1. $l(\theta, x) = \ln p(x,\theta)$，$p(x,\theta)$：似然函数，$\theta = (B,\phi)$：参数
2. (**score function** $k \times 1$) $s(\theta, x) = \frac{\partial l}{\partial \theta}$
3. (**Fisher Information**) $i(\theta) = E_\theta(s(\theta,x)s(\theta,x)^T)$
4. (**Cramér-Rao inequality**) For $E_\theta(\hat\theta) = \theta$, $Var(\hat\theta) \geq i(\theta)^{-1}$ （若 $\hat\theta$ 是 θ 的无偏估计量，它的协方差矩阵不小于 $i(\theta)^{-1}$）
5. (极大似然估计的渐进有效性) 若 $\hat\theta$ 是极大似然估计量 $\hat\theta_{ML}$ 时，则 $\sqrt{n}(\hat\theta_{ML} - \theta) \stackrel{n\to\infty}{\longrightarrow} N(0, i(\theta)^{-1})$
6. 2x2 实对称矩阵 M 的逆，一般公式是：

$$M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} \frac{1}{A - \frac{B^2}{C}} & \frac{1}{\frac{AC}{B} - B} \\ \frac{1}{\frac{AC}{B} - B} & \frac{1}{C - \frac{B^2}{A}} \end{bmatrix}$$

7. Fisher 信息及其逆

$$i = \begin{bmatrix} i_{11} & i_{12} \\ i_{12} & i_{22} \end{bmatrix}, \qquad \text{协方差矩阵} var = i^{-1} = \begin{bmatrix} (i_{11} - \frac{i_{12}^2}{i_{22}})^{-1} & (\frac{i_{11} i_{22}}{i_{12}} - i_{12})^{-1} \\ (\frac{i_{11} i_{22}}{i_{12}} - i_{12})^{-1} & (i_{22} - \frac{i_{12}^2}{i_{11}})^{-1} \end{bmatrix}$$

- x： 事例数， x_0 ： 理论事例数
- $x = Lum * eff * \sigma(s, B, \phi)$
- DB:= $\frac{\partial x_0}{\partial B}$, Dphi:= $\frac{\partial x_0}{\partial \phi}$

1. 情况 1： 一个能量点（用作验算）
   - 似然函数 p

$$p(x; B, \phi) = Poisson(x|x_0(B, \phi)) \approx N(x|x_0, x_0) = \frac{1}{\sqrt{2\pi x_0}} Exp[-\frac{(x - x_0)^2}{2x_0}] \tag{1}$$

   - 对数似然函数的导数

$$\frac{\partial l}{\partial B} = -\frac{1}{2}(\frac{1}{x_0} + 1 - \frac{x^2}{x_0^2}) \cdot \mathrm{DB}, \quad \frac{\partial l}{\partial \phi} = -\frac{1}{2}(\frac{1}{x_0} + 1 - \frac{x^2}{x_0^2}) \cdot \mathrm{Dphi} \tag{2}$$

1 情况 1：一个能量点（用作验算）

- Fisher 信息

$$i_{11}(s,B,\phi)=\int(\frac{\partial l}{\partial B})^2 p(x,\theta)\,\mathrm{d}x,\quad i_{22}(s,B,\phi)=\int(\frac{\partial l}{\partial \phi})^2 p(x,\theta)\,\mathrm{d}x \tag{3}$$

$$i_{12}(s,B,\phi)=i_{21}(\theta,r)=\int(\frac{\partial l}{\partial B}\frac{\partial l}{\partial \phi})p(x,\theta)\,\mathrm{d}x \tag{4}$$

- 均为 x 的高斯型积分，积出如下

$$i_{11}=\frac{1+2x_0}{2x_0^2}\mathrm{DB}^2,\quad i_{22}=\frac{1+2x_0}{2x_0^2}\mathrm{Dphi}^2,\quad i_{12}=\frac{1+2x_0}{2x_0^2}\mathrm{DB}\cdot\mathrm{Dphi} \tag{5}$$

$$i_{11}-\frac{i_{12}^2}{i_{22}}=\frac{i_{11}i_{22}}{i_{12}}-i_{12}=i_{22}-\frac{i_{12}^2}{i_{11}}=0$$

