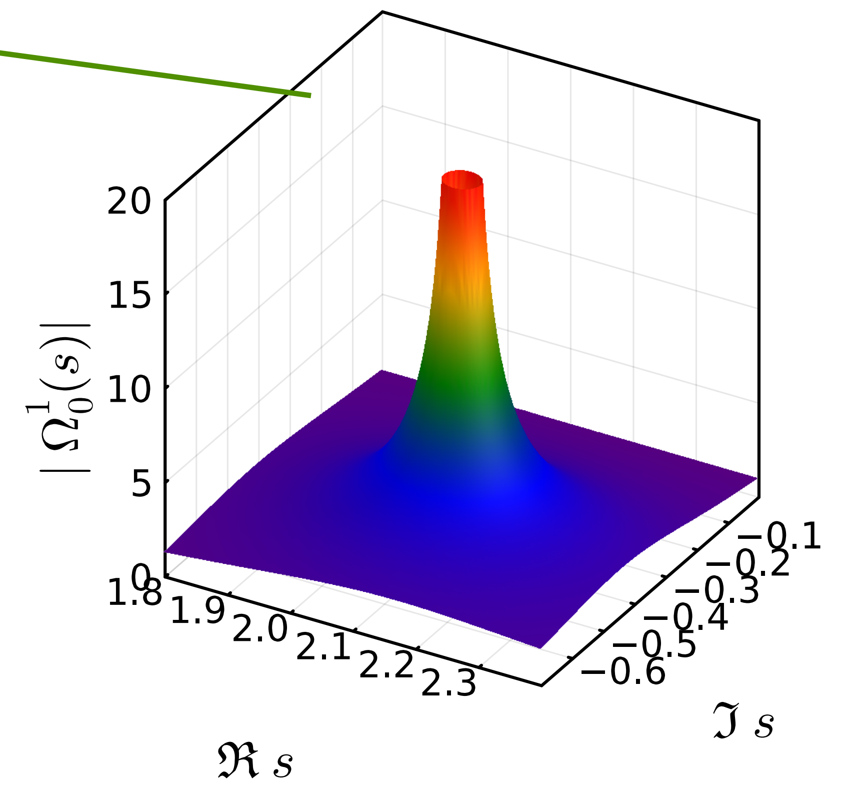
When $\xi = 1$,











$\sqrt{s^{II}_{a_0(980)}}$: $997.1 - i26.1$

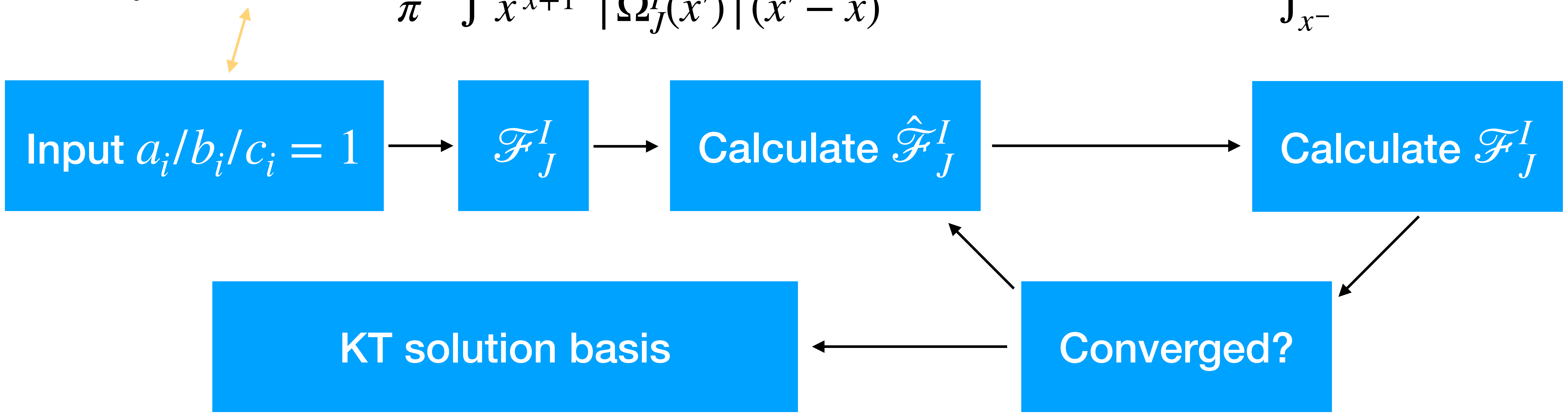
$\sqrt{s^{III}_{a_0(1450)}}$: $1465 - i137$

Khrui-Trieman equation: $1 \to 3$ decaying

$$\mathscr{F}^I_J(x) = \Omega^I_J(x)\{P_n(x) + \frac{x^{n+1}}{\pi}\int \frac{dx'}{x'^{x+1}} \frac{\hat{\mathscr{F}}^I_J(x')\sin\delta^I_J(x')}{|\Omega^I_J(x')|(x'-x)}\}$$

$$\hat{\mathscr{F}}^I_J(x) \propto \int_{x^-}^{x+} dx' \mathscr{F}^I_J(x')$$

Iteration



e.g. $\gamma\gamma$ collisions, $p\bar{p}$ annihilation…

- The solutions are linearly-independent of subtractions $a_i/b_i/c_i$ $\to$ Generic & Open-box;

$$\mathscr{M}(s,t,u;m_{\eta_x}) = \sum_i C_i \mathscr{M}_i(s,t,u;m_{\eta_x})$$

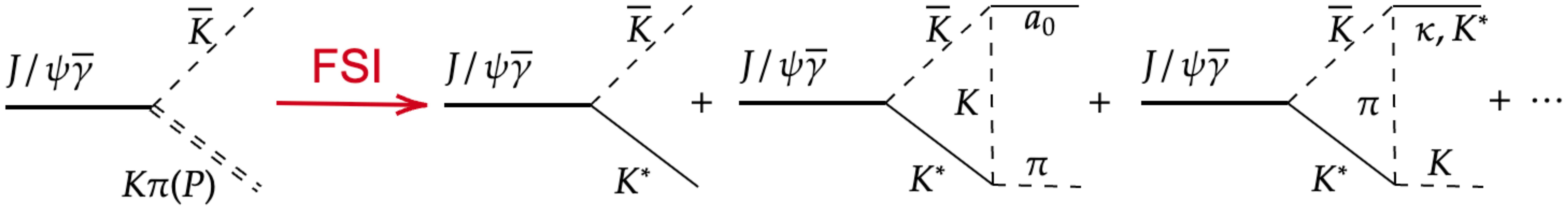
- The subtractions are calculated from CHPT / fitted by experimental data;

KT basis function: $c_0 = 1$

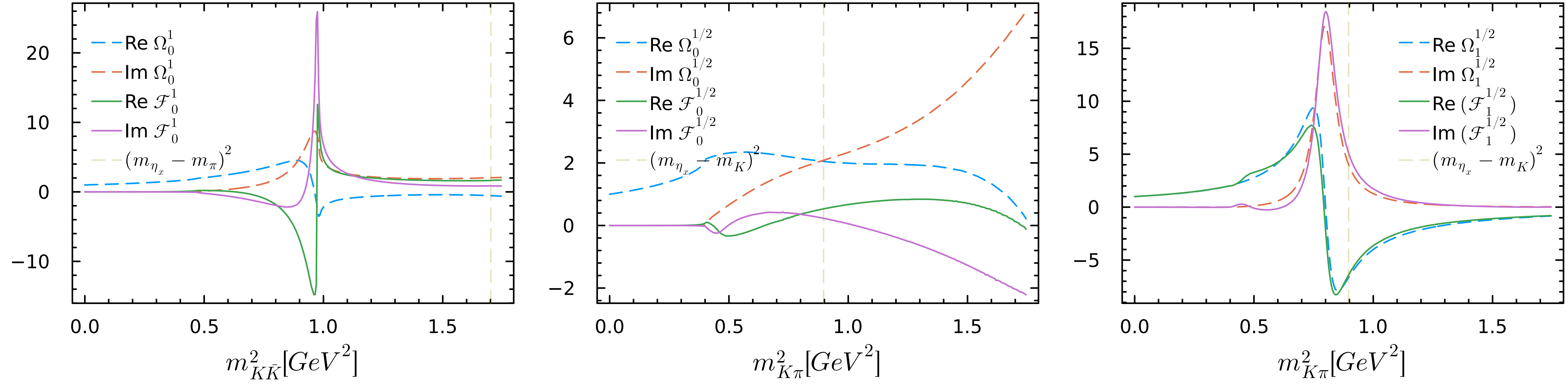
$$\mathscr{F}_0^1(s) = \Omega_0^1(s)\frac{s^3}{\pi}\int\frac{ds'}{s'^3}\frac{\hat{\mathscr{F}}_0^1(s')\sin\delta_0^1(s')}{|\Omega_0^1(s')|(s'-s)} \qquad \mathscr{F}_0^{1/2}(t) = \Omega_0^{1/2}(t)\frac{t^4}{\pi}\int\frac{dt'}{t'^4}\frac{\hat{\mathscr{F}}_0^{1/2}(t')\sin\delta_0^{1/2}(t')}{|\Omega_0^{1/2}(t')|(t'-t)} \qquad \mathscr{F}_1^{1/2}(t) = \Omega_1^{1/2}(t)\{1+\frac{t}{\pi}\int\frac{dt'}{t'}\frac{\hat{\mathscr{F}}_1^{1/2}(t')\sin\delta_1^{1/2}(t')}{|\Omega_1^{1/2}(t')|(t'-t)}\}$$

Dominant channel to $\eta(1440)$!

