

Chiral Representation of the Nucleon Mass at Leading Two-loop Order

Ze-Rui Liang(梁泽锐)

Hebei Normal University

With H.-X. Chen, F.-K. Guo, Z.-H. Guo, D.-L. Yao
arXiv:2502.19168 [hep-ph]

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- I. Introduction
- II. Leading two-loop result of the nucleon mass
- III. Chiral extrapolation of lattice QCD data
- IV. Summary and outlook

1. Introduction

- Decomposition of the nucleon mass (classical view)

$$m_N = \langle N(p) | T_{\mu}^{\mu} | N(p) \rangle$$
$$\sim \langle N(p) | \underbrace{\frac{\beta}{2g} G_{\mu\nu}^a G_a^{\mu\nu}}_{\text{field energy}} + \underbrace{m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s}_{\text{Higgs}} | N(p) \rangle$$

➔ How large is the trace anomaly term?

➔ Sigma terms: $\sigma_{\pi N} \sim \hat{m}(\bar{u} u + \bar{d} d)$, $\sigma_S = m_S \bar{s} s$

- Feynman-Hellman Theorem

$$\sigma_{\pi N} = \hat{m} \frac{\partial m_N}{\partial \hat{m}} = M_{\pi}^2 \frac{\partial m_N}{\partial M_{\pi}^2}, \quad \hat{m} = (m_u + m_d)/2.$$

➔ three-point to two-point problem

➔ High-precision determination of the quark mass dependence of the nucleon mass is crucial.

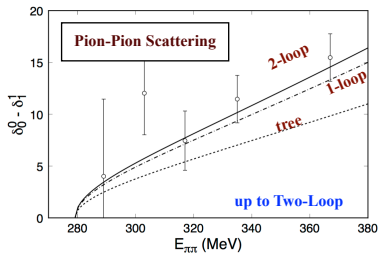
Chiral perturbation theory beyond one-loop is necessary!

Status of ChPT beyond one-loop

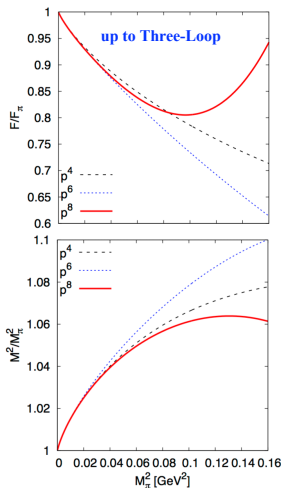
- Pure meson sector: a great triumph has been achieved.

➔ Two-loop calculations become standard and good convergence can be seen.

- ▶ $\pi\pi$ scattering
- ▶ πK scattering
- ▶ $K\ell 4$
- ▶ Electromagnetic form factor
- ▶ Pion decay constant
- ▶ Pion mass
- ▶ ...

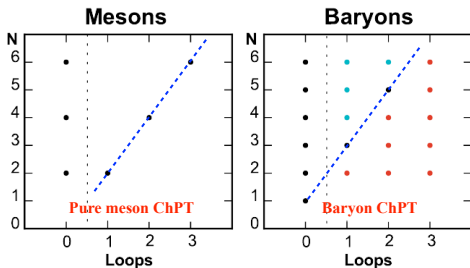


[Bijnens, Colangelo, Ecker, Gasser, Sainio, PLB374 (1996) 210-216]



[J. Bijnens and N. H. Truedsson, JHEP 1711 (2017) 181]

- Baryon sector: multi-loop calculation is much more complex than pure meson sector
 - ✓ Dirac algebra & axial coupling between baryons and Goldstone bosons
→ more diagrams and master integrals
 - ✓ Unequal mass
→ Hard to compute multi-loop integrals (solved by AMflow)
 - ✓ The introduction of non-zero baryon mass in the chiral limit
→ the notable **Power Counting Breaking (PCB)** problem!



Naive power counting rule !!!

red dots denote PCB terms!!!

