

Effective range expansion with the left-hand cut

杜孟林 (UESTC)

University of Electronic Science and Technology of China (电子科技大学)

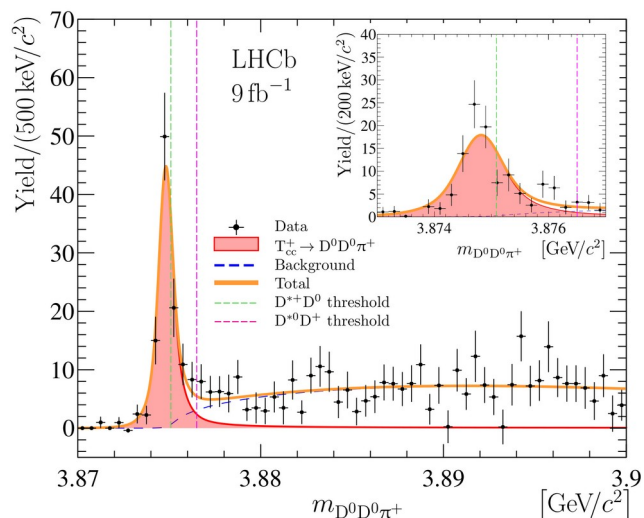
In collaboration with A. Fillin, V. Baru, X.-K. Dong, E. Epelbaum, F.-K. Guo, C. Hanhart,
A. Nefediev, J. Nieves, Q. Wang, and B. Wu

Based on PRL 131,131903 (2023), and arXiv:2408.09375[hep-ph]

第十届 XYZ 研讨会

2025.4.11-15 @ 长沙

Doubly charmed tetraquark (Tcc)

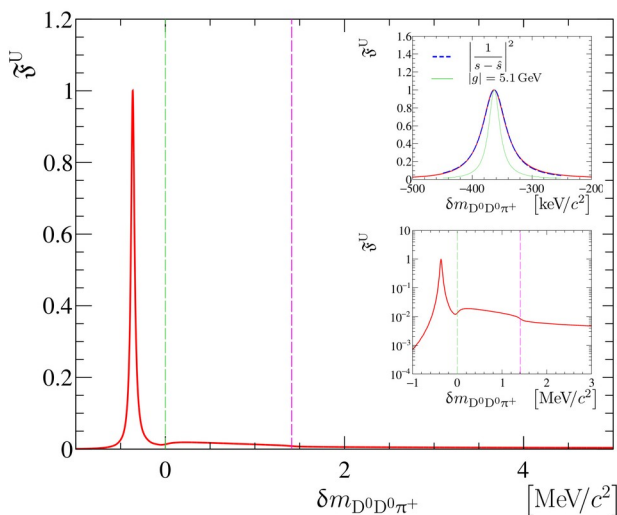


Breit-Wigner fit

LHCb, Nature Phys. 18, (2022) 751

Parameter	Value
N	117 ± 16
δm_{BW}	$-273 \pm 61 \text{ keV}$
Γ_{BW}	$410 \pm 165 \text{ keV}$

☞ $\mathfrak{R} \sim 400 \text{ keV}$.



Unitarized and analytical

LHCb, Nature Commun. 13 (2022), 3351

$$\delta m = m_{T_{cc}^+} - m_{D^{*+}} - m_{D^0}$$

$$\delta m_{\text{pole}} = -360 \pm 40_{-0}^{+4} \text{ keV}$$

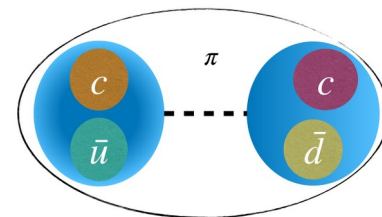
$$\Gamma_{\text{pole}} = 48 \pm 2_{-14}^{+0} \text{ keV}$$

☞ $I = 0$: isoscalar

↪ $D^+ D^0 \pi^+, D^+ D^+$ ✗

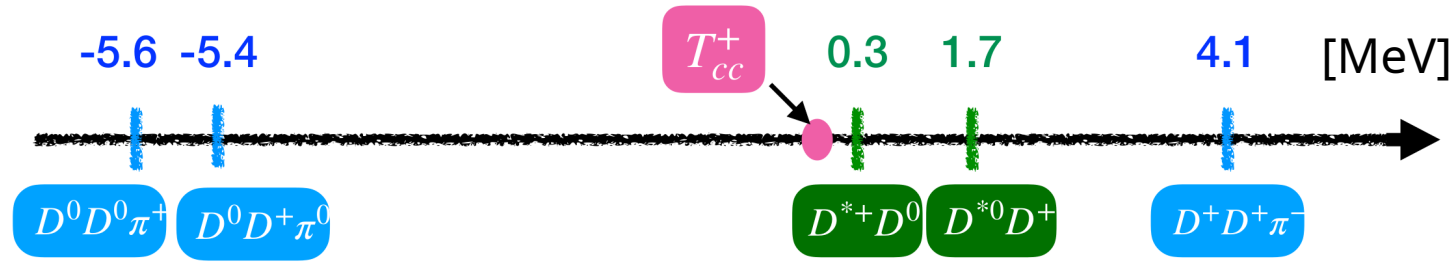
☞ T_{cc}^+ resides near $D^* D$ thresholds

↪ approximate 90% of $D^0 D^0 \pi^+$ events contain a D^{*+} .

