Unification of Particles and Symmetries in An Affine $\widehat{\mathfrak{su}}(8)_{k_U=1}$ Lie Algebra

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References

Pioneering works:

a "Towards a Grand Unified Theory of Flavor", Nucl.Phys.B 156 (1979) 126, Howard Georgi.

b "Doubly Lopsided Mass Matrices from Unitary Unification", Phys. Rev. D 78 (2008) 075001, 0804.1356, Stephen Barr.

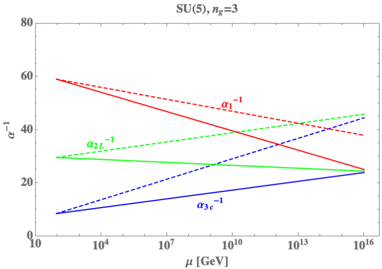
References

Recent papers:

- α "The global $B - L$ symmetry in the flavor-unified $\mathrm{SU}(N)$ theories", JHEP 04 (2024) 046, 2307.07921, **NC**, Ying-nan Mao, Zhaolong Teng.
- β "The Standard Model quark/lepton masses and the Cabibbo-Kobayashi-Maskawa mixing in an $\mathfrak{su}(8)$ theory", JHEP 12 (2024) 137, 2402.10471, **NC**, Ying-nan Mao, Zhaolong Teng.
- γ "The gauge coupling evolutions of an $\mathfrak{su}(8)$ theory with the maximally symmetry breaking pattern", JHEP 10 (2024) 149, 2406.09970, **NC**, Zhanpeng Hou, Ying-nan Mao, Zhaolong Teng.
- ρ "Further study of the maximally symmetry breaking patterns in an $\mathfrak{su}(8)$ theory", 2409.03172, **NC**, Zhiyuan Chen, Zhanpeng Hou, Ying-nan Mao, Zhaolong Teng.
- δ "The unification in an $\widehat{\mathfrak{su}}(8)_{k_U=1}$ affine Lie algebra", JHEP 12 (2024) 137, 2411.12979, **NC**, Zhanpeng Hou, Zhaolong Teng.

Historical reviews

- GUTs were proposed in terms of the simple Lie algebras of $\mathfrak{su}(5)$ with $3 \times \left[\overline{\mathbf{5}_{\mathbf{F}}} \oplus \mathbf{10}_{\mathbf{F}}\right]$ by ['74, Georgi-Glashow] (GG), or $\mathfrak{so}(10)$ with $3 \times \mathbf{16}_{\mathbf{F}}$ by ['75, Fritzsch-Minkowski]. The main ingredients: (i) local Lie algebras $\mathfrak{g}_{\rm SM} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_Y \subset \mathfrak{g}_{\rm GUT}$, AF of the QCD ['73, Gross, Wilczek, Politzer], (ii) the two-generational chiral fermions (with charm quark theorized in '70 by Glashow-Iliopoulos-Maiani, and discovered in late '74).
- The SUSY extension to the $\mathfrak{su}(5)$ can unify three SM gauge couplings at $\mu \sim 10^{16}\,\mathrm{GeV}$ ['81, Dimopoulos-Georgi], with sparticle masses $\sim \mathcal{O}(1)$ TeV.



Historical reviews

- The chiral fermions in the $\mathfrak{su}(5)$ are decomposed as
$\overline{\mathbf{5_F}} = \underbrace{(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}}_{{d_R}^c} \oplus \underbrace{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}}_{\ell_L}$ and
$\mathbf{10_F} = \underbrace{(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}}_{q_L} \oplus \underbrace{(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}}_{{u_R}^c} \oplus \underbrace{(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}}_{{e_R}^c}.$
- The SUSY $\mathfrak{su}(5)$ theory contains Higgs fields of $\mathbf{24_H} \oplus \mathbf{5_H} \oplus \overline{\mathbf{5_H}}$, with the GUT symmetry breaking of $\mathfrak{su}(5) \xrightarrow{\langle \mathbf{24_H} \rangle} \mathfrak{g}_{\rm SM}$.
- The Yukawa couplings come from the holomorphic superpotential of

$$W_Y = Y_D \overline{\mathbf{5_F}} \mathbf{10_F} \overline{\mathbf{5_H}} + Y_U \mathbf{10_F} \mathbf{10_F} \mathbf{5_H} \,. \quad (1)$$

At the EW scale, the Higgs spectrum is a type-II 2HDM of $\Phi_u \equiv (\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{H}} \subset \mathbf{5_H}$ and $\Phi_d \equiv (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}} \subset \overline{\mathbf{5_H}}$.