Triangle singularity



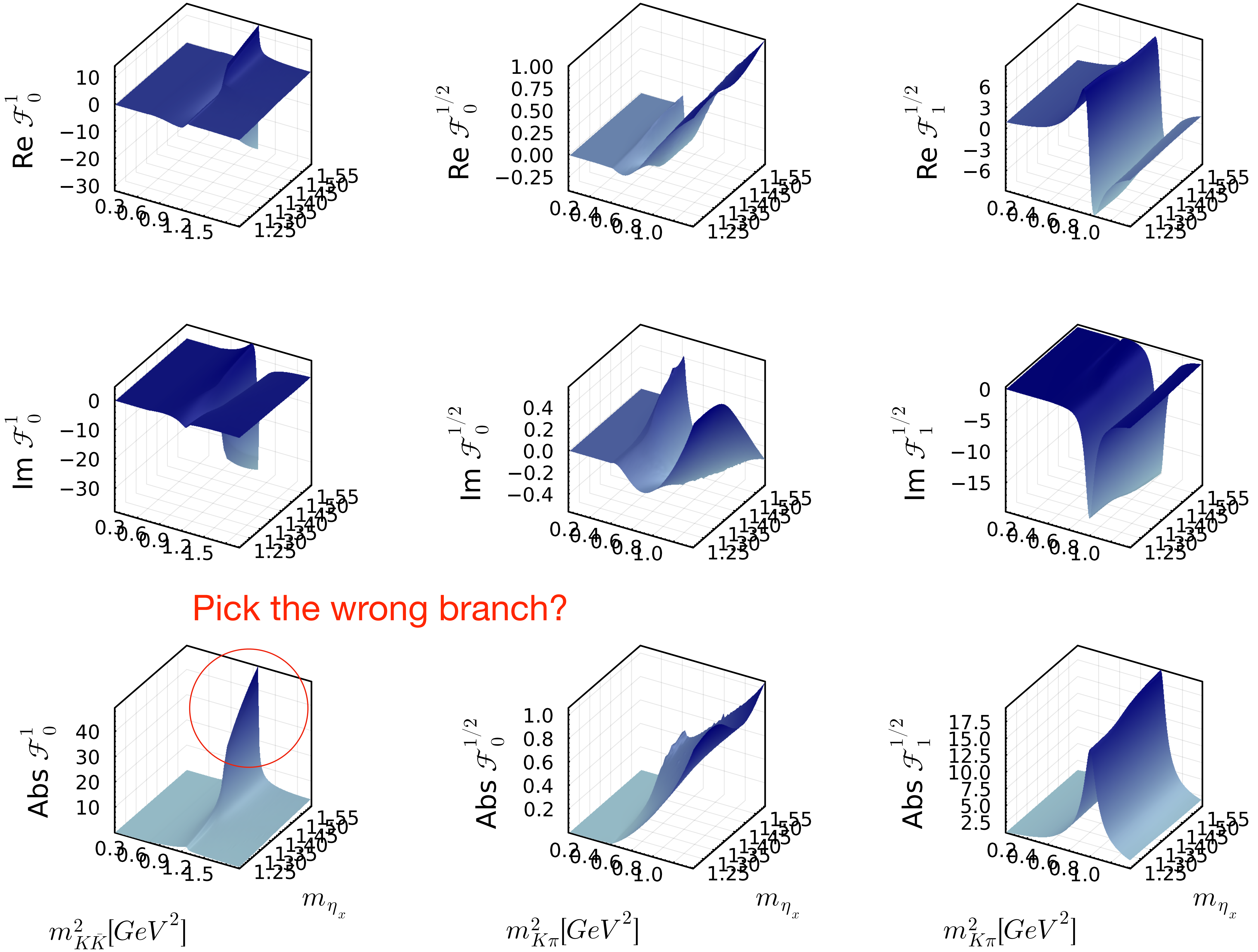
When $m_{K_S^0 K_S^0 \pi^0} \sim 1.44 GeV$,



KT basis function: $c_0 = 1$

And for each $K_S^0 K_S^0 \pi^0$ bin in $1.24 \sim 1.6$ GeV...

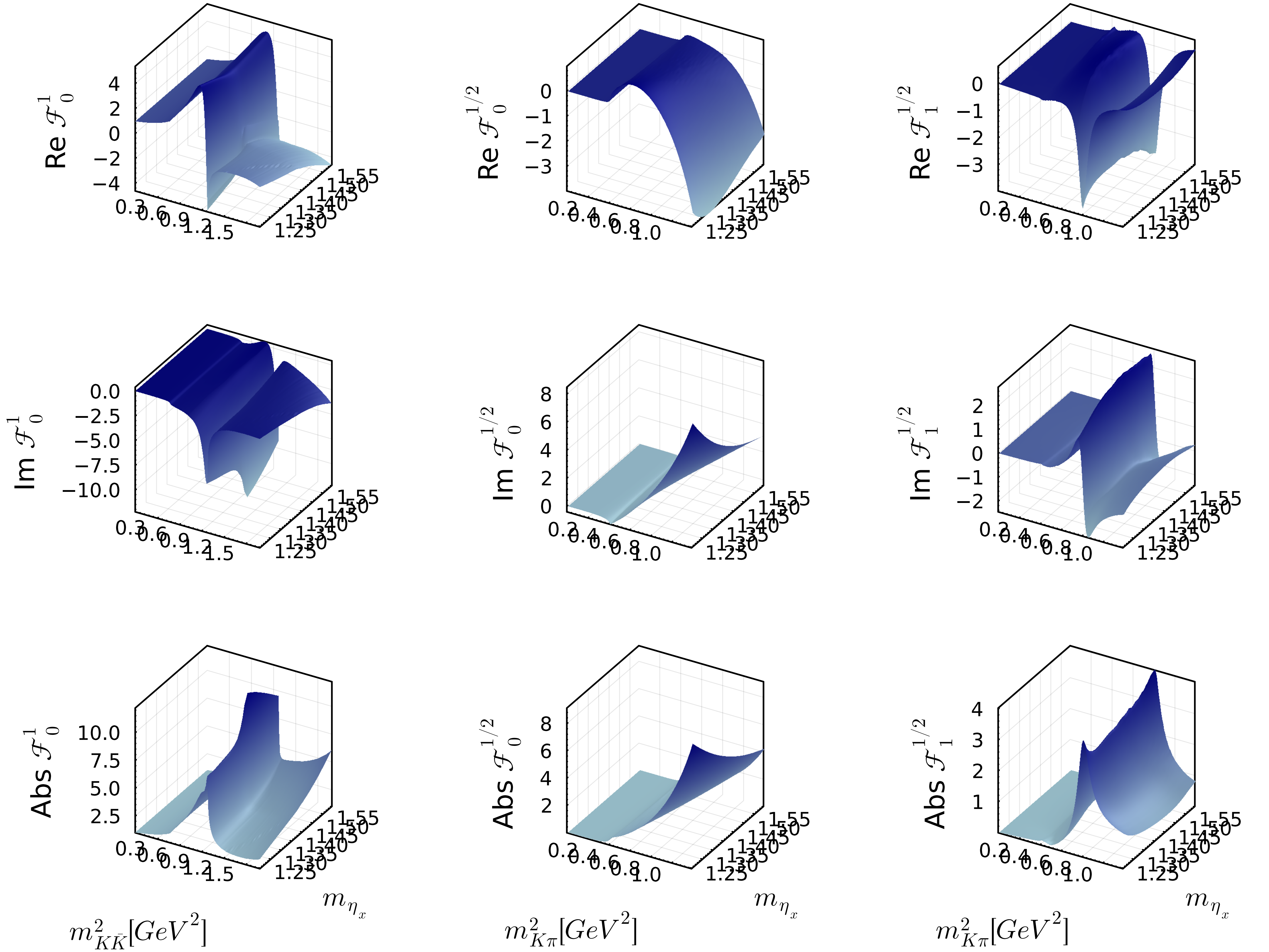
- $K^*K \to a_0\pi$: Triangle singularity
- $K^*\pi \to \kappa\pi$: weak coupling
- $K^*\pi \to K^*\pi$: vertex renormalization



KT basis function: $a_0 = 1$ or else

And for the case $a_0 = 1$…

- $a_0\pi \to \kappa\pi$: sizable coupling



Each case $a_i, b_i, c_i = 1$ corresponds to such a basis!