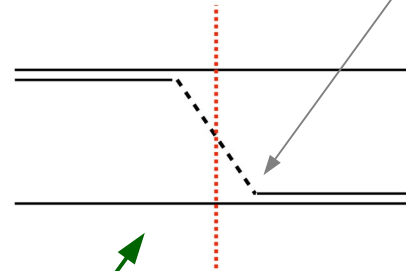
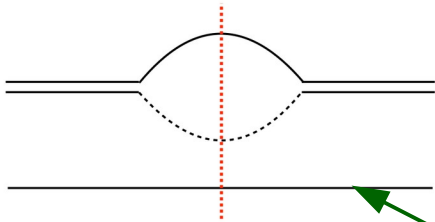


LHCb, Nature Commun. 13 (2022)

The three-body cut



Three-body cuts



LO Chiral Lagrangian (g determined from $D^* \rightarrow D\pi$)

$$\mathcal{L} = \frac{1}{4} g \text{Tr} (\vec{\sigma} \cdot \vec{u}_{ab} H_b H_a^\dagger)$$

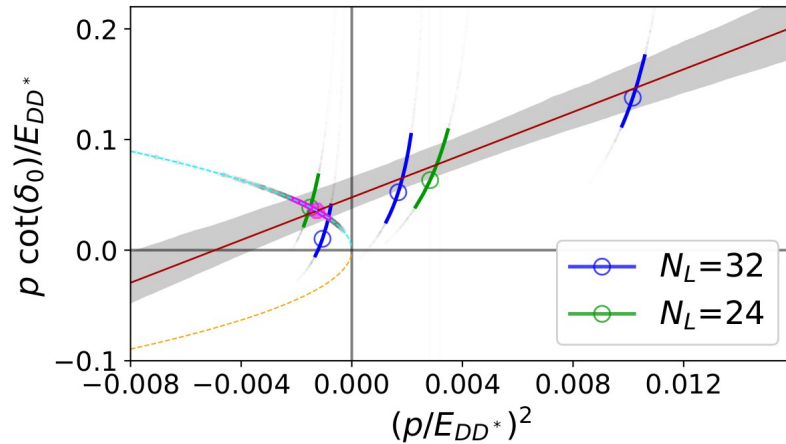
$$U_\alpha(M, p) = P_\alpha - \sum_\beta \int \frac{d^3 \vec{q}}{(2\pi)^3} V_{\alpha\beta}(M, p, q) G_\beta(M, q) U_\beta(M, q)$$

$$\hookrightarrow G_\alpha(M, p) = \frac{1}{m_\alpha^* + m_\alpha + \frac{p^2}{2\mu_\alpha} - M - \frac{i}{2} \Gamma_\alpha(M, p)}$$

Doubly Charm Tetraquark on the Lattice

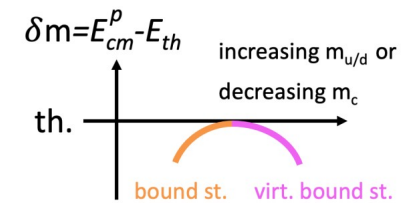
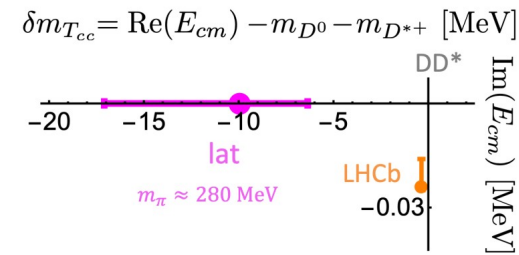
Padmanath *et al*, PRL129,032002(2022)

	m_D (MeV)	m_{D^*} (MeV)	M_{av} (MeV)	$a_{l=0}^{(J=1)}$ (fm)	$r_{l=0}^{(J=1)}$ (fm)	$\delta m_{T_{cc}}$ (MeV)	T_{cc}
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(h)}$)	1927(1)	2049(2)	3103(3)	1.04(29)	$0.96^{(+0.18)}_{(-0.20)}$	$-9.9^{+3.6}_{-7.2}$	Virtual bound st.
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(l)}$)	1762(1)	1898(2)	2820(3)	0.86(0.22)	$0.92^{(+0.17)}_{(-0.19)}$	$-15.0^{(+4.6)}_{(-9.3)}$	Virtual bound st.
Experiment [2,41]	1864.85(5)	2010.26(5)	3068.6(1)	-7.15(51)	$[-11.9(16.9), 0]$	-0.36(4)	Bound st.