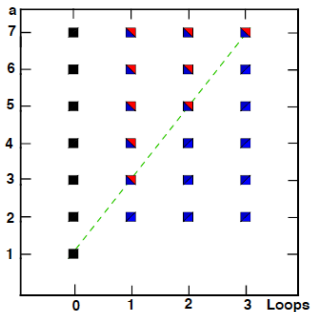
- Various approaches

- ✓ Heavy Baryon (HB) formalism [Jenkins and Manohar,PLB255'91]

- ✓ Infrared Regularisation (IR) prescription [T.Becher and H.Leutwyler,Eur.Phys.J.C9'99]

- ✓ **Extended-On-Mass-Shell (EOMS) scheme**

- [T.Fuchs,J.Gegelia,G.Japaridze and S.Scherer,PRD68'03]



philosophy

HB & IR: throwing away

EOMS: reorganisation

- Previous works on the nucleon mass in ChPT

➔ 1999: HB formalism up to $O(p^5)$ [McGovern & Birse, PLB446(1999)]

➔ 2007/2008: IR prescription up to $O(p^6)$ [expanded]

[Schindeler, Djukanovic, Gegelia & Scherer, PLB649(2007), NPA803(2008)]

➔ 2024: EOMS up to $O(p^6)$ [expanded]

[Conrad, Gasparyan & Epelbaum, PoS CD2021, 074 (2024), EPJ Web Conf. 303, 02001(2024)]

[Chen, Hu, Mo & Jia, arXiv:2406.040124 [hep-ph]]

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 \\ + k_7 M^6 \ln^2 \frac{M}{\mu} + k_8 M^6 \ln \frac{M}{\mu} + k_9 M^6 .$$

- Our work: **full EOMS** chiral result up to $O(p^5)$

- ➔ All the necessary analytical two-loop structure
- ➔ Avoid too many unknown LECs

2. Leading two-loop ChPT result

- The dressed propagator of the nucleon

$$\text{Full propagator} = \text{Bare propagator} + \text{Self energy} + \dots$$

$$iS_N(p) = \frac{i}{p \cdot \gamma - m - \Sigma_N(p \cdot \gamma)} = \frac{iZ_N}{p \cdot \gamma - m_N} + \text{non-pole piece}$$

- Physical mass as the pole: $m_N - m - \Sigma(m_N, m) = 0$

$$\text{Self energy} = \text{tree} + \text{1-loop} + \text{2-loop} + \dots$$

$$m_N = m_4 + \hbar \Delta m_N^{(1)} + \hbar^2 \Delta m_N^{(2)}$$

Mass insertion contribution by setting:

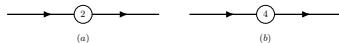
$$m \rightarrow m_4 = m + m_{\text{tree}}$$

$$\Delta m_N^{(1)} = \Sigma_N^{(1)}(m_4, m_4)$$

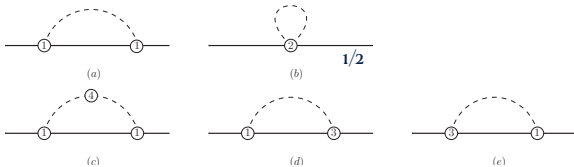
$$\Delta m_N^{(2)} = \Sigma_N^{(1)}(m_4, m_4) \Sigma_N^{(1)'}(m_4, m_4) + \Sigma_N^{(2)}(m_4, m_4)$$

Tree and one-loop diagrams

- Tree and one-loop diagrams up to $O(p^5)$



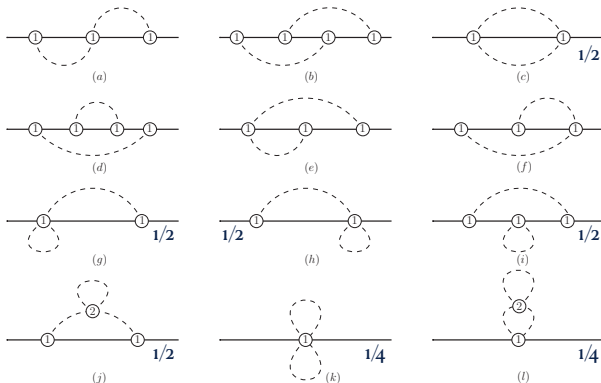
$$m_{\text{tree}} = -4c_1 M^2 - 2e_m M^4$$



$$\Delta m_N^{(1)} = m_N^{(1a)} + m_N^{(1b)} + m_N^{(1c)} + 2m_N^{(1d)}$$

Two-loop diagrams

- Two-loop diagrams at $O(p^5)$



$$\Delta m_N^{(2)} = m_N^{(2a)} + m_N^{(2b)} + m_N^{(2c)} + m_N^{(2d)} + 2m_N^{(2e)} + 2m_N^{(2g)} + m_N^{(2j)}$$