Discussion

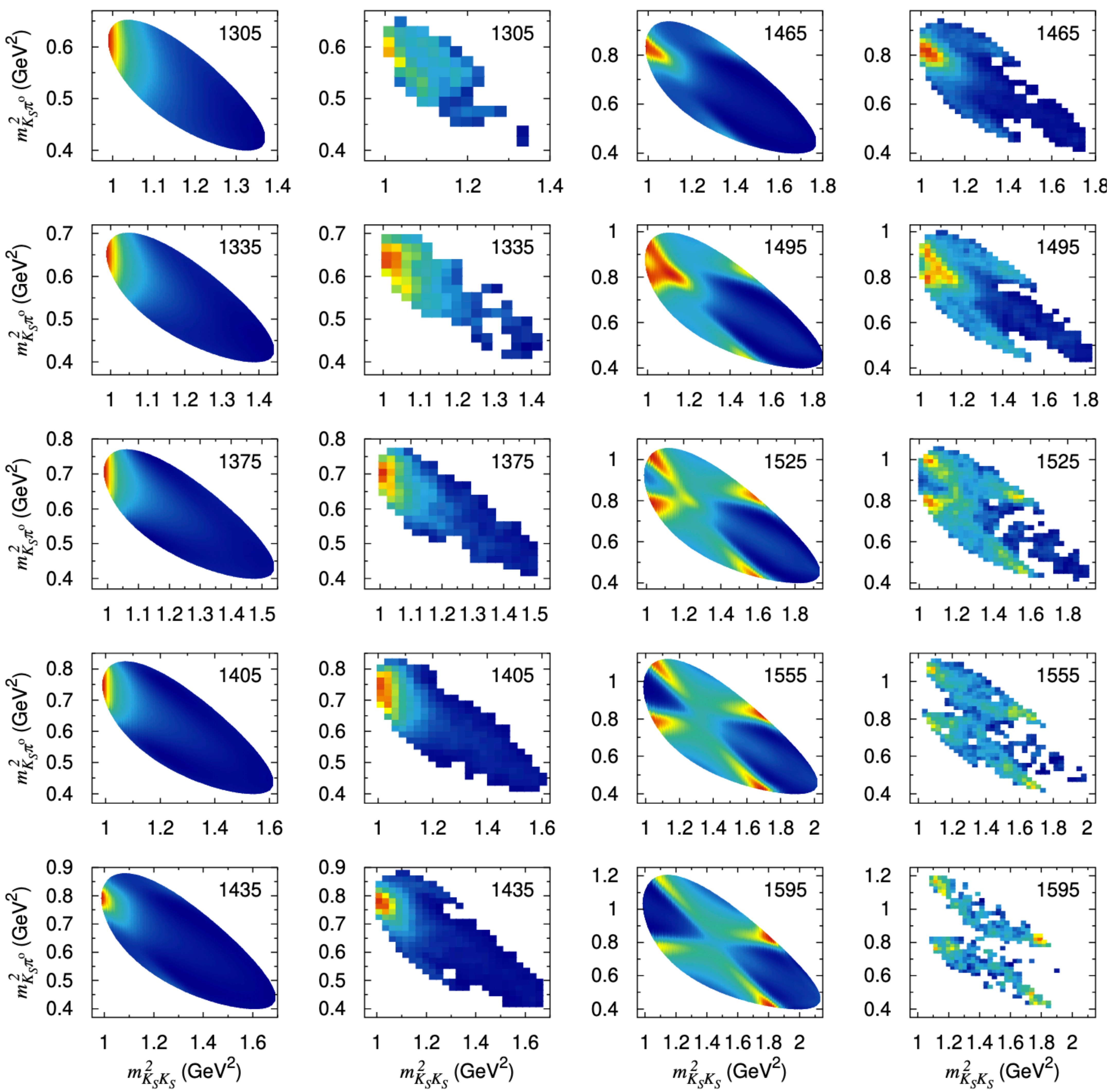
Monte-Carlo Dalitz plots

S.X.Nakamura et al., PRD.109.014021;PRD.107.L091505 BESIII, JHEP03(2023)121

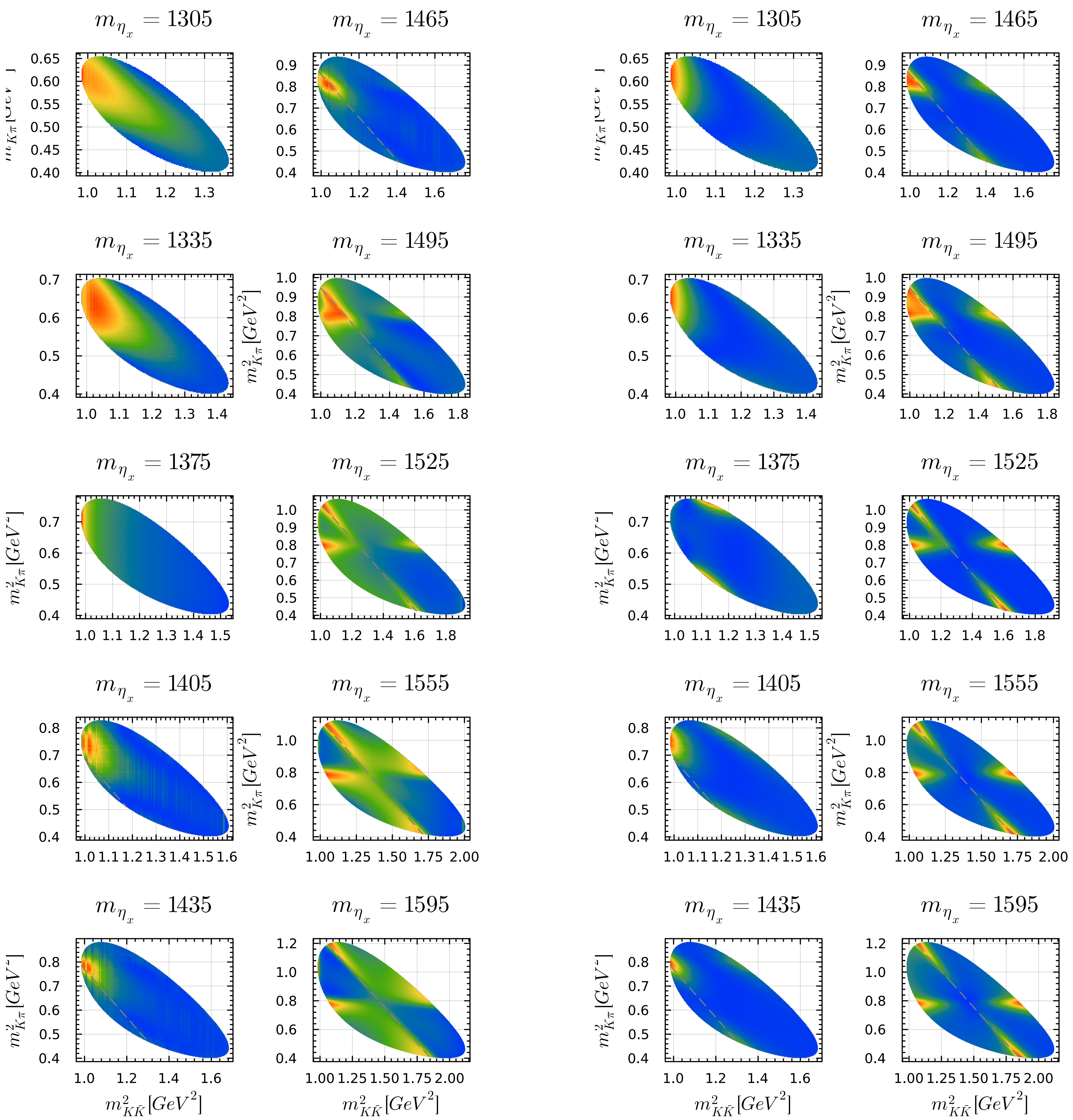
Fixed-x description

↪ scale-dependent subtractions?

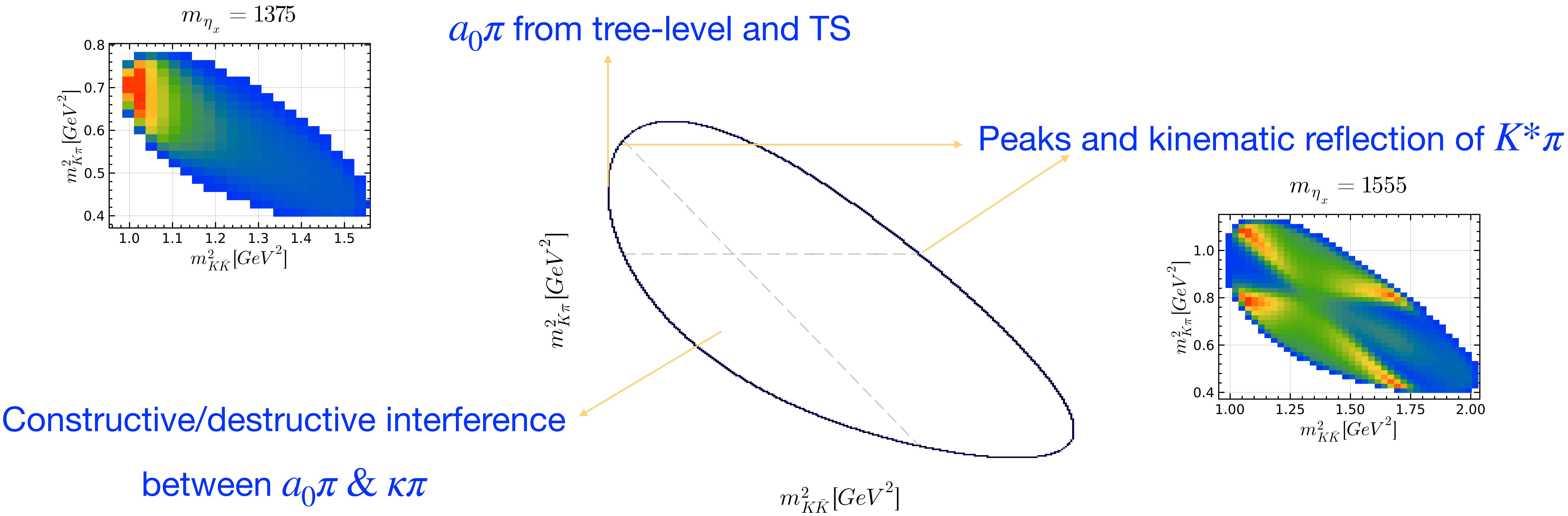
S.X. Nakamura's | MC | S.X. Nakamura's | MC



$K\bar{K}[S] + K\pi[S,P]$ | $K\pi[P]$



Analysis from Monte-Carlo data



On real axis (Data), our model is consistent with S.X. Nakamura, PRD109.014021;107.L091505

On complex plane (continuation), we will try to gain some insights into the poles.