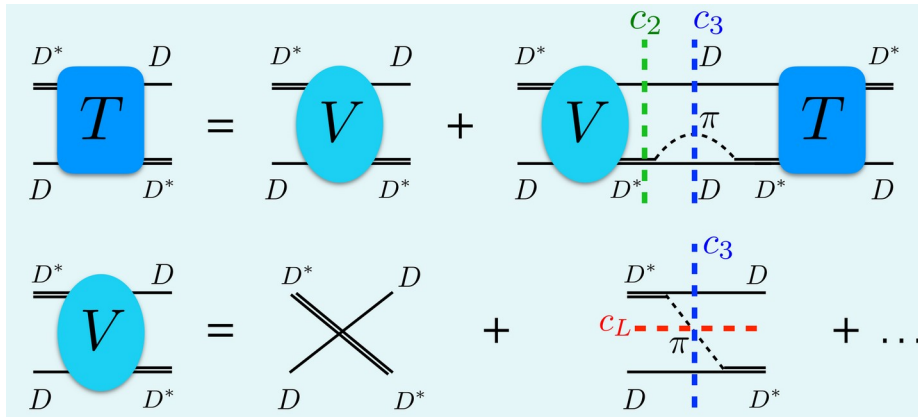


$$t = \frac{E_{cm}}{2} \frac{1}{p \cot \delta - ip},$$

$$p \cot \delta = \frac{1}{a_0} + \frac{1}{2} r_0 p^2,$$

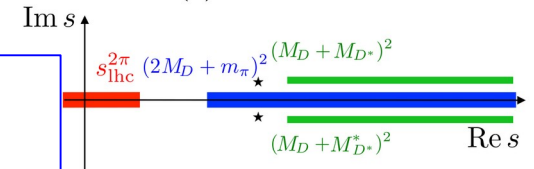


The three-body cut vs. left-hand cut



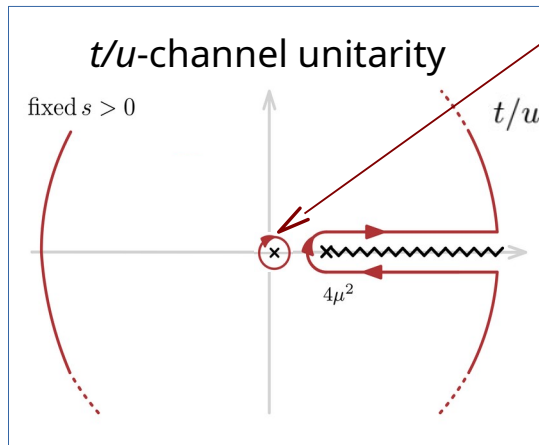
three-body cut

$$E > M_D + M_D + M_\pi$$



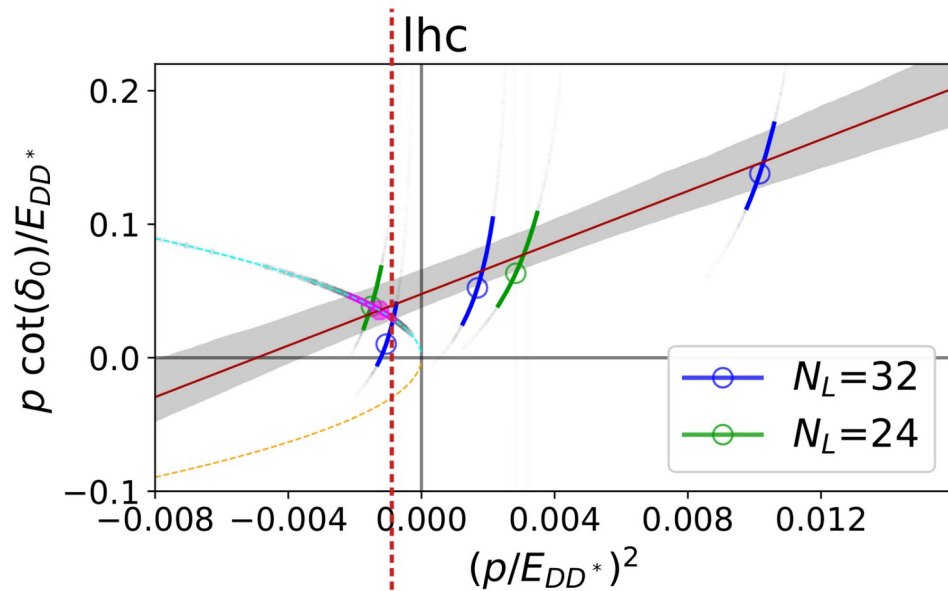
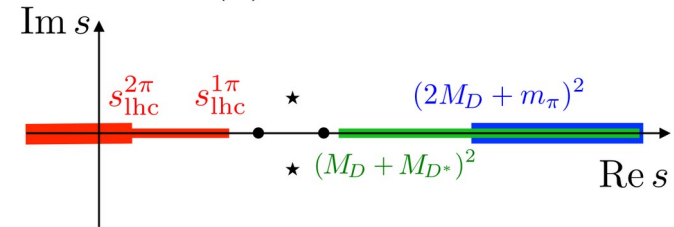
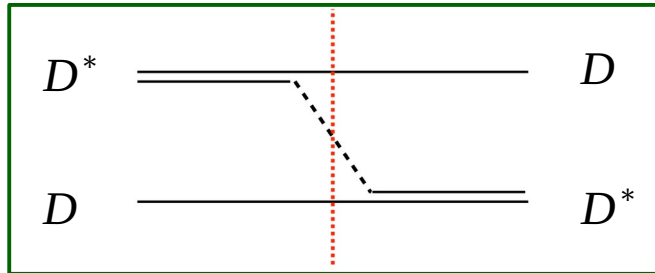
left-hand cut

$$\int_{-1}^1 d \cos \theta G_\pi(E, p, p)$$



$$G_\pi^{-1}(E, k, k') \xrightarrow[\text{on shell: } k=k'=p]{\cos \theta = \pm 1} E_{D^*}(p^2) - E_D(p^2) - \omega_\pi(4p^2/0) = 0$$

The left-hand cut



$$m_\pi = 280 \text{ MeV}$$

☞ two-body branch point:

$$E = M_D + M_{D^*}$$

$$\Rightarrow p_{\text{rhc}2}^2 = 0$$

☞ three-body branch point:

$$E = M_D + M_D + m_\pi$$

$$\Rightarrow \left(\frac{p_{\text{rhc}3}}{E_{DD^*}} \right)^2 = +0.019$$

☞ left-hand cut branch point:

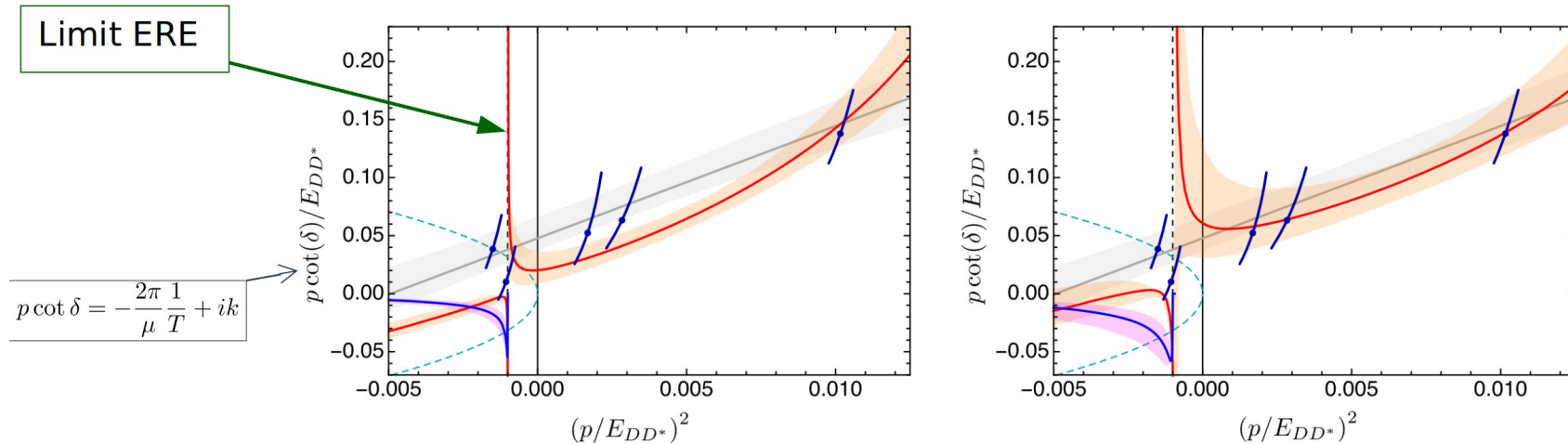
$$\Rightarrow \left(\frac{p_{\text{lhc}}^{1\pi}}{E_{DD^*}} \right)^2 = -0.001$$