- Two-loop integrals

$$I_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = \iint \frac{d^d \ell_1}{(i\pi^{d/2})} \frac{d^d \ell_2}{(i\pi^{d/2})} \frac{1}{\mathcal{D}_1^{\nu_1} \mathcal{D}_2^{\nu_2} \mathcal{D}_3^{\nu_3} \mathcal{D}_4^{\nu_4} \mathcal{D}_5^{\nu_5}}$$

- ν_i integers; scalar integrals: all $\nu_i \geq 0$; tensor integrals: $\exists \nu_i \leq 0$
- Number of irreducible scalar product: $N = L \times E + L(L + 1)/2$

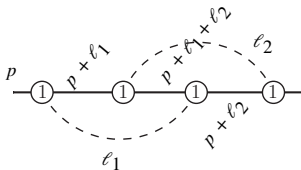
$$\mathcal{D}_1 = \ell_1^2 - M^2$$

$$\mathcal{D}_2 = \ell_2^2 - M^2$$

$$\mathcal{D}_3 = (p + \ell_1 + \ell_2)^2 - m^2$$

$$\mathcal{D}_4 = (p + \ell_1)^2 - m^2$$

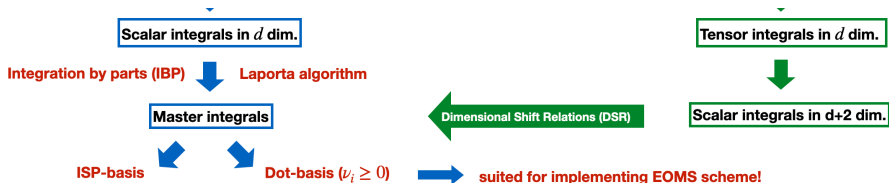
$$\mathcal{D}_5 = (p + \ell_2)^2 - m^2$$



IBP reduction and master integrals

- IBP reduction: e.g. diagram (2c)

$$m_N^{(2c)} = \frac{-3\kappa^2}{32m F^4} \left[2M^2 \left(2M^2 I_{11100} + 2I_{01100} + 2I_{10100} - I_{11000} \right) + I_{110(-1)0} + I_{1100(-1)} - 4I_{111(-1)(-1)} \right]$$



- Master integrals: 3 one-loop + 10 two-loop (dot-basis)

$$J_{\text{master}}^{1\text{-loop}} = \{J_{10}, J_{01}, J_{11}\} .$$

$$I_{\text{master}}^{2\text{-loop}} = \{I_{11111}, I_{11110} \leftrightarrow I_{11101}, I_{10111} \leftrightarrow I_{01111}, I_{21100} \leftrightarrow I_{12100}, I_{11100}, I_{00111}, I_{10101} \leftrightarrow I_{01110}, I_{10110} \leftrightarrow I_{01101}, I_{10100} \leftrightarrow I_{01100}, I_{00110} \leftrightarrow I_{00101}\} .$$

➔ Differential equation method [Lotikov, PLB254 (1991), PLB267(1991)] [Henn, PRL110(2013)]

- **Pedagogic example:** $g_A = 0$



$$m_N = m + 4c_1 M^2 - e_m M^4 + m_N^{1\text{-loop}} + m_N^{2\text{-loop}}$$

$$\text{UV}[m_N^{2\text{-loop}}] = \frac{3(6M^4 m + 8M^2 m^3 - 3m^5)}{32F^4 \Lambda^2 \epsilon^2} + \frac{132M^4 m - 16M^2 m^3 - 9m^5}{128F^4 \Lambda^2 \epsilon}$$

$$+ \frac{3m^3(3m^2 - 8M^2)}{16F^4 \Lambda^2 \epsilon} \log \frac{m^2}{\mu^2} - \frac{9M^4 m}{8F^4 \Lambda^2 \epsilon} \log \frac{M^2}{\mu^2} \quad \text{Non-local UV divergence}$$

$$\text{UV}[m_N^{1\text{-loop}}] = -\frac{3(d(c_3 - 2c_1) + c_2)M^4}{F^2 d} \left[-\frac{1}{\Lambda \epsilon} + \frac{1}{\Lambda} \log \frac{M^2}{\mu^2} + \mathcal{O}(\epsilon) \right]$$

- ➔ Local UV div. → tree-level counter term
- ➔ Non-local UV div. → one-loop diagram as the sub-diagram of two-loop diagram

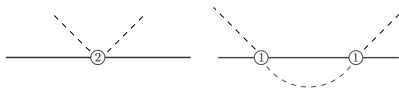
- UV renormalisation at two-loop level**