Historical reviews

- Besides of the well-acknowledged challenges within the *minimal* GUTs, there are two longstanding problems within the SM that have never been solved with/without the susy extension, which are: (i) the SM flavor puzzle, (ii) the PQ quality problem of the QCD axion.
- My understanding: the flavor sector is the "rough water" sector of the SM (according to S. Weinberg) where every BSM builder must confront with in your preferred framework.
- The formulation of the QM solved several fundamental puzzles in the late $19^{\rm th}$ century: (i) blackbody radiation [1900, Max Planck], (ii) photoelectric effects [1905, Albert Einstein], (iii) hydrogen spectrum [1913, Niels Bohr], by hypothesizing the quantized energies/angular momenta of particles.

The flavor puzzle: origin

- The SM flavor puzzle: (i) inter-generational mass hierarchies, (ii) intra-generational mass hierarchies with non-universal splitting patterns, and (iii) the CKM mixing pattern of the quarks and the PMNS mixing pattern of the neutrinos.
- Why/how $n_g = 3$? Both the SM and the *minimal* GUTs exhibit the simple repetitive structure in terms of their chiral irreducible anomaly-free fermion sets (IRAFFSs).

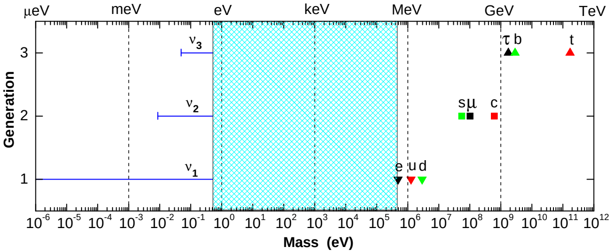


Figure: The SM fermion mass spectrum, [1909.09610, Z.Z. Xing].

The flavor puzzle: Yukawa couplings

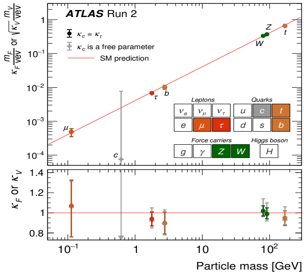


Figure: The LHC measurements of the SM Higgs boson, [2207.00092].

- The hierarchical/non-universal Yukawa couplings of the *single* SM Higgs boson $y_f = \sqrt{2} m_f / v_{\rm EW}$ for all SM quarks/leptons.
- Symmetry dictates interactions ['80, Chen-Ning Yang].

The flavor puzzle

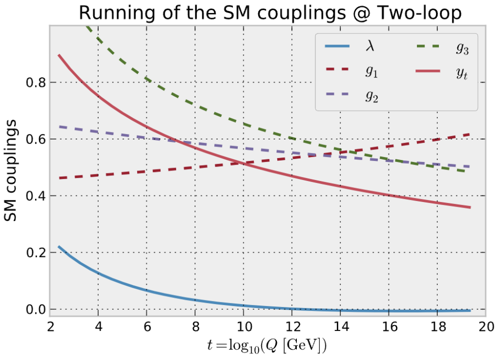


Figure: The RGEs of SM couplings, by PyR@TE.

- The RGEs cannot generate large mass hierarchies ['78, Froggatt, Nielsen].