$$\left(\frac{\tilde{p}_{\text{lhc}}^{1\pi}}{E_{DD^*}} \right)^2 = -0.190$$

Phase shift with the left-hand cut: LSE

$$M_D = 1927 \text{ MeV}, M_{D^*} = 2049 \text{ MeV}, m_\pi = 280 \text{ MeV}$$

Du et al., PRL 131,131903 (2023)



Related recent works on FV w/ LHC...

Plane-wave basis to treat long-range interactions

Project to irep. of the cubic to avoid the lhc associated to the partial wave projection

Generalization of the Lüscher + K -matrix

Three-body framework (automatically includes lhc)

Modify the Lüscher formula via “modified effective range expansion”

Meng and Epelbaum, JHEP (2021)

Meng et al., PRD (2024)
Mai and Döring, EPJA (2017), PRL (2019)

Hansen and Raposo, JHEP (2024)

Dawid et al., PRD (2023)

Hansen et al., PRD (2024)

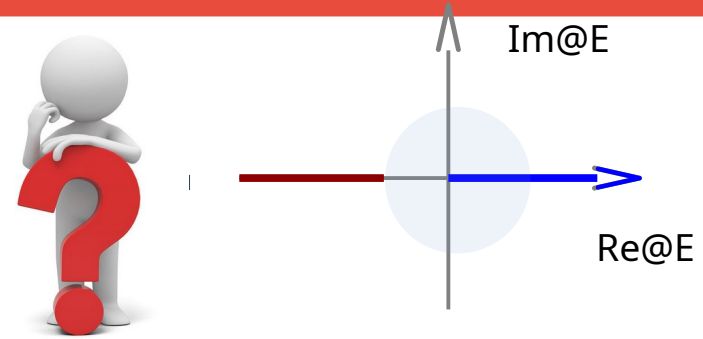
Bubna et al., JHEP (2024)

The N/D method

ERE

$$f(k^2) = \frac{1}{k \cot \delta - ik}$$

$$k \cot \delta = \frac{1}{a} + \frac{1}{2}rk^2 + \mathcal{O}(k^4)$$



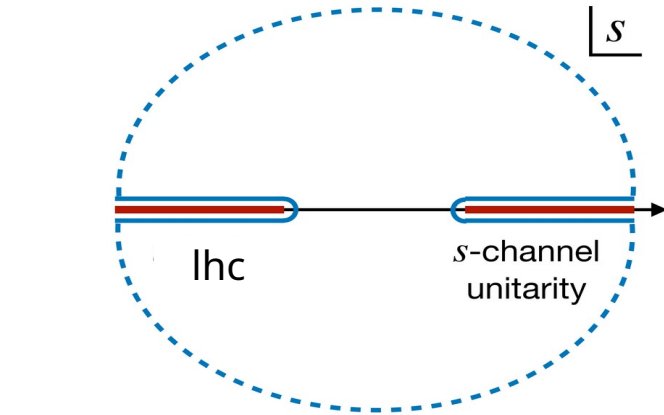
$$T(s) = \frac{N(s)}{D(s)}$$

$$\text{Im}D = \text{Im}\frac{N}{T} = N\text{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\text{thr}} \\ 0, & s < s_{\text{thr}} \end{cases}$$

$$\text{Im}N = \begin{cases} \text{Im}TD, & s < s_{\text{lh}} \\ 0, & s > s_{\text{lh}} \end{cases}$$

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s - s_0)^n}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\rho(s') N(s')}{(s' - s)(s' - s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s - s_0)^{n-\ell}}{\pi} \int_{-\infty}^{s_{\text{left}}} ds' \frac{\text{Im}T(s') D(s')}{(s' - s_0)^{n-\ell}(s' - s)}.$$



N=1

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + P(s) + G(s)$$

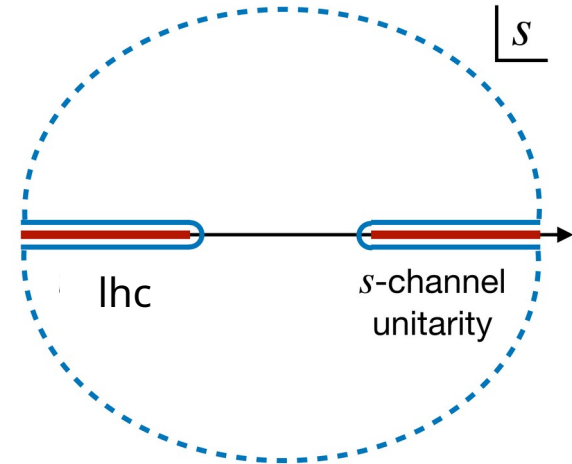
$$T(s) = \frac{1}{D(s)}$$

The N/D method

$$T(s) = \frac{N(s)}{D(s)}$$

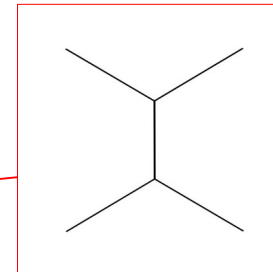
$$\text{Im}D = \text{Im}\frac{N}{T} = N\text{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\text{thr}} \\ 0, & s < s_{\text{thr}} \end{cases}$$

$$\text{Im}N = \begin{cases} \text{Im}TD, & s < s_{\text{lhc}} \\ 0, & s > s_{\text{lhc}} \end{cases}$$