➔ **Step 1:** split the bare LECs in one-loop diagram

$$\text{UV}[m_N^{\text{1-loop}}] = - \frac{3(d(c_3 - 2c_1) + c_2)M^4}{F^2 d} \left[- \frac{1}{\Lambda \epsilon} + \frac{1}{\Lambda} \log \frac{M^2}{\mu^2} + \mathcal{O}(\epsilon) \right]$$

$$X = - \frac{\beta_X^{(1)}}{\epsilon \Lambda} + X^r + \mathcal{O}(\epsilon), \quad X \in \{c_1, c_2, c_3\}$$

$$\beta_{c_1}^{(1)} = 0, \quad \beta_{c_2}^{(1)} = \frac{m}{2F^2}, \quad \beta_{c_3}^{(1)} = \frac{m}{4F^2}$$



Pion-nucleon scattering up to $\mathcal{O}(\epsilon)$

➔ **Step 2:** split the bare LECs in the tree diagrams

$$m_N = \underbrace{m + 4c_1 M^2 - e_m M^4}_{\text{tree}} + m_N^{\text{1-loop}} + m_N^{\text{2-loop}}$$

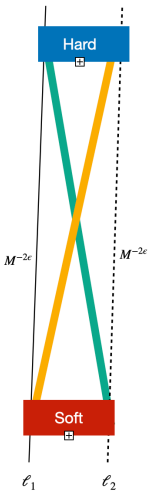
$$X = \frac{\beta_X^{(22)}}{\epsilon^2 \Lambda^2} - \frac{\beta_X^{(21)}}{\epsilon \Lambda^2} - \frac{\beta_X^{(11)}}{\epsilon \Lambda} + X^r, \quad X \in \{m, c_1, e_m\} \quad \text{Two-loop UV renormalised parameters!}$$

Chiral expansion of loop integrals

- Dimensional counting analysis**

i.e. Equivalent to strategy of regions

[Gegelia, Japaridze and Turashvili, Theor. Math. Phys. 101(1994)]
 [Beneke and Smirnov, NPB522(1998)]



$$I_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = \iint \frac{d^d \ell_1}{(i\pi^{d/2})} \frac{d^d \ell_2}{(i\pi^{d/2})} \frac{1}{\mathcal{D}_1^{\nu_1} \mathcal{D}_2^{\nu_2} \mathcal{D}_3^{\nu_3} \mathcal{D}_4^{\nu_4} \mathcal{D}_5^{\nu_5}}$$

$\ell_i \rightarrow M^{\alpha_i} \tilde{\ell}_i \quad \alpha_i \text{ non-negative real numbers}$

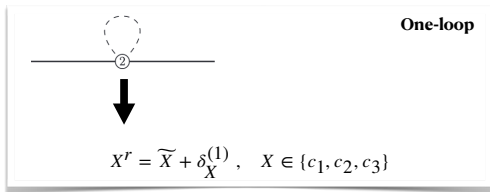
$$I_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = \sum_{\alpha_1, \alpha_2} M^{\varphi(\alpha_1, \alpha_2)} \times I_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{(\alpha_1, \alpha_2)}$$

$(\alpha_1, \alpha_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$

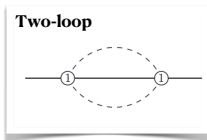
$$I_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = I_{\dots}^{(0,0)} + M^{-2\epsilon} I_{\dots}^{(0,1)} + M^{-2\epsilon} I_{\dots}^{(1,0)} + M^{-4\epsilon} I_{\dots}^{(1,1)}$$

Pure infrared regular part	infrared regular-singular part	Pure infrared singular part
Local PCB term	Non-local PCB term	PCB-conserving term (IR)

- EOMS scheme:** absorption of PCB terms by LECs

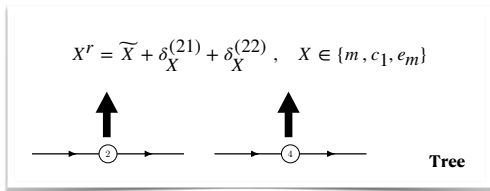


1-loop local PCB term



non-local PCB term

2-loop local PCB term



- ✓ $\delta_X^{(21)}$: cancel 1-loop local PCB term
- ✓ $\delta_X^{(22)}$: cancel 2-loop local PCB term
- ✓ $\delta_X^{(1)}$: cancel 2-loop non-local PCB term