Summary & Outlook

- The nature of iso-scalar pseudo-scalar states and their dynamics involved are still beyond our knowledge;

- The 2-body $K\bar{K}\pi$ FSIs have been established dispersively (almost model-independently) and the 3-body ones are almost ready$\Rightarrow$ $\eta\pi\pi$,3π etc

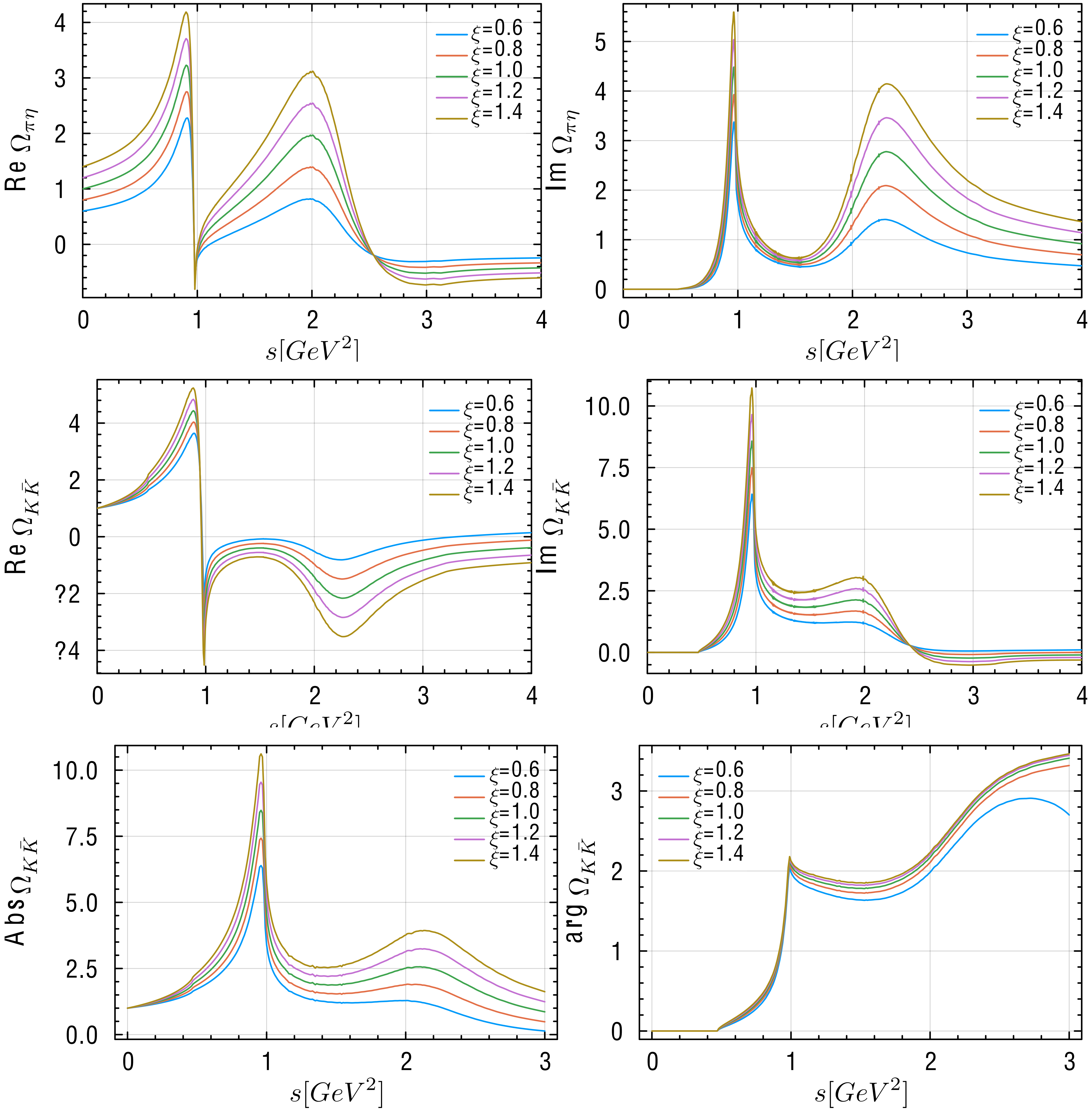
- The above treatment proceeds similarly for generic 3-body scatterings in a modern & sophisticated perspective $\Rightarrow$ $f_1(1285), f_1(1420), a_1(1260)$

- The comprehensive understanding of those states relies on the inclusions of more robust experimental data (upcoming) and more fundamental theories such as χPT (setting up)

Thank you!

Spares

ξ-dependence in $\pi\eta - K\bar{K}$ coupling



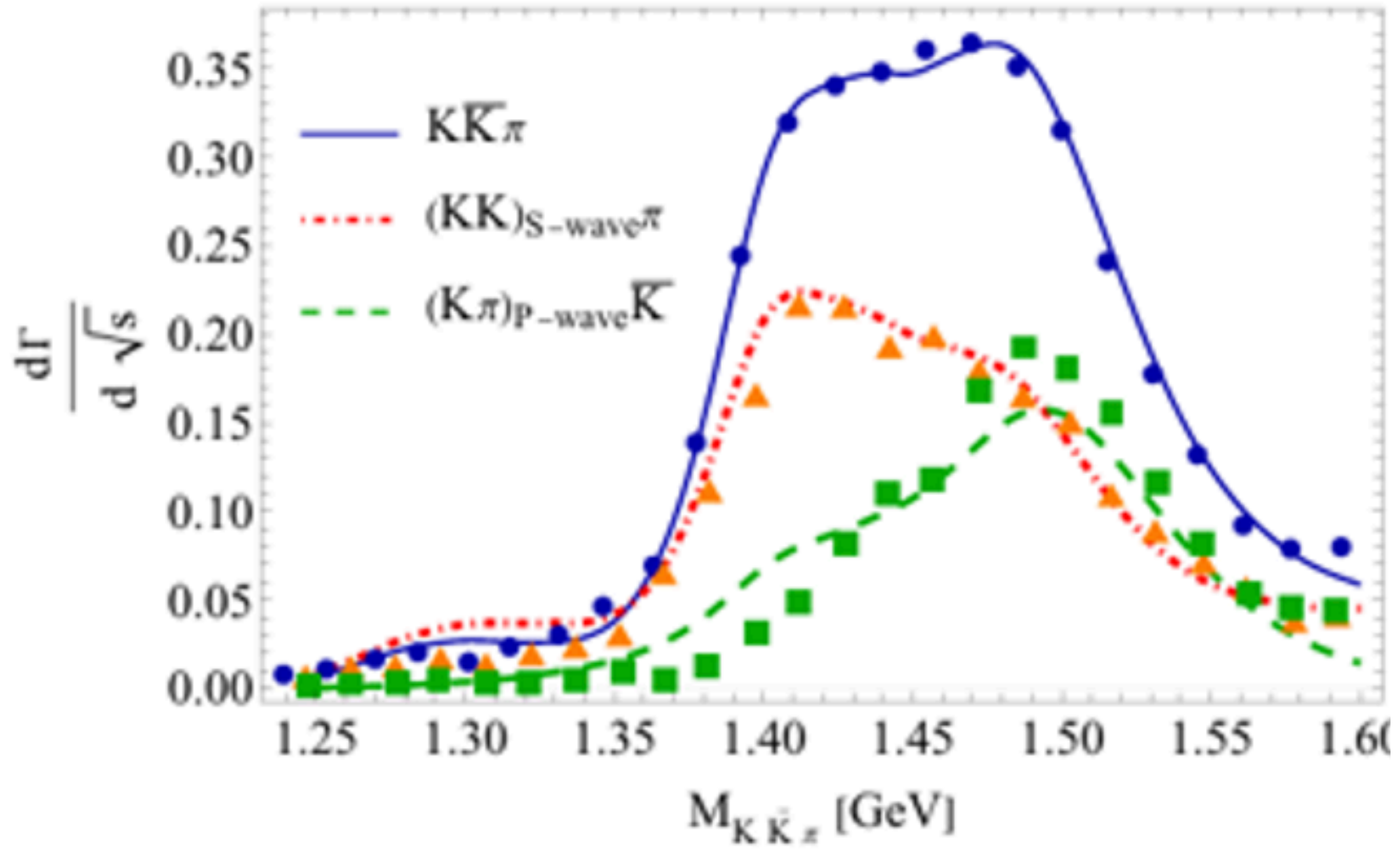
The pole of $a_0(980)$ determined from phase shifts below $K\bar{K}$ almost does not change!

