# Heavy-flavor mesons in a strong electric field

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# - Introduction -

- Cornell potential is very widely used in describing the confinement feature of heavy-flavor mesons, which consist of a Coulomb potential and a linear potential (isotropic, tends to bind the hadrons together).
- There is already a well-known linear potential in classic physics, which is the vector potential of electric field (anisotropic, tends to split hadrons apart).

# - General Formalism -

- The mesonic wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ satisfies two-body Schroedinger Eq.

$$\left[\sum_{i=1}^{d}\left(\frac{\hat{p}_{1i}^2}{2m_1}+\frac{\hat{p}_{2i}^2}{2m_2}\right)+\sigma|\mathbf{r}_1-\mathbf{r}_2|-q_1\varepsilon x_1-q_2\varepsilon x_2-E\right]\Psi=0.$$

- The Coulomb part in Cornell potential is neglected, and only linear confining part is left (long-range scenario).
- Redefine the independent coordinates

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}, \qquad \mathbf{r} \equiv \mathbf{r}_1-\mathbf{r}_2$$

- The Schroedinger Eq. becomes

$$\left(\sum_{i=1}^{d}\frac{\hat{p}_i^2}{2m}+\sigma\, r-q\varepsilon x\right)\Psi=\left(E-\sum_{i=1}^{d}\frac{\hat{P}_i^2}{2M}+Q\varepsilon X\right)\Psi.$$

- Reduced mass: $m=\frac{m_1 m_2}{m_1+m_2}$, charge: $q=\frac{m}{m_1}q_1-\frac{m}{m_2}q_2$

# - General Formalism -

- Take the variable separation for the wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r})\Phi(\mathbf{R})$

$$\left(\sum_{i=1}^{d} \frac{\hat{P}_i^2}{2M} - Q\varepsilon X\right)\Phi = E_g\Phi,$$

$$\left(\sum_{i=1}^{d} \frac{\hat{p}_i^2}{2m} + \sigma r - q\varepsilon x\right)\psi = E_r\psi,$$

Total energy: $E = E_r + E_g$

Relative energy

- The masses and charges of $c$ and $b$

$$m_c = 1.29 \text{ GeV}, \quad q_c = \frac{2}{3}e;$$

$$m_b = 4.7 \text{ GeV}, \quad q_b = -\frac{1}{3}e;$$

TABLE I. The reduced masses and charges

| Meson | $c\bar{c}$ | $c\bar{b}$ | $b\bar{c}$ | $b\bar{b}$ |
|---|---|---|---|---|
| **m/GeV** | 0.645 | 1.012 | 1.012 | 2.35 |
| **q/e** | $\frac{2}{3}$ | 0.45 | −0.45 | $-\frac{1}{3}$ |

TABLE II. The relative energies from experiments and Eq.(4)

| State | $\psi(2S)$ | $B_C(2S)$ | $\Upsilon(2S)$ | $\Upsilon(3S)$ |
|---|---|---|---|---|
| $E_r^{Exp}$(GeV) | 1.106 | 0.881 | 0.623 | 0.955 |
| $E_r^{Th}$ (GeV) | 0.735 | 0.632 | 0.477 | 0.835 |

# - One dimensional case -

- Schroedinger Eq. at $d = 1$

$$\left(\frac{\hat{p}_x^2}{2m} + \sigma\,|x| + |q|\varepsilon x\right)\psi(x) = E_r\psi(x),$$

$$\Rightarrow \begin{cases} \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - (\sigma - |q|\varepsilon)x\right]\psi_-(x) = E_r\psi_-(x), & x < 0; \\ \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + (\sigma + |q|\varepsilon)x\right]\psi_+(x) = E_r\psi_+(x), & x > 0. \end{cases}$$

Effective string tension

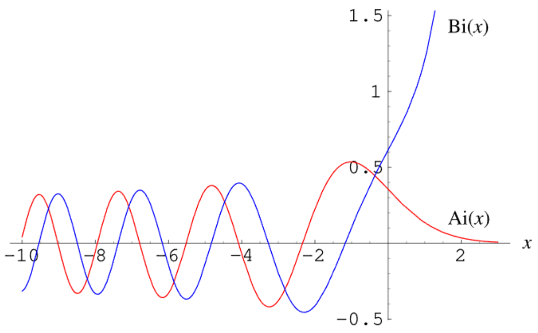
$$\begin{cases} \sigma_- \equiv \sigma - |q|\varepsilon & x < 0 \\ \sigma_+ \equiv \sigma + |q|\varepsilon & x > 0 \end{cases}$$