$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s - s_0)^n}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\rho(s') N(s')}{(s' - s)(s' - s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s - s_0)^{n-\ell}}{\pi} \int_{-\infty}^{s_{\text{left}}} ds' \frac{\text{Im}T(s') D(s')}{(s' - s_0)^{n-\ell}(s' - s)}.$$

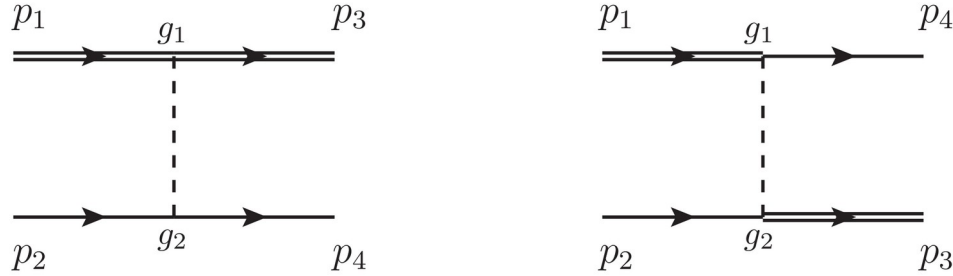


$$\frac{1}{T_\ell^{\text{II}}} = \frac{1}{T_\ell} + 2i\rho \implies T^{\text{II}} = \frac{1}{\frac{D}{N} + 2i\rho} = \frac{N}{D + 2i\rho N}$$

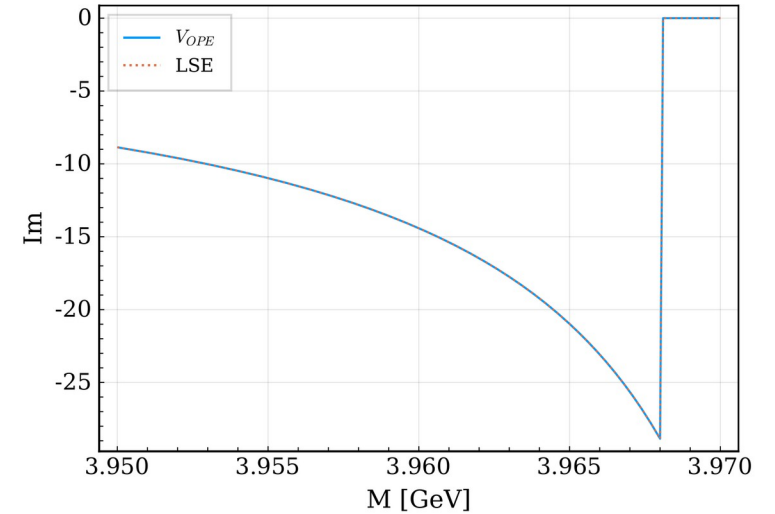
Along the lhc, $i\rho$ and D is real, N has imaginary part.

$$D + 2i\rho N \neq 0$$

The left-hand cut



$$\text{Im } f(k^2) = c \text{Im } L(k^2) = -\frac{c}{4k^2} \pi, \quad \text{for } k^2 < k_{\text{thc}}^2$$



Solving LSE could be time-consuming.

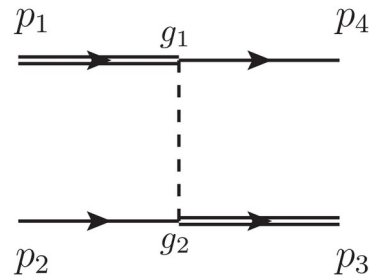
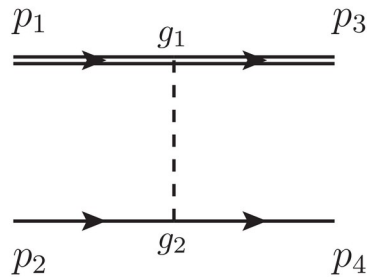
For a t -channel exchange at low-energies, an S -wave amplitude reads

$$L_t(s) = \frac{1}{2} \int \frac{1}{t - m_5^2} d\cos\theta = -\frac{s}{\lambda(s, m_1^2, m_2^2)} \log \left(\frac{s - 2(m_1^2 + m_2^2) + m_5^2 + \frac{(m_1^2 - m_2^2)^2}{s}}{m_5^2} \right),$$

with m_5 the mass of changed particle. Likewise, the u -channel exchanged S -wave amplitude reads

$$L_u(s) = \frac{1}{2} \int \frac{1}{u - m_5^2} d\cos\theta = -\frac{s}{\lambda(s, m_1^2, m_2^2)} \left(\log(s + m_5^2 - 2(m_1 + m_2)^2) - \log(m_5^2 - \frac{(m_1^2 - m_2^2)^2}{s}) \right).$$

The left-hand cut: nonrelativistic



Exchanged-particle: relativistic

$$\eta = |m_1 - m_2|/(m_1 + m_2)$$

$$\mu_{\text{ex}}^2 = m_{\text{ex}}^2 - (m_1 - m_2)^2$$

$$\mu_+^2 = 4\mu\mu_{\text{ex}}^2/(m_1 + m_2)$$

$$L_t(k^2) \equiv \frac{1}{2} \int_{-1}^{+1} \frac{d \cos \theta}{t - m_{\text{ex}}^2} = -\frac{1}{4k^2} \log \frac{m_{\text{ex}}^2/4 + k^2}{m_{\text{ex}}^2/4},$$

$$L_u(k^2) \equiv \frac{1}{2} \int_{-1}^{+1} \frac{d \cos \theta}{u - m_{\text{ex}}^2} \approx -\frac{1}{4k^2} \log \frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2},$$

$$\frac{1}{2} \int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_3)^2}{t - m_{\text{ex}}^2} d \cos \theta = -\frac{m_{\text{ex}}^2}{2} \int_{-1}^{+1} \frac{d \cos \theta}{t - m_{\text{ex}}^2} - 1$$

$$\frac{1}{2} \int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_4)^2}{u - m_{\text{ex}}^2} d \cos \theta \approx -\frac{\mu_{\text{ex}}^2}{2} \int_{-1}^{+1} \frac{d \cos \theta}{u - m_{\text{ex}}^2} - 1$$