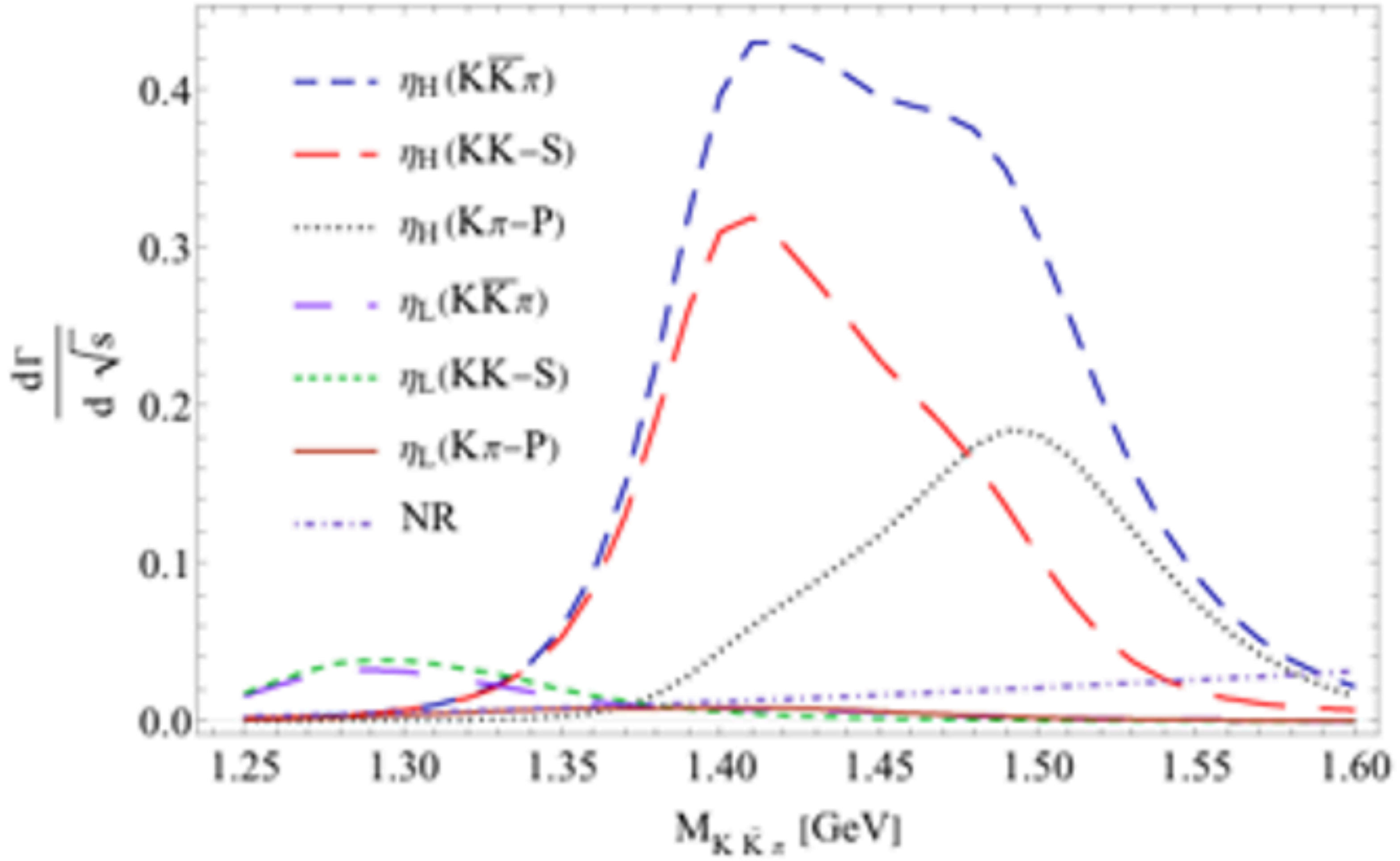
Fitting scheme up to one-loop level

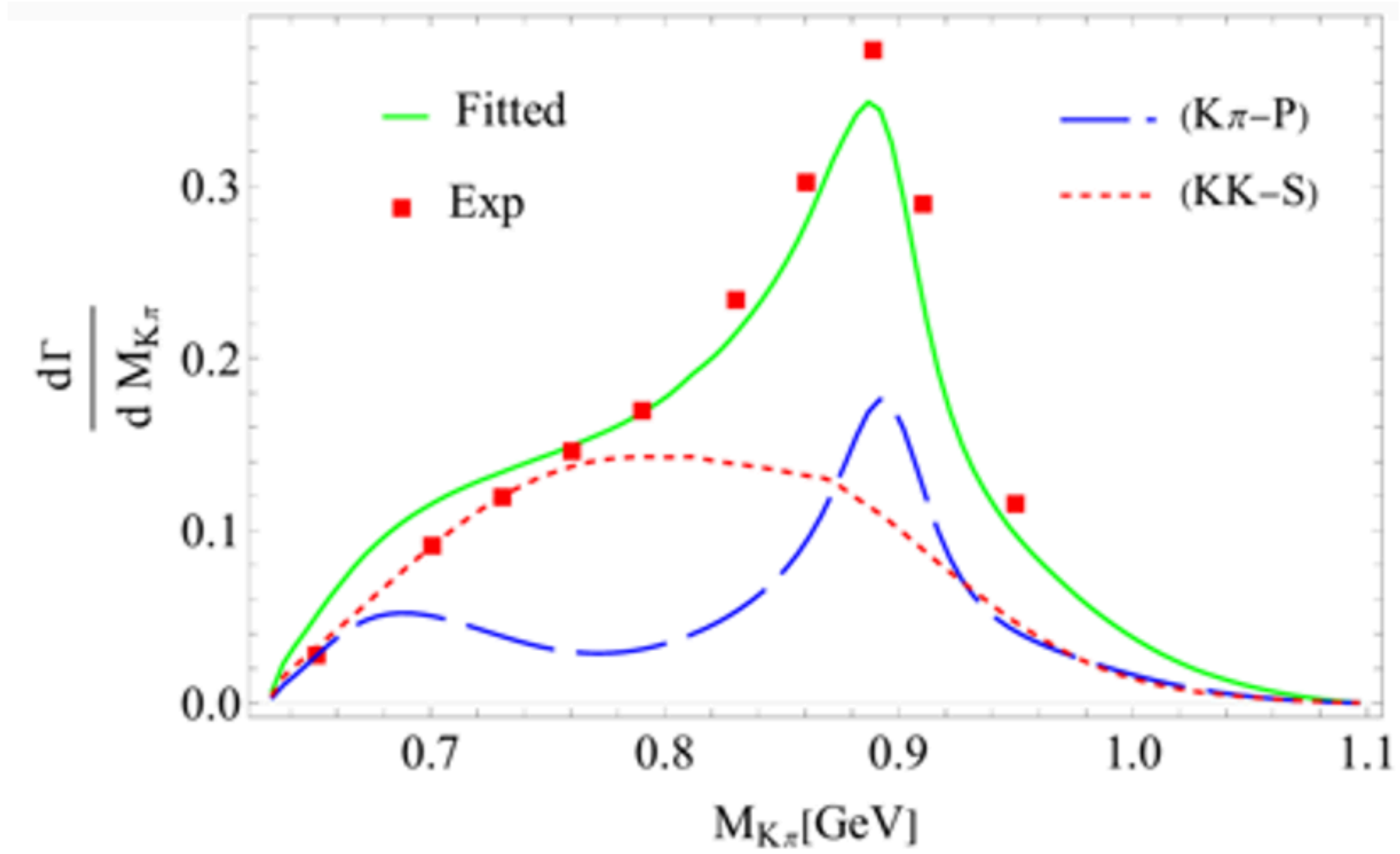
Y.Cheng et al., arXiv:2407.10234

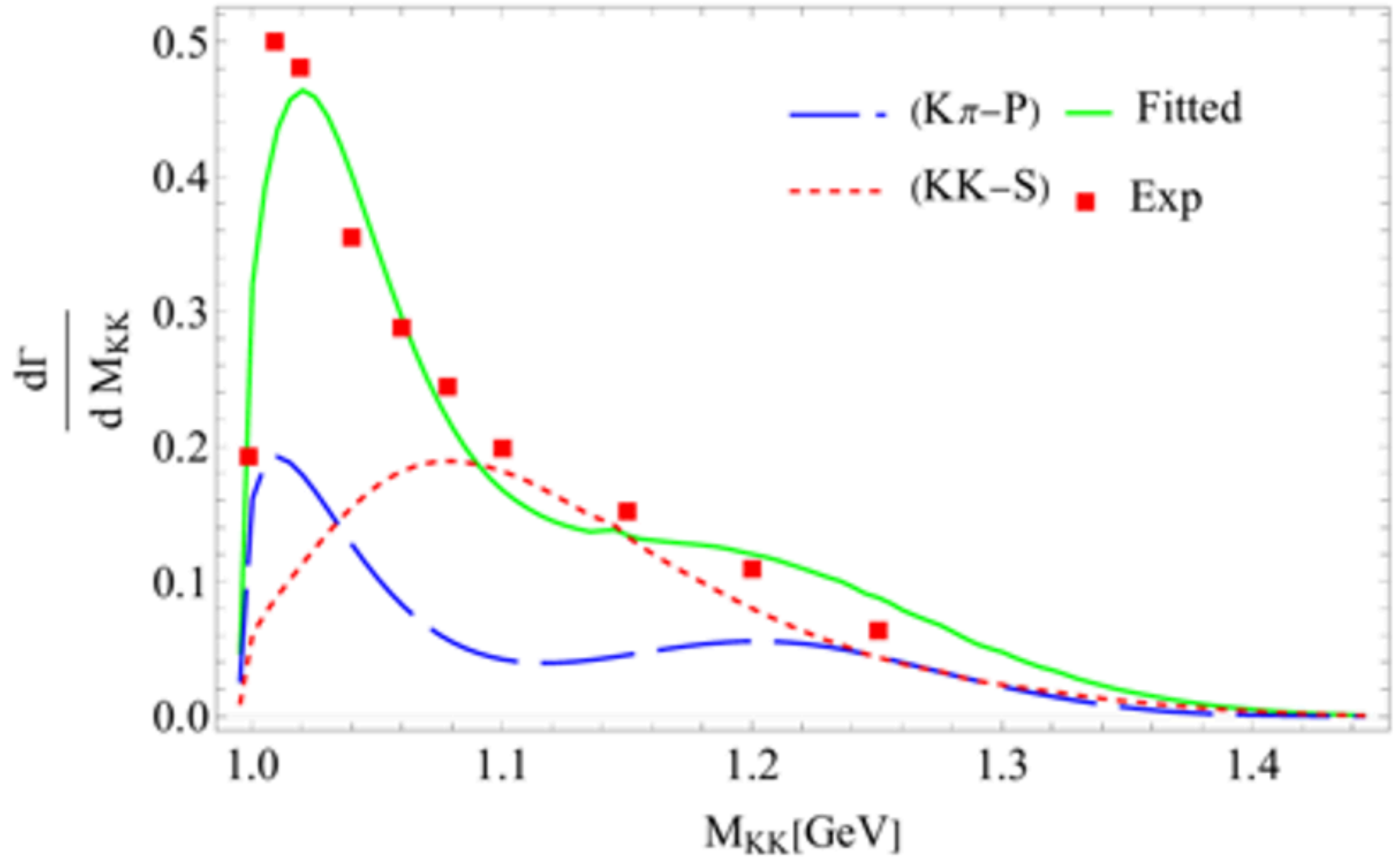
The preliminary Khrui-Trieman study implies that