- **The nucleon mass up to $O(p^5)$**

$$m_N = \underbrace{\widetilde{m}}_{\mathcal{O}(p^0)} - \underbrace{4\widetilde{c}_1 M^2}_{\mathcal{O}(p^2)} + \underbrace{\widetilde{m}_N^{(1a)}}_{\mathcal{O}(p^3)} - \underbrace{2\widetilde{c}_m M^4 + \widetilde{m}_N^{(1b)}}_{\mathcal{O}(p^4)} + \underbrace{\widetilde{m}_N^{(1c)} + 2\widetilde{m}_N^{(1d)} + \widetilde{m}_N^{\text{2-loop}} + \widetilde{m}_N^{\text{sub-diag.}}}_{\mathcal{O}(p^5)}$$

$$m_N^{\text{sub-diag.}} \sim \sum_X \left\{ \left(-\frac{\beta_X^{(1)}}{\epsilon\Lambda} \right) \times [\text{linear term in } \epsilon \text{ from one loops}] \right\}$$

- **Merit:**

- ➔ Faithful structure: respect original analytic property
- ➔ Controllable accuracy: possess correct power counting rule
- ➔ Renormalisation scale independent

- EOMS truncated at $O(p^5)$

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 .$$

$$k_1^{\text{EOMS}} = -4c_1 ,$$

$$k_2^{\text{EOMS}} = -\frac{3g^2\pi}{2F^2\Lambda} ,$$

$$k_3^{\text{EOMS}} = -\frac{3g^2}{2F^2 m \Lambda} + \frac{3(8c_1 - c_2 - 4c_3)}{2F^2\Lambda} ,$$

$$k_4^{\text{EOMS}} = -2e_m + \frac{3g^2}{2F^2 m \Lambda} + \frac{3c_2}{8F^2\Lambda} ,$$

$$k_5^{\text{EOMS}} = \frac{3g^2(16g^2 - 3)\pi}{4F^4\Lambda^2} ,$$

$$k_6^{\text{EOMS}} = \frac{19\pi g^4}{4F^4\Lambda^2} + \frac{6\pi g(d_{18} - 2d_{16})}{F^2\Lambda} + g^2 \left[\frac{6\pi}{F^4\Lambda^2} + \frac{3\pi(F^2 + 8m^2(2\ell_4 - 3\ell_3))}{16F^4\Lambda m^2} \right] .$$

Caution: the chiral expansion unavoidably enforces a truncation at a certain order, which changes the analytic property (as a function of M) of the original expression.

- IR truncated at $O(p^5)$

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 .$$

$$k_1^{\text{IR}} = -4c_1 ,$$

$$k_2^{\text{IR}} = -\frac{3g^2\pi}{2F^2\Lambda} ,$$

$$k_3^{\text{IR}} = -\frac{3g^2}{2F^2m\Lambda} + \frac{3(8c_1 - c_2 - 4c_3)}{2F^2\Lambda} ,$$

$$k_4^{\text{IR}} = -2e_m - \frac{3g^2}{4F^2m\Lambda} + \frac{3c_2}{8F^2\Lambda} ,$$

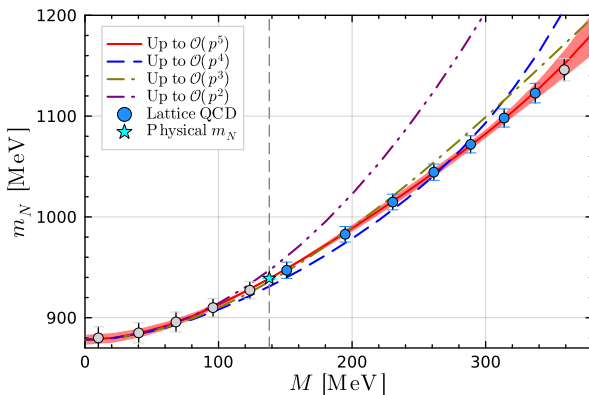
$$k_5^{\text{IR}} = \frac{3g^2(16g^2 - 3)\pi}{4F^4\Lambda^2} ,$$

$$k_6^{\text{IR}} = \frac{3\pi g^4}{F^4\Lambda^2} + \frac{6\pi g(d_{18} - 2d_{16})}{F^2\Lambda} + \frac{3\pi g^2(F^2 + 8m^2(2\ell_4 - 3\ell_3))}{16F^4\Lambda m^2} .$$

- ➔ Non-analytic terms in quark mass: (k_2, k_3, k_5) same, k_6 different;
- ➔ Scheme independent terms: k_3 & k_5 , guaranteed by the RGE method.