- i^{-1} 的矩阵元均为 $1/0$，即误差无穷大，说明只有一个截面无法拟合出两个参数

1. 情况 2：两个能量点 s_1 和 s_2 (s_2 对应 3773)

- 似然函数 p（变量的下标 1 和 2 对应 s_1 和 s_2）

$$p(x_1, x_2; B, \phi) = \frac{1}{\sqrt{2\pi x_{01}}} Exp[-\frac{(x_1 - x_{01})^2}{2x_{01}}] * \frac{1}{\sqrt{2\pi x_{02}}} Exp[-\frac{(x_2 - x_{02})^2}{2x_{02}}] \tag{6}$$

- Fisher 信息

$$i_{11}(s, B, \phi) = \int\int (\frac{\partial l}{\partial B})^2 p(x_1, x_2; \theta)\, \mathrm{d}x_1\, \mathrm{d}x_2, \quad i_{22}(s, B, \phi) = \int\int (\frac{\partial l}{\partial \phi})^2 p(x_1, x_2; \theta)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{7}$$

$$i_{12}(s, B, \phi) = i_{21}(\theta, r) = \int\int (\frac{\partial l}{\partial B}\frac{\partial l}{\partial \phi}) p(x_1, x_2; \theta)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{8}$$

1. 情况 2：两个能量点 s_1 和 s_2 (s_2 对应 3773)
   - 积分得到

$$
\begin{aligned}
i_{11} &= \frac{1+2x_{01}}{2x_{01}^2}\mathrm{DB_1}^2 + \frac{1+2x_{02}}{2x_{02}^2}\mathrm{DB_2}^2 \\
i_{22} &= \frac{1+2x_{01}}{2x_{01}^2}\mathrm{Dphi_1}^2 + \frac{1+2x_{02}}{2x_{02}^2}\mathrm{Dphi_2}^2 \\
i_{12} &= \frac{1+2x_{01}}{2x_{01}^2}\mathrm{DB_1}\cdot\mathrm{Dphi_1} + \frac{1+2x_{02}}{2x_{02}^2}\mathrm{DB_2}\cdot\mathrm{Dphi_2}
\end{aligned} \tag{9}
$$

- B 的方差的倒数

$$i_{11}-\frac{i_{12}^2}{i_{22}}=\frac{1}{2}\left(\frac{(1+2x_{01})\mathrm{DB_1}^2}{x_{01}^2}+\frac{(1+2x_{02})\mathrm{DB_2}^2}{x_{02}^2}\right)-\frac{\left(\frac{(1+2x_{01})\mathrm{DB_1}\cdot\mathrm{Dphi_1}}{x_{01}^2}+\frac{(1+2x_{02})\mathrm{DB_2}\cdot\mathrm{Dphi_2}}{x_{02}^2}\right)^2}{2\left(\frac{(1+2x_{01})\mathrm{Dphi_1}^2}{x_{01}^2}+\frac{(1+2x_{02})\mathrm{Dphi_2}^2}{x_{02}^2}\right)}$$

- 通分得到

$$i_{11}-\frac{i_{12}^2}{i_{22}}=\frac{(\mathrm{DB_2\ Dphi_1-DB_1\ Dphi_2})^2(1+2x_{01})(1+2x_{02})}{2\left(\mathrm{Dphi_1}^2(1+2x_{01})x_{02}^2+\mathrm{Dphi_2}^2(1+2x_{02})x_{01}^2\right)}$$

1. 记 continuum 振幅为 A_c，ψ'' 振幅不含 B 的部分为 A_R，则 $x_0 \sim |A_c + A_R\sqrt{B} * e^{i\phi}|^2$
2. 记 ψ'' 质量为 s_R，则 $s = s_R$ 时，A_R 为纯虚数，$A_R = |A_R| i$
3. 此时 x_0 为

$$\begin{aligned} x_0 &\sim |A_c + A_R\sqrt{B} * e^{i\phi}|^2 \\ (s = s_R) \quad &= |A_c + |A_R|\sqrt{B} * ie^{i\phi}|^2 \\ &\sim \sin\phi + (\phi \text{ independent terms}) \end{aligned} \tag{10}$$