- Most corrections above two-loops shall be able to be absorbed into the vertex, propagators…
- The one-loop approximation (TS mechanism) may be reasonable









Isobaric approach

- $\eta(1295)$ & $\eta(1440)$ intermediators

🙂 3-body spectrum

🙂 2-body spectra

🙂 Dalitz plots

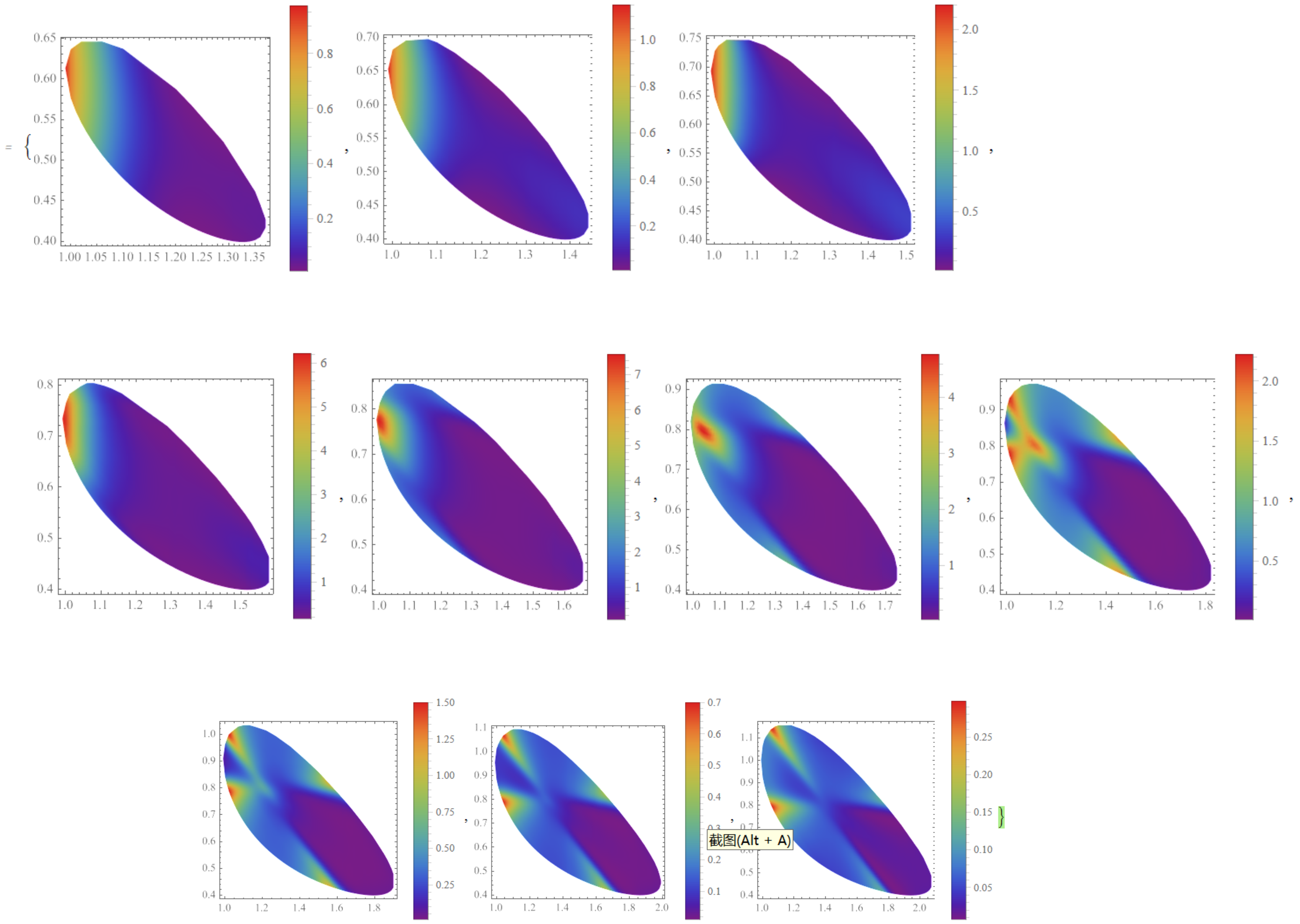
} BESIII spectra (3-body spectrum, 2-body spectra)

} BESIII MC (2-body spectra, Dalitz plots)

A comprehensive dispersive analysis is on the way!

MC fitting in isobaric approach

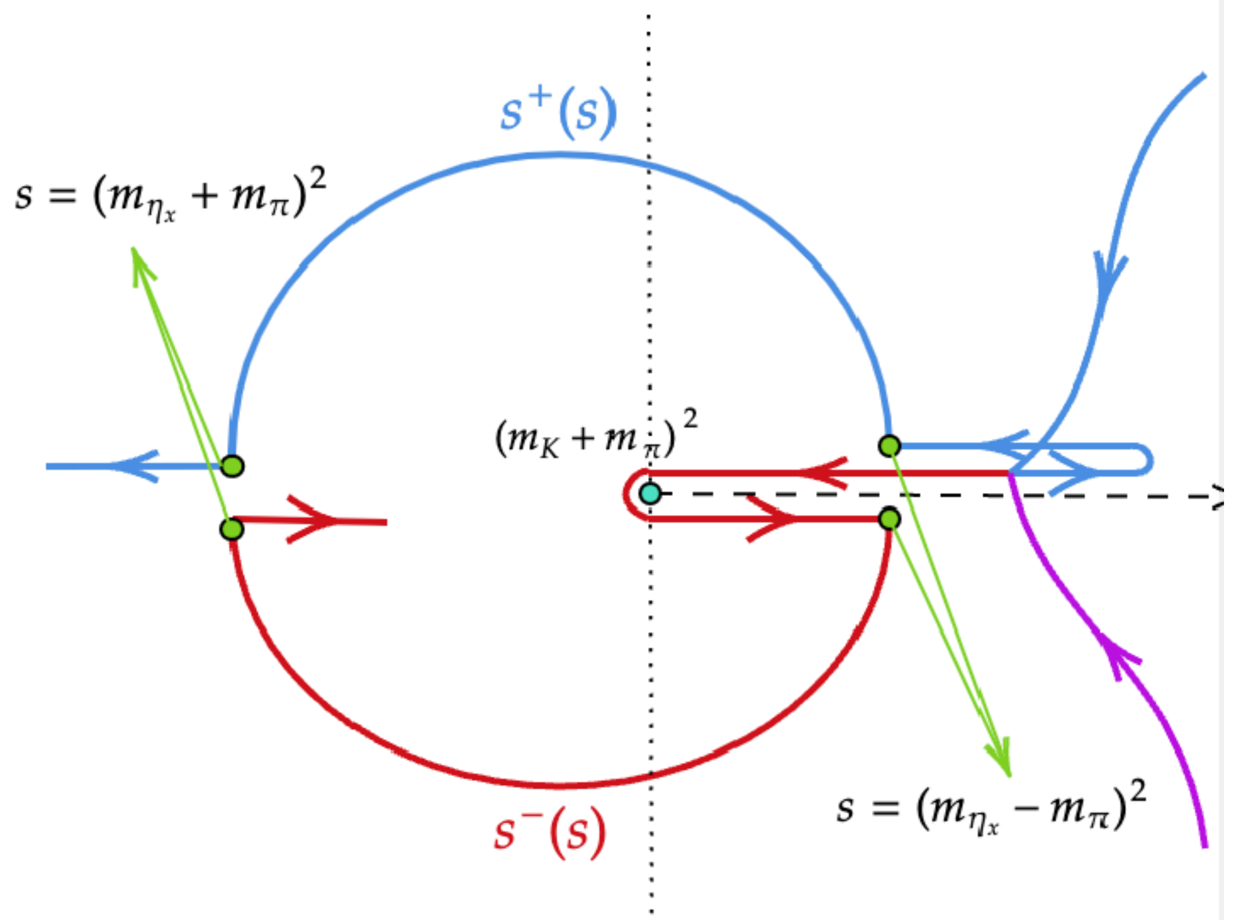
1300~30~1600 MeV



Pseudo-threshold singularity and its nature

$$\kappa_{K\bar{K}}(s) = \frac{\sqrt{\lambda(s,m_K^2,m_K^2)}\sqrt{(m_{\eta_x}-m_\pi)^2-s+i\epsilon}\sqrt{(m_{\eta_x}+m_\pi)^2-s+i\epsilon}}{s}$$

$$\kappa_{\pi K}(t) = \frac{\sqrt{\lambda(t,m_\pi^2,m_K^2)}\sqrt{(m_{\eta_x}-m_K)^2-t+i\epsilon}\sqrt{(m_{\eta_x}+m_K)^2-t+i\epsilon}}{t}$$