$\mathcal{F}_\ell/2$

$$f(k^2) = \frac{n(k^2)}{d(k^2)} \quad \text{Im } d(k^2) = -k n(k^2), \quad \text{for } k^2 > 0,$$

$$\text{Im } n(k^2) = d(k^2) \text{Im } f(k^2), \quad \text{for } k^2 < k_{\text{hnc}}^2.$$

The N/D method: nonrelativistic

$$n(k^2) = n_m(k^2) + \frac{(k^2)^m}{\pi} \int_{-\infty}^{k_{\text{lh}}^2} \frac{d(k'^2) \text{Im} f(k'^2)}{(k'^2 - k^2)(k'^2)^m} dk'^2$$

$\propto \text{Im} L$

No singularity along lh

$$n(k^2) = n'_m(k^2) + \frac{P(k^2)}{\pi} \int_{-\infty}^{k_{\text{lh}}^2} \frac{\text{Im} f(k'^2)}{k'^2 - k^2} dk'^2 = n'_m(k^2) + P(k^2) \tilde{g} L(k^2)$$

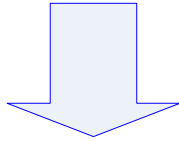
$$\begin{aligned} n(k^2) &= n_0 + n_1 k^2 + \frac{k^2}{\pi} \int_{-\infty}^{k_{\text{lh}}^2} \frac{(d_0 + d_1 k'^2) \text{Im} f(k'^2)}{(k'^2 - k^2) k'^2} dk'^2 \\ &= n_0 + n_1 k^2 - c L_0 + (d_0 + d_1 k^2) \frac{c}{\pi} \int_{-\infty}^{k_{\text{lh}}^2} \frac{\text{Im} f(k'^2)}{k'^2 - k^2} dk'^2 \\ &= n'_0 + n_1 k^2 + (d_0 + d_1 k^2) c L(k^2) \end{aligned}$$

$$n(k^2) = \tilde{n}(k^2) + \tilde{g}(L(k^2) - L_0)$$

$$L_0 = L(k^2 = 0) = -1/\mu_{\text{ex}}^2$$

The N/D method: nonrelativistic

$$d(k^2) = d_n(k^2) - \frac{(k^2 - k_0^2)^n}{\pi} \int_0^\infty \frac{k' n(k'^2) dk'^2}{(k'^2 - k^2)(k'^2 - k_0^2)^n}$$



$$\begin{aligned} d(k^2) &= \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi} \int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2} dk'^2 \\ &= \tilde{d}(k^2) - ik n(k^2) - \tilde{g}d^R(k^2) \end{aligned}$$

$$d_u^R(k^2) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

It is worth stressing that $d(k^2)$ is free of lhc, as the lhc associated with $n(k^2)$ below the threshold is counterbalanced by $d^R(k^2)$, which is crucial to ensure that $f(k^2)$ exhibits the correct lhc behavior. Along the rhc, both $n(k^2)$ and $d^R(k^2)$ are real such that $\text{Im}d(k^2) = -k n(k^2)$.

Effective range expansion with the left-hand cut

$$\frac{1}{f(k^2)} = \frac{\tilde{d}(k^2) - \tilde{g}d^R(k^2)}{\tilde{n}(k^2) + \tilde{g}(L(k^2) - L_0)} - ik$$

$$d_u^R(k^2) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

$$L(k^2) = -\frac{1}{4k^2} \log \frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2}$$

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^n \tilde{d}_i k^{2i} - \tilde{g}d^R(k^2)}{1 + \sum_{j=1}^m \tilde{n}_j k^{2j} + \tilde{g}(L(k^2) - L_0)} - ik$$

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g}d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1} \xrightarrow{\tilde{g} \rightarrow 0} \frac{1}{f(k^2)} = \frac{1}{a} + \frac{1}{2} r k^2 - ik$$

Scattering length

$$a = f(k^2 = 0) = \left[\tilde{d}_0 + \frac{\tilde{g}}{\mu_+} (1 - \eta) \right]^{-1}$$