# - One dimensional case -

- The solutions of the above equation are Airy functions

$$\begin{cases} \psi_- = C_{1-}Ai\left(a_-\left(x+\frac{E_r}{\sigma_-}\right)\right) + C_{2-}Bi\left(a_-\left(x+\frac{E_r}{\sigma_-}\right)\right) \\ \psi_+ = C_{1+}Ai\left(a_+\left(x-\frac{E_r}{\sigma_+}\right)\right) + C_{2+}Bi\left(a_+\left(x-\frac{E_r}{\sigma_+}\right)\right) \end{cases}$$

Airy function




- To fix the eigenenergy $E_r$ and the coefficient, introduce the smooth condition and normalization

$$\psi_-(0) = \psi_+(0),\ \psi'_-(0) = \psi'_+(0);$$
$$\int_{x<0} |\psi_-(x)|^2 \mathrm{d}x + \int_{x>0} |\psi_+(x)|^2 \mathrm{d}x = 1.$$

# - One dimensional case -

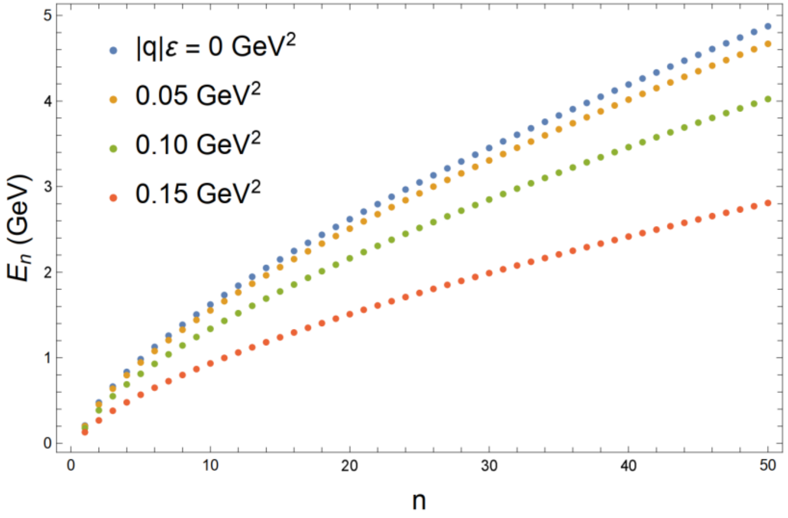
- For $x > 0$, since $a_+$ is positive definite

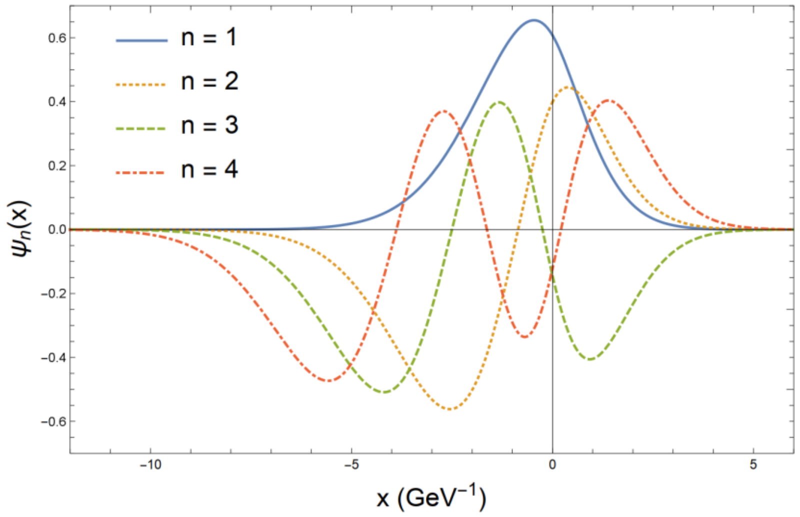
$$\lim_{x\to\infty} \psi_+(x) = 0 \quad \Longrightarrow \quad C_{2+} = 0$$