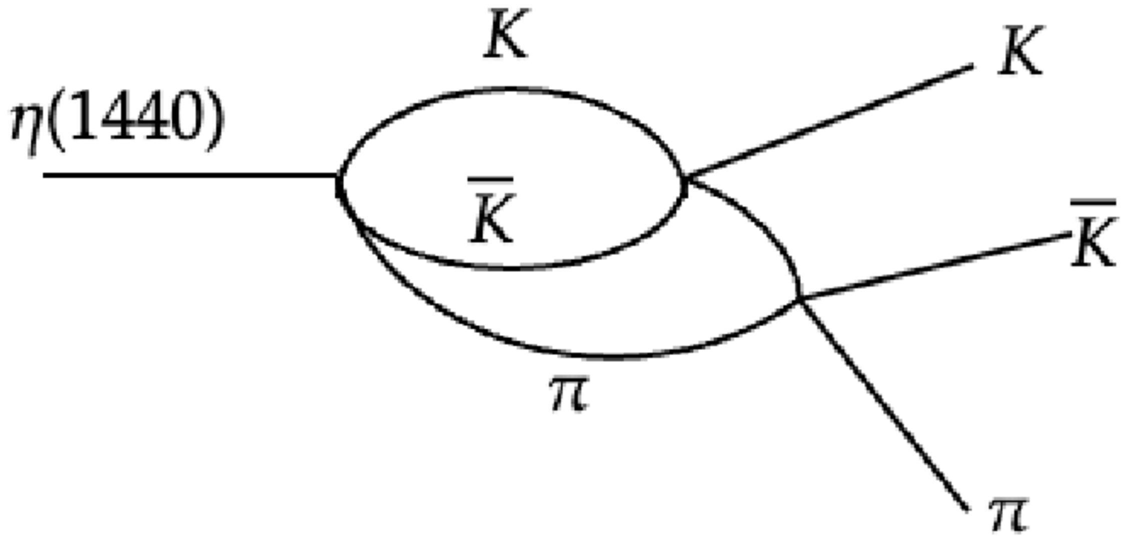


The singular behaviour of $\hat{\mathcal{F}}_J^I(x)$ at pseudo-threshold is $\frac{\tilde{\mathcal{F}}_J^I(x)}{\kappa^{2J+1}(x)} \propto \frac{1}{\sqrt{a_x - x}^{2J+1}}$:

1. manifests both when solving $\mathcal{F}_J^I(x)$ and $\hat{\mathcal{F}}_J^I(x)$
2. S-wave $(J=0)$ $\Rightarrow$ integrable numerically
3. above S-wave $(J>0)$ $\Rightarrow$ very hard to integrate numerically

The integral $H(x) = \frac{x^n}{\pi}\int \frac{dx'}{x'^n}\frac{\hat{\mathcal{F}}(x')\sin\delta(x')}{|\Omega(x')|(x'-x)}$ J.Gasser, NPB850(2011)96-147

1. is finite on physical sheet, i.e., $H(a_x + i\epsilon)$
2. disc $H(a_x) = H(a_x + i\epsilon) - H(a_x - i\epsilon) = \infty$
3. can be evaluated both analytically and numerically



Avoiding the pseudo-threshold singularity

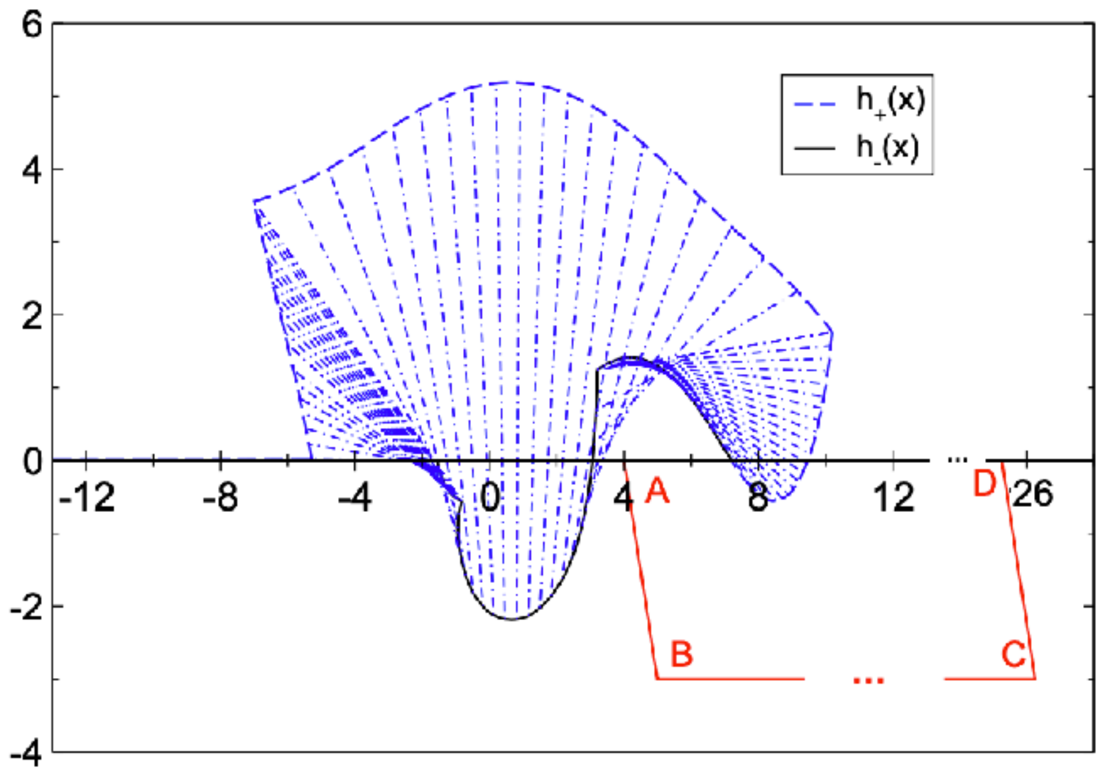
$$H(x+i\epsilon)=\frac{x^n}{\pi}\int\frac{dx'}{x'^n}\frac{\tilde{\mathcal{F}}^I_J(x')}{\kappa^{2J+1}(x')(x'-x-i\epsilon)}\frac{\sin\delta^I_J(x')}{|\Omega^I_J(x')|}$$

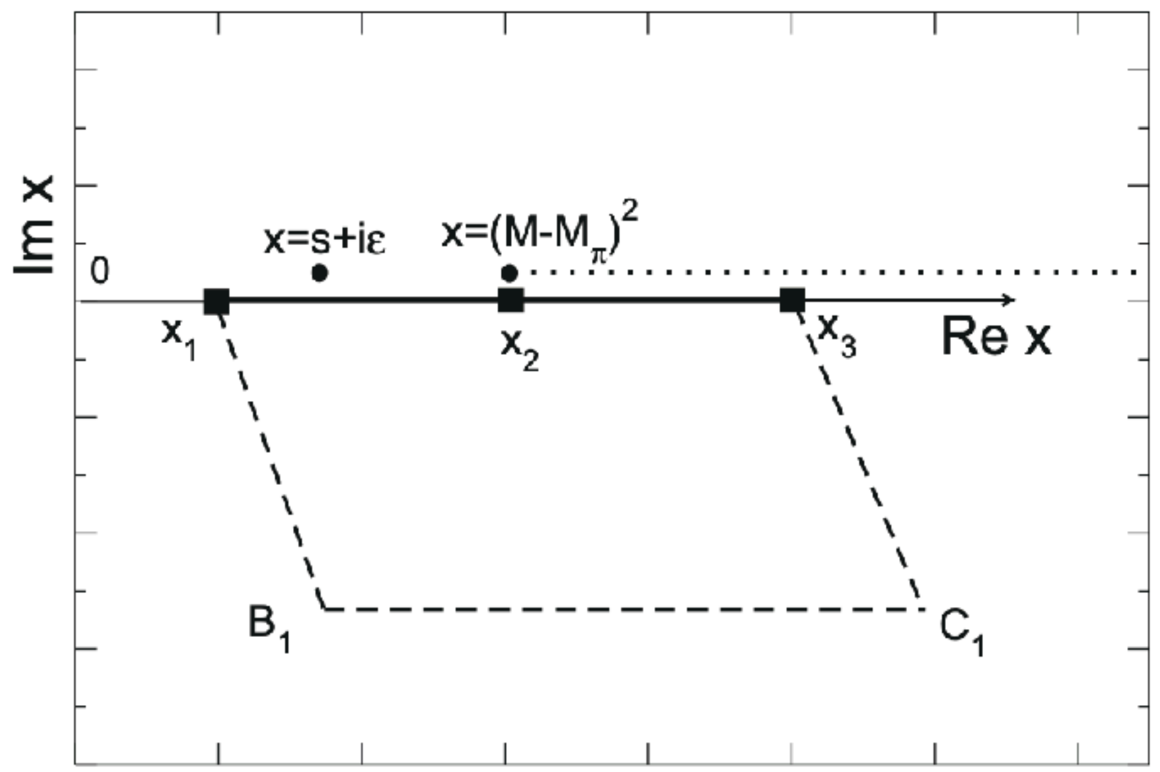
- Analytical approach G.Colangelo et al.,EPJC(2018)78:947