4. 上式说明 x_0 中含 ϕ 的项正比于 $\sin\phi$，那么 $\mathrm{Dphi} \sim \cos\phi$，若再取 $\phi = 270°$

$$\mathrm{Dphi}|_{s=s_R,\ \phi=270°} = 0, \quad \forall B \tag{11}$$

5. 上式与 B 取值无关。$s_2 = s_R$，$\phi_2 = 270°$ 时，Fisher 信息中 s_1 的项消失

$$i_{11} - \frac{i_{12}^2}{i_{22}} = \frac{\mathrm{DB}_2^2(1 + 2x_{02})}{2x_{02}^2}\Big|_{s_2=s_R,\ \phi_2=270°} \tag{12}$$

6. Lum $\to \infty$

$$i_{11} - \frac{i_{12}^2}{i_{22}} \longrightarrow \frac{(\mathrm{DB}_2\ \mathrm{Dphi}_1 - \mathrm{DB}_1\ \mathrm{Dphi}_2)^2}{{\mathrm{Dphi}_1}^2 x_{02} + {\mathrm{Dphi}_2}^2 x_{01}} \overset{s=s_R,\ \phi=270°}{=} \frac{\mathrm{DB}_2^2}{x_{02}} \tag{13}$$

HESS

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}, \qquad var = h^{-1} = \begin{bmatrix} (h_{11} - \frac{h_{12}^2}{h_{22}})^{-1} & (\frac{h_{11}h_{22}}{h_{12}} - h_{12})^{-1} \\ (\frac{h_{11}h_{22}}{h_{12}} - h_{12})^{-1} & (h_{22} - \frac{h_{12}^2}{h_{11}})^{-1} \end{bmatrix}$$

$$h_{11} = \frac{\partial^2 \chi^2}{\partial B^2}, \quad h_{11} = \frac{\partial^2 \chi^2}{\partial \phi^2}, \quad h_{12} = \frac{\partial^2 \chi^2}{\partial B \partial \phi} \tag{14}$$

HESS

$$h_{11} = \frac{2}{x_1}[(\frac{\partial x_{01}}{\partial B})^2 - (x_1 - x_{01})(-\frac{\partial^2 x_{01}}{\partial B^2})] + \frac{2}{x_2}[(\frac{\partial x_{02}}{\partial B})^2 - (x_2 - x_{02})(-\frac{\partial^2 x_{02}}{\partial B^2})] \quad (15)$$

$$h_{22} = \frac{2}{x_1}[(\frac{\partial x_{01}}{\partial \phi})^2 - (x_1 - x_{01})(-\frac{\partial^2 x_{01}}{\partial \phi^2})] + \frac{2}{x_2}[(\frac{\partial x_{02}}{\partial \phi})^2 - (x_2 - x_{02})(-\frac{\partial^2 x_{02}}{\partial \phi^2})] \quad (16)$$

$$h_{12} = \frac{2}{x_1}[(\frac{\partial x_{01}}{\partial B}\frac{\partial x_{01}}{\partial \phi}) - (x_1 - x_{01})(-\frac{\partial^2 x_{01}}{\partial B \partial \phi})] + \frac{2}{x_2}[(\frac{\partial x_{02}}{\partial B}\frac{\partial x_{02}}{\partial \phi}) - (x_2 - x_{02})(-\frac{\partial^2 x_{02}}{\partial B \partial \phi})] \quad (17)$$

$$h_{11} - \frac{h_{12}^2}{h_{22}} \xrightarrow{(x-x_0)\to 0} \frac{2(\mathrm{DB}_2\ \mathrm{Dphi}_1 - \mathrm{DB}_1\ \mathrm{Dphi}_2)^2}{\mathrm{Dphi}_1{}^2 x_2 + \mathrm{Dphi}_2{}^2 x_1}$$

上式与 (13) 式有相同的形式