The origin of generations

- The main conjecture [‘79, Georgi, ‘80, Nanopolous]: no three repetitive generations in the UV (GUT), but they emerge in the IR (SM).
- This was first considered by [‘79, Georgi] based on a unified Lie algebra of $\mathfrak{su}(N)$, with the anti-symmetric chiral fermions of

$$\{f_L\}_{\mathrm{SU}(N)} = \sum_k n_k \, [N\,, k]_{\mathbf{F}} \,, \; n_k \in \mathbb{Z}\,. \tag{2}$$

  No exotic fermions in the spectrum with the $[N\,, k]_{\mathbf{F}}$.
- The anomaly-free condition can be expressed in terms of a Diophantine equation

$$\sum_k n_k \, \mathrm{Anom}([N\,, k]_{\mathbf{F}}) = 0\,, \tag{3}$$

$$\mathrm{Anom}([N\,, k]_{\mathbf{F}}) = \frac{(N-2k)\,(N-3)!}{(N-k-1)!\,(k-1)!}\,. \tag{4}$$

Georgi's counting of the SM generations

- To decompose the $\mathfrak{su}(N)$ irreps into the $\mathfrak{su}(5)$, e.g., $\mathbf{N_F} = (N-5) \times \mathbf{1_F} \oplus \mathbf{5_F}$, and other irreps are decomposed by tensor products, ['79, Georgi].
- All fermion irreps in Eq. (2) are decomposed into the $\mathfrak{su}(5)$ irreps of $(\mathbf{1_F}\,, \mathbf{5_F}\,, \mathbf{10_F}\,, \overline{\mathbf{10_F}}\,, \overline{\mathbf{5_F}})$, and we denote their multiplicities as $(\nu_{\mathbf{1_F}}\,, \nu_{\mathbf{5_F}}\,, \nu_{\mathbf{10_F}}\,, \nu_{\overline{\mathbf{10_F}}}\,, \nu_{\overline{\mathbf{5_F}}})$.
- Their multiplicities should satisfy $\nu_{\mathbf{5_F}} + \nu_{\mathbf{10_F}} = \nu_{\overline{\mathbf{5_F}}} + \nu_{\overline{\mathbf{10_F}}}$ from the anomaly-free condition.
- The total SM fermion generations are determined by the net $\overline{\mathbf{5_F}}$'s or net $\mathbf{10_F}$'s

$$n_g = \nu_{\overline{\mathbf{5_F}}} - \nu_{\mathbf{5_F}} = \nu_{\mathbf{10_F}} - \nu_{\overline{\mathbf{10_F}}}\,. \tag{5}$$

- The SM $\overline{\mathbf{5_F}}$'s are from the $\overline{[N\,,1]}_{\mathbf{F}}$, and the SM $\mathbf{10_F}$'s are from the $[N\,,k\geq 2]_{\mathbf{F}}$.

Georgi's counting of the SM generations in GUTs

- The net $\mathbf{10_F}$'s from a particular $\mathfrak{su}(N)$ irrep [2209.11446]

$$\nu_{\mathbf{10_F}}\,[N\,,k]_{\mathbf{F}} - \nu_{\overline{\mathbf{10_F}}}\,[N\,,k]_{\mathbf{F}} = \frac{(N-2k)(N-5)!}{(k-2)!\,(N-k-2)!}\,. \tag{6}$$

- The usual rank-2 GG models can only give $\nu_{\mathbf{10_F}}\,[N\,,2]_{\mathbf{F}} - \nu_{\overline{\mathbf{10_F}}}\,[N\,,2]_{\mathbf{F}} = 1$. This means one can only repeat the set of anomaly-free fermion irreps to form multiple generations in rank-2 GG models.
- Alternatively, to embed multiple generations non-trivially in the GUTs, one must consider at least the rank-3 GG models. The leading candidate group must be $\mathfrak{su}(7)$, ['79, Frampton], since the $[6\,,3]_{\mathbf{F}}$ irrep of $\mathfrak{su}(6)$ is self-conjugate.