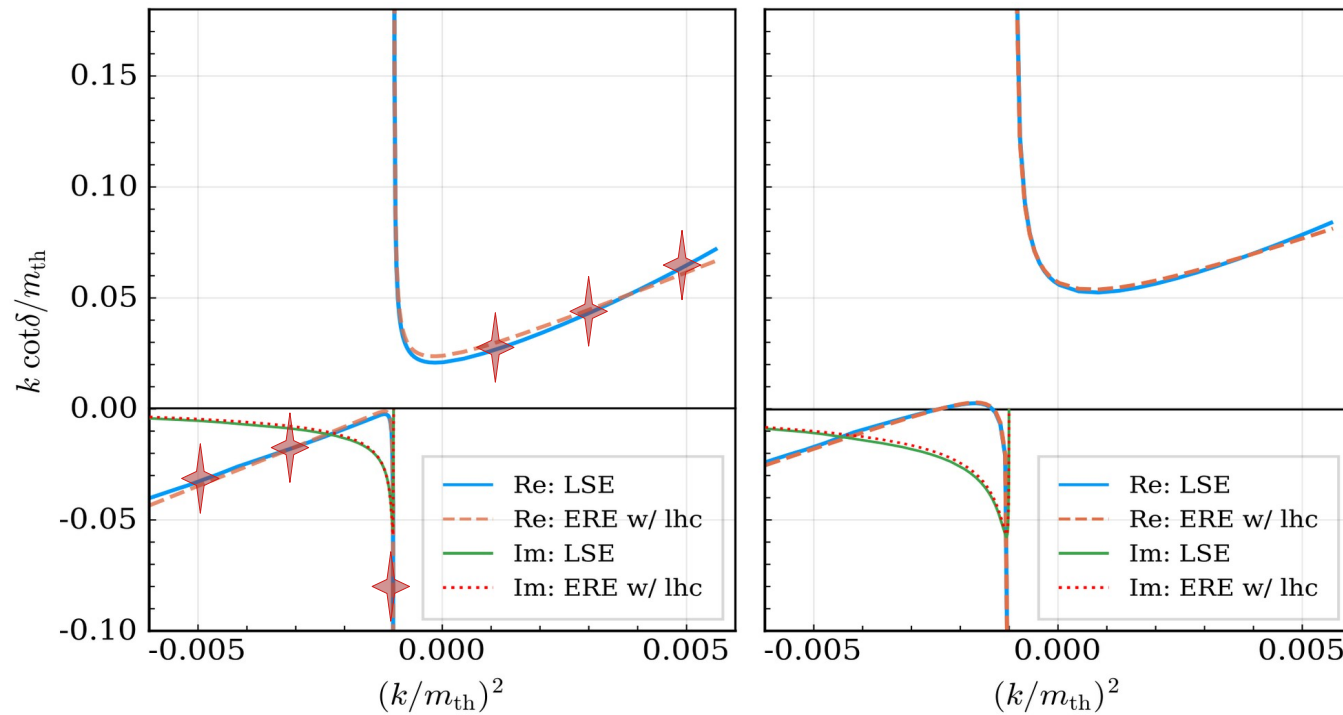
Effective range

$$r = \left. \frac{d^2(1/f + ik)}{dk^2} \right|_{k=0} = 2\tilde{d}_1 - \frac{8\tilde{g}}{3\mu_+^3} (1 - \eta^3) - \frac{4\tilde{g}}{\mu_+^4 a_u} (1 - \eta^4)$$

Example: Tcc on the Lattice [3 parameters]

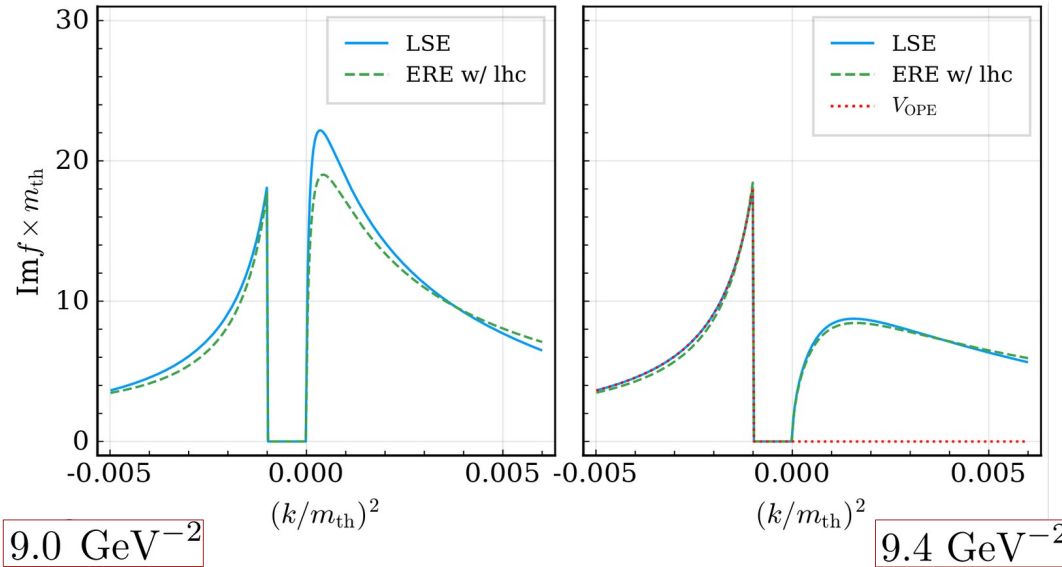
$f_{[0,1]}$

Du *et al.*, 2408.09375 [hep-ph]



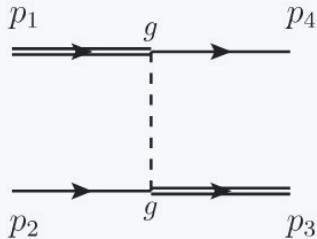
$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1}$$

Couplings to the exchanged-particle



$$g_P = -\frac{2\pi\tilde{g}}{\mu d^{0,\text{lhc}} \mathcal{F}_\ell}$$

$$g_{D^* D \pi}^2 / (4F^2) = 9.2 \text{ GeV}^{-2}$$



$$-\frac{2\pi}{\mu} \text{Im} f = \text{Im} T = \text{Im} V_{\text{OPE}}(k^2) = g_P \frac{-\pi}{4k^2} \mathcal{F}_\ell, \quad \text{for } k^2 < k_{\text{lhc}}^2$$

$$\text{Im} n(k^2) = -\tilde{g} \frac{\pi}{4k^2}, \quad \text{for } k^2 < k_{\text{lhc}}^2$$

$$d_u^{0,\text{lhc}} = \tilde{d}_0 - \frac{\tilde{d}_1 \mu_+^2}{4} + \frac{\mu_+}{2} \left(1 + \frac{\tilde{g}}{\mu_{\text{ex}}^2} \right) + \frac{\tilde{g} \log[2/(1+\eta)]}{\mu_+}$$

The amplitude zero

At leading order, i.e., $\tilde{n}(k^2) = 1$,

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1}$$

For a general u -channel exchange,

$$1 + \tilde{g} \left[L_u(k_{u,\text{zero}}^2) + \frac{1}{\mu_{\text{ex}}^2} \right] = 0,$$

for the case $|\Delta| \ll m_{\text{th}}$ such that $\eta \ll 1$, $\Rightarrow$ the t -channel exchange

$$k_{t,\text{zero}}^2 = -\frac{m_{\text{ex}}^2}{4} \left[1 + \frac{1}{y} W(-e^{-y} y) \right]$$

where $y \equiv 1 + m_{\text{ex}}^2/\tilde{g}$ and W is the Lambert W function.

$$y = 1 + \frac{1 + \frac{4}{3} a_t m_{\text{ex}} (1 - \log 4) - \frac{4\pi a_t m_{\text{ex}}^2}{\mu g_P \mathcal{F}_\ell}}{2 + a_t m_{\text{ex}} (1 - m_{\text{ex}} r_t/4)}.$$