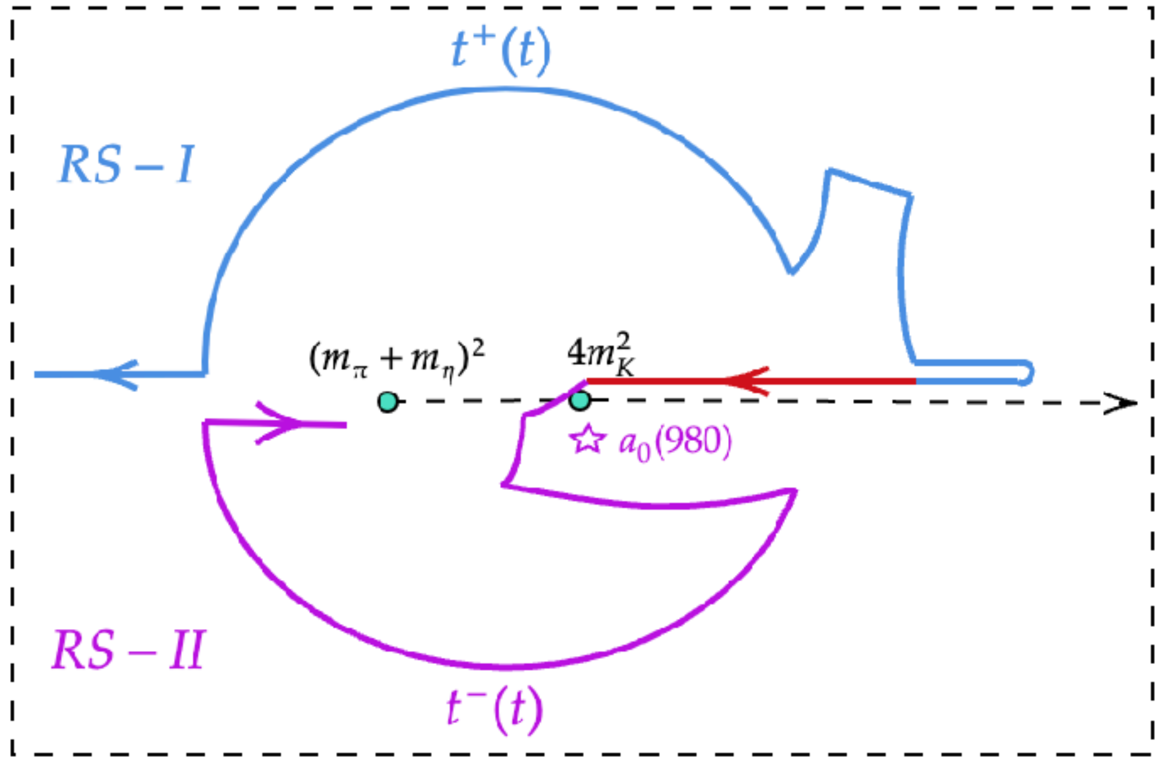
$$\mathcal{M}^H_1(s)=\Omega_1(s)\{\int_{s_1}^{s_3}ds'\frac{\bar\phi(s')H_1(s')-h(s')\bar\phi(s_2)H_1(s_2)}{(s'-s-i\epsilon)(s_2-s')^{3/2}}+\bar\phi(s_2)H_1(s_2)G(s)\}$$

J.Gasser and A.Rusetsky,EPJC(2018)78:906

- Contour deformation without crossing the pole positions (δ^I_J diverges at pole)







- Contour deformation even crossing the pole positions:

$\frac{\sin\delta^I_J(x')}{|\Omega^I_J(x')|}$ is free of the singularities $\Rightarrow$ { the singularity is avoided; integrate on elastic complex region now!

Analytical continuation of $\sin\delta_J^I / |\Omega_J^I|$ (1)

For 2-body elastic scattering,

$$f_l(s) = \frac{e^{i2\delta_l(s)} - 1}{2i\sigma(s)} = \frac{1}{\sigma(s)} \cdot \frac{1}{\cot\delta_l(s) - i}$$

with $\cot\delta_l(s)$ real and satisfying Schwartz reflection theorem and can be expanded by conformal polynomials on a certain analytical region.
The S-matrix is then,

$$\hat{S}(s) = \begin{cases} 1 + 2i\sigma f_l(s) = \frac{\cot\delta_l(s)+i}{\cot\delta_l(s)-i}, & \Im s \geq 0 \\ [\frac{\cot\delta_l(s^*)+i}{\cot\delta_l(s^*)-i}]^* = \frac{\cot\delta_l(s)-i}{\cot\delta_l(s)+i}, & \Im s < 0. \end{cases}$$

By utilizing $\frac{\sin\delta_l(s)}{|\Omega_l(s)|} = \frac{e^{i\delta_l(s)}\sin\delta_l(s)}{\Omega_l(s)} = \frac{1}{\Omega_l(s)} \cdot \frac{1}{\cot\delta_l(s)-i}$ and $\Omega_l^{(II)}(s) = \frac{\Omega^{(I)}(s)}{\hat{S}(s)}$, one derives,

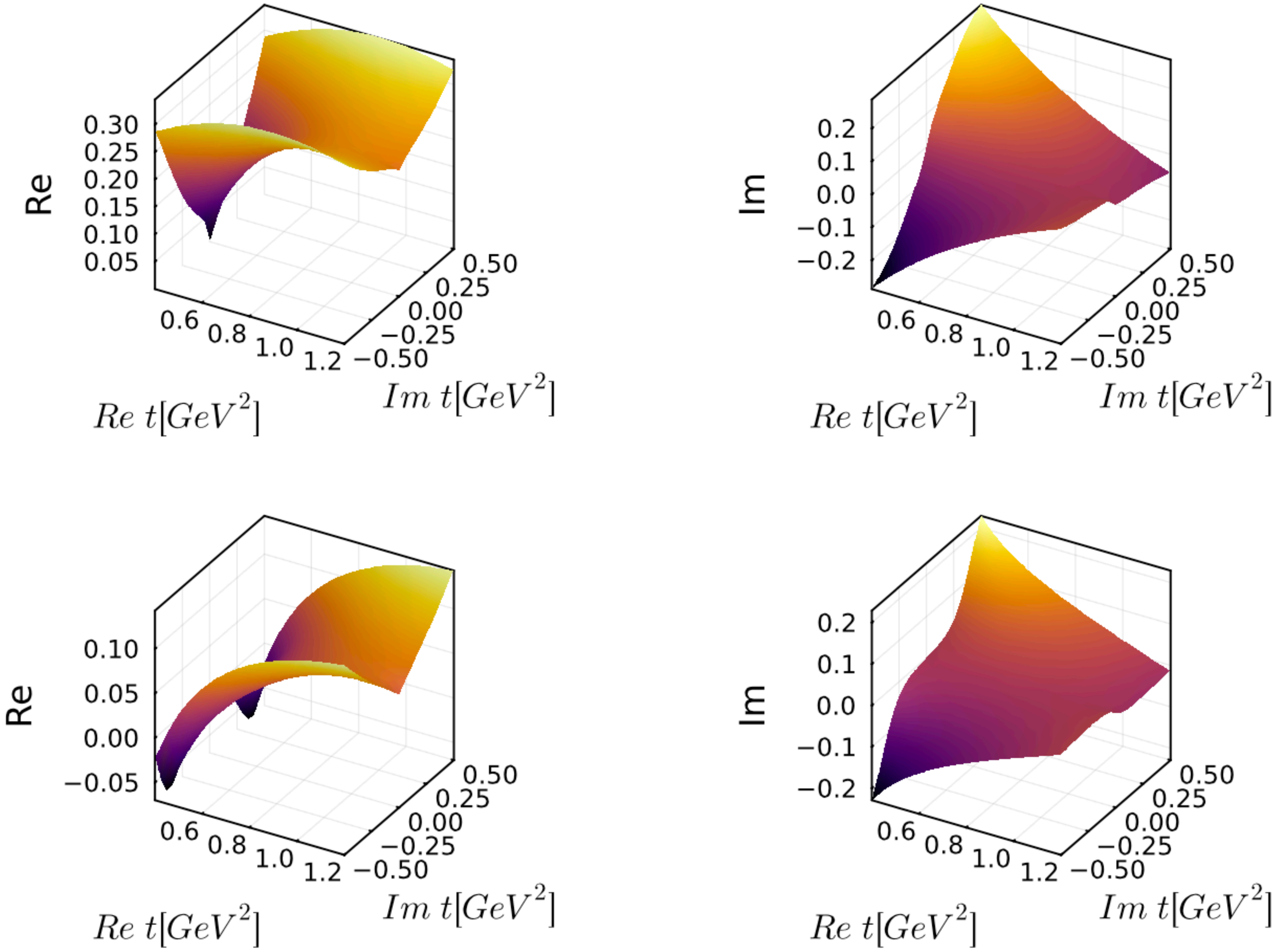
$$\frac{\sin\delta_l(s)}{|\Omega_l(s)|} = \begin{cases} \frac{1}{\Omega_l^{(I)}(s)} \frac{1}{\cot\delta_l(s)-i}, & \Im s \geq 0 \\ \frac{1}{\Omega_l^{(I)}(s)} \frac{1}{\cot\delta_l(s)+i}, & \Im s < 0 \end{cases} .$$

The convention of $\cot\delta_l(s)$ may differentiate from the literature by an extra minus sign on the lower half plane but the conclusion shall not change!

Analytical continuation of $\sin \delta_J^I / |\Omega_J^I|$ *(2)*

The complex function $\dfrac{\sin \delta_{0,1}^{1/2}(s)}{|\Omega_{0,1}^{1/2}(s)|}$ of $K\pi$ scatterings are plotted below,

$$\frac{\sin \delta_{0,1}^{1/2}(t)}{|\,\Omega_{0,1}^{1/2}(t)|}$$



The dispersive integral on any deformed integral-path on the lower half plane (even crossing the pole position) has been checked to be consistent with that integrated from the real axis!