Georgi's counting of the SM generations in GUTs

- $\mathfrak{so}(2N)$ cannot embed multiple generations non-trivially, e.g., spinor irrep of $\mathbf{64_F}$ in the $\mathfrak{so}(14)$ is decomposed into $\mathfrak{so}(10)$ as

$$\mathbf{64_F} \to \ldots \to 2 \times \left[\mathbf{16_F} \oplus \overline{\mathbf{16_F}} \right] \,, \tag{7}$$

  $\mathbf{16_F} = \mathbf{1_F} \oplus \overline{\mathbf{5_F}} \oplus \mathbf{10_F}$, and $\overline{\mathbf{16_F}} = \mathbf{1_F} \oplus \mathbf{5_F} \oplus \overline{\mathbf{10_F}}$. One can only obtain $n_g = 2 - 2 = 0$ at the EW scale.
- The realistic symmetry breaking patterns of the $\mathfrak{su}(N)$ usually do not follow the $\mathfrak{su}(N) \to \ldots \to \mathfrak{su}(5) \to \mathfrak{g}_{\rm SM}$ sequence, which is dangerous in terms of the proton decay. n_g is independent of the symmetry breaking patterns.
- The zeroth stage would better be achieved by the $\mathfrak{su}(N)$ adjoint Higgs field [‘74, L.F.Li] as $\mathfrak{su}(N) \to \mathfrak{su}(k_1)_S \oplus \mathfrak{su}(k_2)_W \oplus \mathfrak{u}(1)_X \,, k_1 = [\frac{N}{2}]$ in the non-SUSY theories, since we wish to set the proton decay scale as high as possible. Currently, $\tau_p \gtrsim 10^{34}\,{\rm yr}$ [‘20, SuperK].

Georgi's counting of the SM generations in GUTs

- Georgi's "third law" of GUT ['79]: no repetition of a particular irrep of $\mathfrak{su}(N)$, i.e., $n_k = 0$ or $n_k = 1$ in Eq. (2). Georgi's minimal solution is

$$\{f_L\}_{\mathrm{SU}(11)} = [11\,,4]_{\mathbf{F}} \oplus [11\,,8]_{\mathbf{F}} \oplus [11\,,9]_{\mathbf{F}} \oplus [11\,,10]_{\mathbf{F}}\ , \tag{8}$$

  with $\dim_{\mathbf{F}} = 561$.
- The $\left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\right]$ is one chiral irreducible anomaly-free fermion set (IRAFFS) of the $\mathrm{SU}(5)$, so do one-generational SM fermions. They simply repeat.
- My understanding: no $3 \times \left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\right]$ rather than no $3 \times \overline{\mathbf{5_F}} \oplus 3 \times \mathbf{10_F}$ in order to prevent the simple repetitions of one chiral IRAFFS, such as in an $\mathfrak{su}(5)$ chiral theory.

The $\mathfrak{su}(8)$ theory

- A chiral IRAFFS is a set of left-handed anti-symmetric fermions of $\sum_{\mathcal{R}} m_{\mathcal{R}} \mathcal{F}_L(\mathcal{R})$, with $m_{\mathcal{R}}$ being the multiplicities of a particular fermion representation of $\mathcal{R}$. The anomaly-free condition reads $\sum_{\mathcal{R}} m_{\mathcal{R}} \,\mathrm{Anom}(\mathcal{F}_L(\mathcal{R})) = 0$. We also require the following conditions:
  1. $\mathrm{GCD}\{m_{\mathcal{R}}\} = 1$.
  2. The fermions in a chiral IRAFFS can no longer be removed, which would otherwise bring non-vanishing gauge anomalies.
  3. No singlet, self-conjugate, or adjoint fermions in a chiral IRAFFS.
- My "third law" [2307.07921]: *only distinctive chiral IRAFFSs without simple repetitions and can lead to $n_g = 3$ at the EW scale are allowed in an $\mathfrak{su}(N)$ chiral theory.*

The $\mathfrak{su}(8)$ theory

- The $\mathfrak{su}(8)$ theory with rank-2 and rank-3 chiral IRAFFSs of

$$\{f_L\}^{n_g=3}_{\mathfrak{su}(8)} = \Big[\overline{\mathbf{8_F}}^{\omega} \oplus \mathbf{28_F}\Big] \bigoplus \Big[\overline{\mathbf{8_F}}^{\dot\omega} \oplus \mathbf{56_F}\Big]\,,\ \dim_{\mathbf{F}} = 156\,,$$
$$\Omega = (\omega\,,\dot\omega)\,,\ \omega = (3\,,\mathrm{IV}\,,\mathrm{V}\,,\mathrm{VI})\,,\ \dot\omega = (\dot{1}\,,\dot{2}\,,\dot{\mathrm{VII}}\,,\dot{\mathrm{VIII}}\,,\dot{\mathrm{IX}})\,. \quad (9)$$