- For $x < 0$, if $a_-$ is negative ($\sigma > |q|\varepsilon$)

$$\lim_{x\to-\infty} \psi_-(x) = 0 \quad \Longrightarrow \quad C_{2-} = 0$$

Combined with smooth condition and normalization

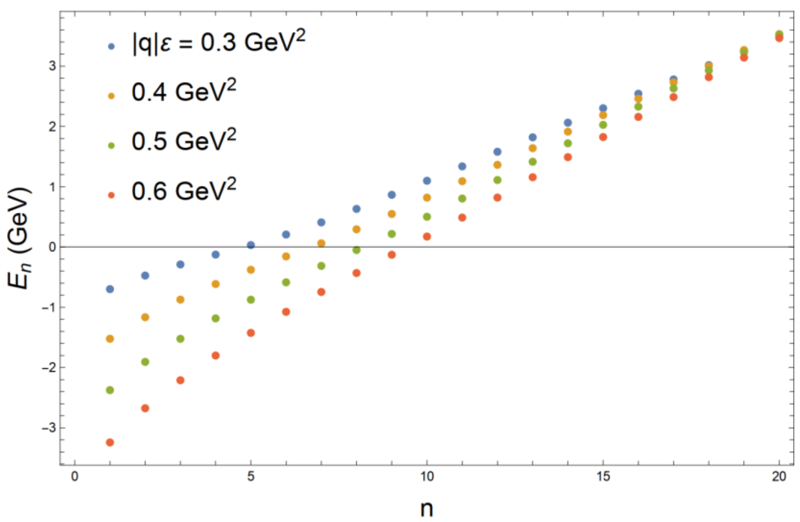


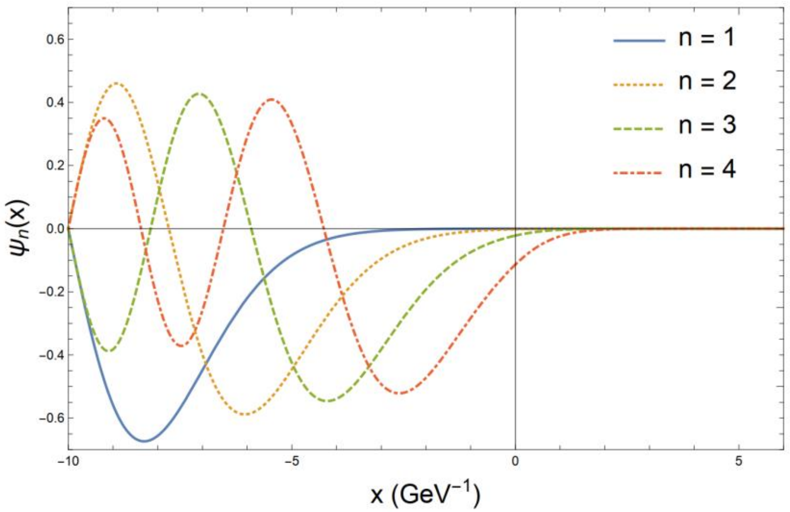



# - One dimensional case -

If $a_-$ is positive ($\sigma < |q|\varepsilon$), both $C_{1-}$ and $C_{2-}$ can be nonzero, and we need a boundary at $-b$ ($b > 0$) to bind the mesons

$$C_{1-}Ai\left(a_-\left(\frac{E_r}{\sigma_-}-b\right)\right)+C_{2-}Bi\left(a_-\left(\frac{E_r}{\sigma_-}-b\right)\right)=0.$$




Negative ground state relative energy and strongly oscillating wave function around the boundry $-b$ signals deconfinement

# - Two dimensional case -

- Schroedinger Eq. at $d = 2$

$$\left(\frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \sigma\, r + |q|\varepsilon x\right)\psi(x,y) = E_r \psi(x,y)$$