3. Chiral extrapolation

- Full EOMS

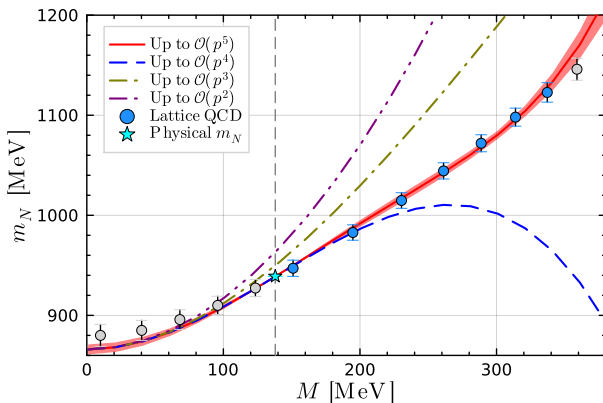


✓ Original analyticity guarantees the successfulness of chiral extrapolation

✓ Relativistic renormalisation scheme possesses good convergency

m	878.2(5.0) MeV
c_1	-0.90(0.07) GeV ⁻¹
e_m	-7.41(1.71) GeV ⁻²

- EOMS truncated at order p^5



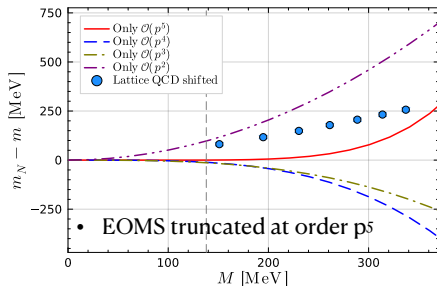
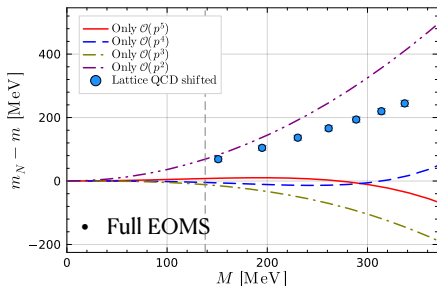
✓ Agree well in the fitting range, but deviate beyond;

✓ Weird behaviour of the pion mass dependence.

m	878.2(5.0) MeV
c_1	-0.90(0.07) GeV ⁻¹
e_m	-7.41(1.71) GeV ⁻²

Convergency property

- Assess the convergency of the chiral expansion



- Two-loop contribution is small, approximately 10 MeV, for pion mass < 300 MeV;
- At physical point:

$$m_N = \left\{ 878.2 + \underbrace{68.8}_{\mathcal{O}(p^2)} + \underbrace{[-11.4]}_{\mathcal{O}(p^3)} + \underbrace{[-4.6]}_{\mathcal{O}(p^4)} + \underbrace{7.9}_{\mathcal{O}(p^5)} \right\} \text{ MeV} = 938.9 \text{ MeV}$$

4. Summary and outlook

❖ Summary

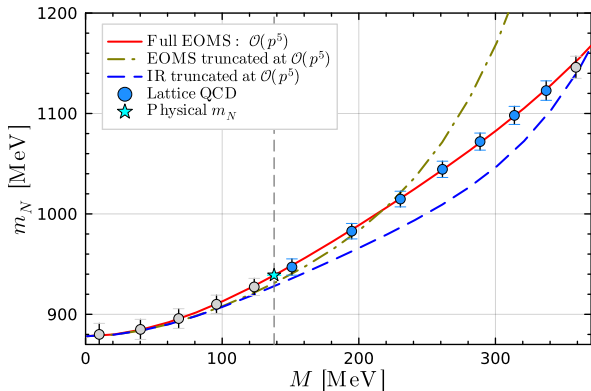
1. The applicability of EOMS at two-loop order is verified explicitly.
2. The notable PCB issue can be addressed by using the dimensional counting analysis approach.
3. The $O(p^5)$ contribution from the full EOMS is small around 10 MeV, implying that the chiral series in the full EOMS scheme converges very well.
4. The EOMS-truncated and IR-truncated results change the analytic structure.

❖ Outlook

Step into a promising era of two-loop BChPT, where the nucleon property can be explored with high precision!

Thanks!

- Full EOMS, EOMS truncated, IR truncated



- ➡ IR-truncated: the discarded terms due to the $\mathcal{O}(p^5)$ truncation contribute sizeably.
- ➡ EOMS-truncated: starts to deviate at pion mass ~ 220 MeV.