Application to the NN scattering

$$\frac{1}{f_{[3,1]}(k^2)} = \frac{\tilde{d}_0 + \tilde{d}_1 k^2 + \tilde{d}_2 k^4 - \tilde{g} d^R(k^2)}{1 + \tilde{n}_1 k^2 + \tilde{g}(L(k^2) - L_0)} - ik$$

$$\frac{g_{\pi N}^2}{4\pi} = 12.42^{+3.76}_{-4.71}$$

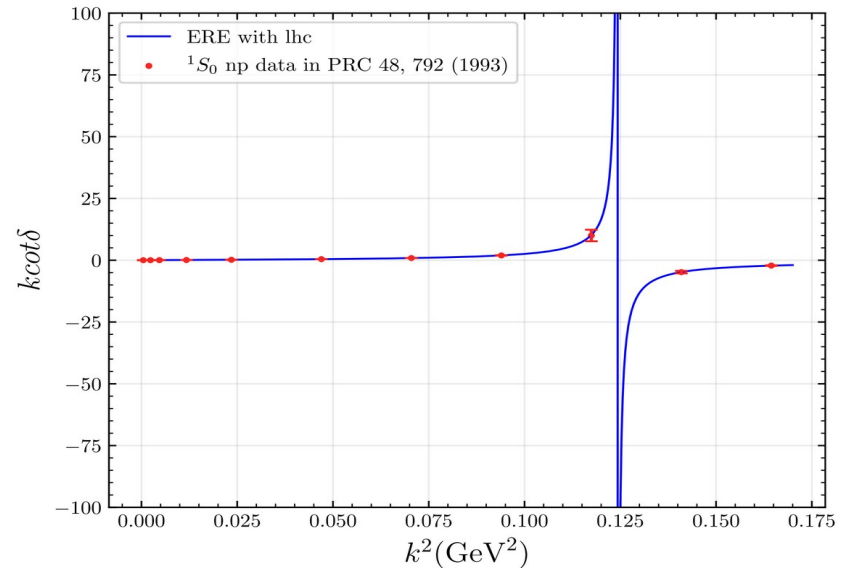
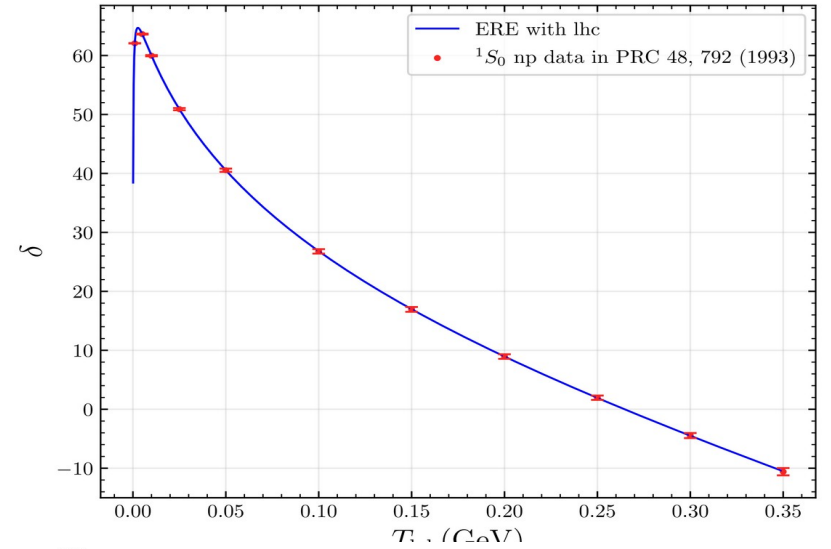
TABLE I. Some deduced values for the πNN coupling constant. The quoted uncertainty are those quoted by the authors and usually do not include systematic uncertainties.

Source	Year	System	$g_{\pi NN}^2/4\pi$
Karlsruhe-Helsinki [3]	1980	πp	14.28(18) ^a
Kroll [4]	1981	pp	14.52(40) ^a
Nijmegen [6]	1993	pp, np	13.58 (5) ^a
VPI [8]	1994	pp, np	13.70
Nijmegen [7]	1997	pp, np	13.54 (5) ^a
Timmermans [12]	1997	$\pi^+ p$	13.45(14) ^a
VPI [9]	1994	GMO, πp	13.75(15) ^a
Uppsala [2]	1998	$np \rightarrow pn$	14.52(26)
Pavan <i>et al.</i> [11]	1999	πp	13.73 (9)
Schröder <i>et al.</i> , corrected [14,10]	1999	GMO, $\pi^+ p$	13.77(18)
Present work	2001	GMO, $\pi^+ p$	14.11(20)

^aStatistical uncertainty only.

$$d_0, d_1, n_1, n_2, g_0 \quad 11.33 \pm 4.53$$

$$d_0, d_1, d_2, n_1, n_2, g_0 \quad 14.71 \pm 8.88$$



Summary

★ Unphysical pion masses on the Lattice

$$M_D = 1927 \text{ MeV}, M_{D^*} = 2049 \text{ MeV}, m_\pi = 280 \text{ MeV}$$

↪ the three-body cut above the two-body cut ($\sqrt{s_{\text{thc}}} = 3968 \text{ MeV}$)

↪ The traditional ERE valid only in a very limited range

↪ An accurate extraction of the pole requires the OPE implemented

★ The ERE with the left-hand cut

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1}$$

↪ correct behavior of the left-hand cut

↪ can be used to extract the couplings of the exchanged particle to the scattering particles

↪ amplitude zeros caused by the interplay between the short- and long-range interactions

Thank you very much for your attention!

Thank you very much for your attention!

Without $d^R(k^2)$

$$d(k^2) = \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi} \int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2} dk'^2$$

$$= \tilde{d}(k^2) - ik n(k^2) - \cancel{\tilde{g} d^R(k^2)}$$

$$d_u^R(k^2) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

lhc

$$f^{-1} = \frac{\frac{1}{a} + \frac{1}{2}rk^2}{1 + \tilde{g}(L(k^2) - L_0)} - ik$$

$$d(k^2) = \frac{1}{a} + \frac{1}{2}rk^2 - ik - ik\tilde{g}(L(k^2) - L_0)$$