- First consider limit $\varepsilon \to 0$, and define $\psi(x,y) \equiv e^{i\,l\,\theta} R(r)$ and $R(r) = u(r)/\sqrt{r}$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{\frac{1}{4} - l^2}{r^2}\right) + \sigma\, r\right] u(r) = E_r u(r)$$

For a given orbital angular momentum $l$, the equation can be reduced to

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \sigma\, r\right] u(r) = 0.$$

# - Two dimensional case -

- For a finite $\varepsilon$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{4r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+(\sigma+|q|\varepsilon\cos\theta)\,r\right]u(r,\theta)=E_r u\,(r,\theta)$$

- If we assume $u\,(r,\theta)\;\equiv\;\Theta(r,\theta)\upsilon(r)$ and $\frac{\partial^2}{\partial r^2}\Theta\,(r,\theta)$ is small, it can be separated into two coupled equations

$$\begin{aligned}\left[-\frac{\hbar^2}{2mr^2}\frac{\partial^2}{\partial\theta^2}+|q|\varepsilon r\cos\theta-\epsilon(r)\right]\Theta(r,\theta)&=0\\ \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{4r^2}\right)+\sigma\,r+\epsilon(r)-E_r\right]\upsilon(r)&=0\end{aligned}$$

The former equation's solution is Mathieu function

$$S(\frac{8mr^2\epsilon(r)}{\hbar^2},\frac{4|q|\varepsilon mr^3}{\hbar^2},\frac{\theta}{2})\text{ and }C(\frac{8mr^2\epsilon(r)}{\hbar^2},\frac{4|q|\varepsilon mr^3}{\hbar^2},\frac{\theta}{2})$$

# - Two dimensional case -

- The requirement of $2\pi$-periodicity constrains the eigenenergy $\epsilon_{2n}$ to

$$\frac{\hbar^2}{8mr^2} b_{2n}\left(\frac{4|q|\varepsilon m r^3}{\hbar^2}\right) \quad (n = 1, 2, \ldots),$$
$$\frac{\hbar^2}{8mr^2} a_{2n}\left(\frac{4|q|\varepsilon m r^3}{\hbar^2}\right) \quad (n = 0, 1, 2, \ldots),$$

with $b_{2n}(p)$ and $a_{2n}(p)$ are the characteristic values of $S(a, p, \frac{\theta}{2})$ and $C(a, p, \frac{\theta}{2})$, which give the eigenfunctions $\Theta_{2n}(r, \theta)$ in form of elliptic cosine and sine functions

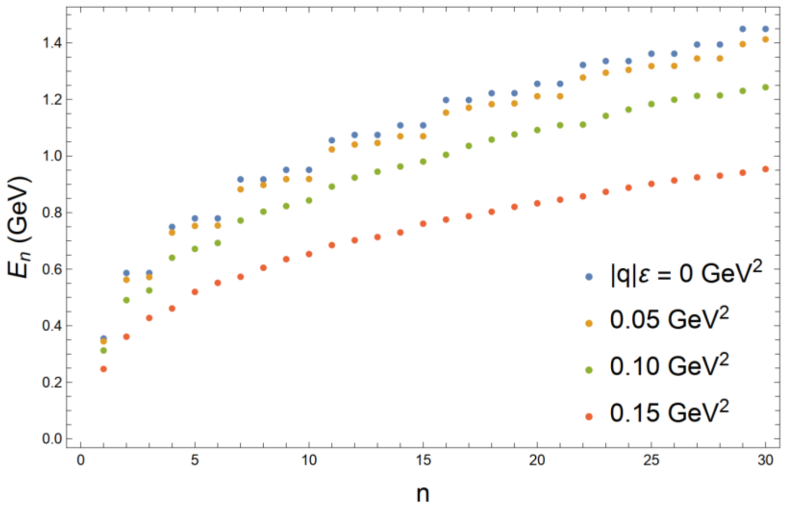
$$se_{2n}\left(\frac{\theta}{2}, \frac{4|q|\varepsilon m r^3}{\hbar^2}\right) \quad (n = 1, 2, \ldots),$$
$$ce_{2n}\left(\frac{\theta}{2}, \frac{4|q|\varepsilon m r^3}{\hbar^2}\right) \quad (n = 0, 1, 2, \ldots),$$

# - Two dimensional case -

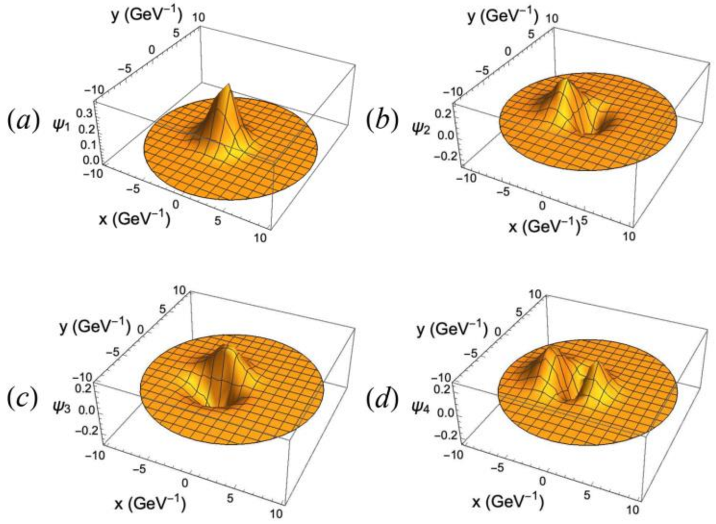
- Then the latter equation

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{4r^2}\right)+\sigma\, r+\epsilon_{2n}(r)-E_r\right]v(r)=0.$$

- Numerically solving the above equation gives




Two degenerate states broken by $\varepsilon$




$|q|\varepsilon\ =\ 0.1\ \text{GeV}^2$

# - Three dimensional case -

- Schroedinger Eq. at $d = 3$

$$\left(\frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + \sigma\, r + |q|\varepsilon x\right)\psi(x,y,z) = E_r\psi(x,y,z)$$

- Detail of $d = 3$ case is similar to $d = 2$ case, we can just go to numerical result

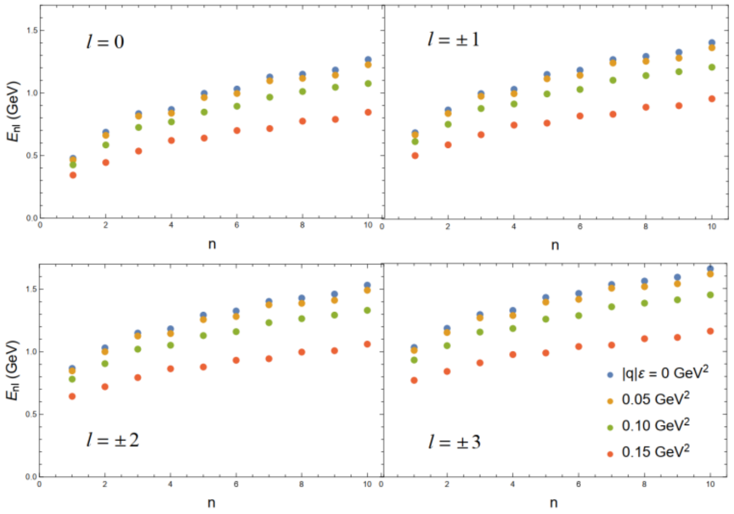



FIG. 7. The lowest eigenenergies $E_{nl}$ of bottomium with $n = 1, 2, \ldots, 10$ for different subcritical electric fields and orbital angular momenta $l = 0, \pm 1, \pm 2, \pm 3$ along the electric field.

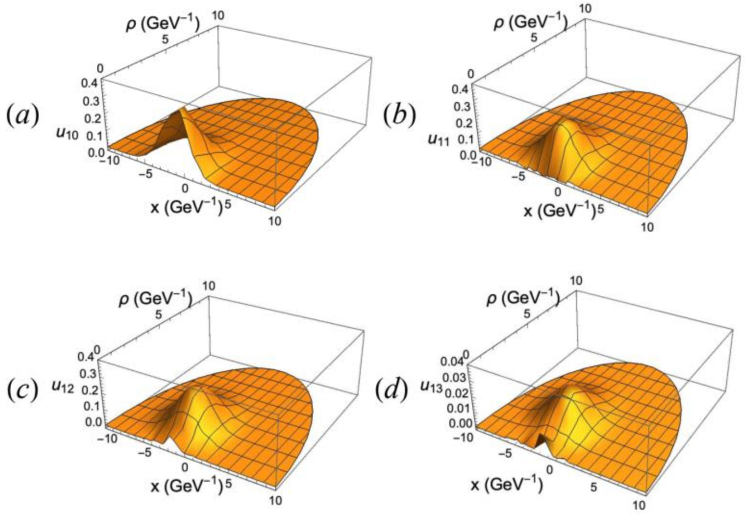


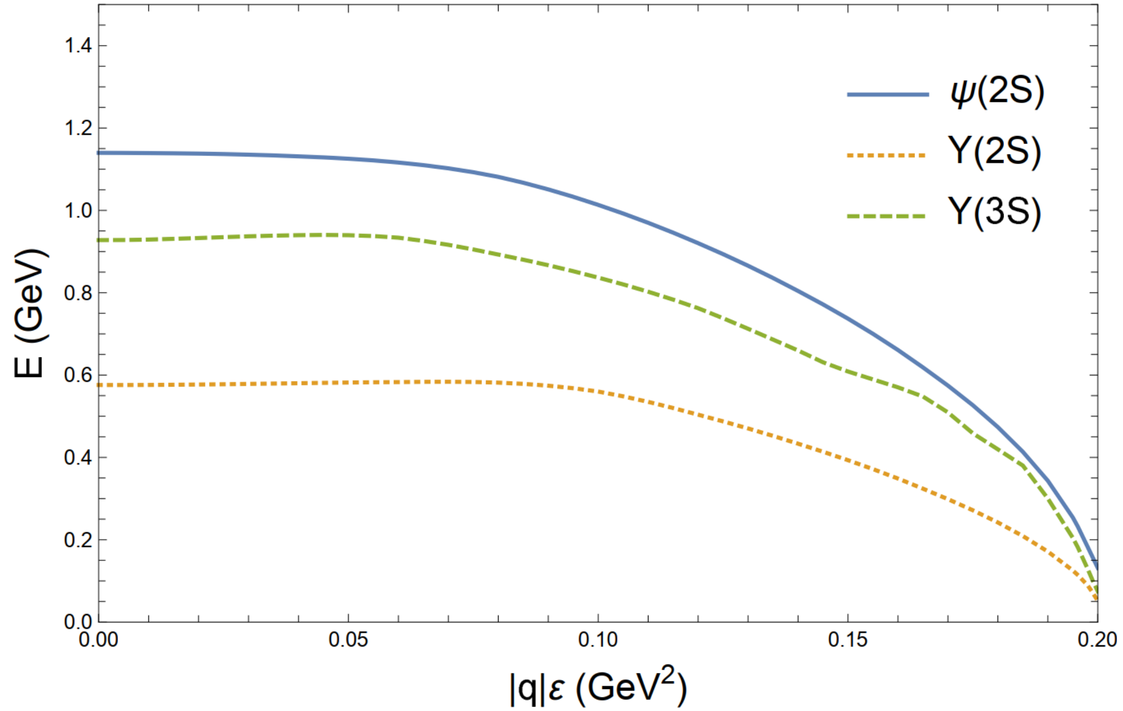

FIG. 8. The lowest eigenfunctions $u_{nl}$ of bottomium for the subcritical electric field $|q|\varepsilon = 0.1$ GeV$^2$ and orbital angular momenta $l = 0, 1, 2, 3$ along the electric field..

# - Three dimensional case -

- In a more relativistic situation, the potential gives

$$V(r) = 0.2\,r - \frac{0.4105}{r} + \beta e^{-1.982r}\mathbf{s_q}\cdot\mathbf{s_{\bar{q}}},$$

$\beta = 2.06$ for charmonium and $\beta = 0.318$ for bottomium




The increasing feature reflects the competition between effects of small electric field and Coulomb potential

Around the critical point $|q|\varepsilon_c \equiv \sigma$ of deconfinement, the relative energy decreases greatly for every quarkonium, but there is still a small positive redundant energy left at $|q|\varepsilon_c$ due to the spin-spin interactions