- Georgi's decompositions:

$$\begin{aligned} \overline{\mathbf{8_F}}^{\Omega} &= 3\times \mathbf{1_F}^{\Omega} \oplus \overline{\mathbf{5_F}}^{\Omega}\,,\\ \mathbf{28_F} &= 3\times \mathbf{1_F} \oplus 3\times \mathbf{5_F} \oplus \mathbf{10_F}\,,\\ \mathbf{56_F} &= \mathbf{1_F} \oplus 3\times \mathbf{5_F} \oplus 3\times \mathbf{10_F} \oplus \overline{\mathbf{10_F}}\,. \end{aligned} \quad (10)$$

Six $(\mathbf{5_F}\,,\overline{\mathbf{5_F}})$ pairs, one $(\mathbf{10_F}\,,\overline{\mathbf{10_F}})$ pair from the $\mathbf{56_F}$, and $3\times\big[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\big]_{\rm SM}$.

The $\mathfrak{su}(8)$ theory

- The global symmetries of the $\mathfrak{su}(8)$ theory:

$$\widetilde{\mathcal{G}}_{\rm global}\,[\mathfrak{su}(8)] = \left[\widetilde{\rm SU}(4)_\omega \otimes \widetilde{\rm U}(1)_{T_2} \otimes \widetilde{\rm U}(1)_{\rm PQ_2}\right]$$
$$\bigotimes \left[\widetilde{\rm SU}(5)_{\dot\omega} \otimes \widetilde{\rm U}(1)_{T_3} \otimes \widetilde{\rm U}(1)_{\rm PQ_3}\right],$$
$$[\mathfrak{su}(8)]^2 \cdot \widetilde{\rm U}(1)_{T_{2\,,3}} = 0\,, \quad [\mathfrak{su}(8)]^2 \cdot \widetilde{\rm U}(1)_{\rm PQ_{2\,,3}} \neq 0\,. \tag{11}$$

- The Higgs fields and the Yukawa couplings:

$$-\mathcal{L}_Y = Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega} + Y_{\mathcal{T}} \mathbf{28_F} \mathbf{28_F} \mathbf{70_H}$$
$$+ Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot\omega} \mathbf{56_F} \overline{\mathbf{28_H}}_{,\dot\omega} + \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \overline{\mathbf{28_H}}^{\dagger}_{,\dot\omega} \mathbf{63_H} + H.c.\,. \tag{12}$$

NB: $\mathbf{56_F}\mathbf{56_F}\mathbf{28_H} = 0$ [‘08, S. Barr], $d=5$ operator suppressed by $1/M_{\rm pl}$ is possible, with $M_{\rm pl} = (8\pi G_N)^{1/2} = 2.4 \times 10^{18}\,{\rm GeV}$.

- *Gravity breaks global symmetries.*

Global symmetries in the $\mathfrak{su}(8)$ theory

Fermions	$\overline{\mathbf{8_F}}^{\Omega=\omega,\dot{\omega}}$	$\mathbf{28_F}$	$\mathbf{56_F}$	
$\widetilde{\mathrm{U}}(1)_T$	$-3t$	$+2t$	$+t$	
$\widetilde{\mathrm{U}}(1)_{\mathrm{PQ}}$	p	q_2	q_3	
Higgs	$\overline{\mathbf{8_H}}_{,\omega}$	$\overline{\mathbf{28_H}}_{,\dot{\omega}}$	$\mathbf{70_H}$	$\mathbf{63_H}$
$\widetilde{\mathrm{U}}(1)_T$	$+t$	$+2t$	$-4t$	0
$\widetilde{\mathrm{U}}(1)_{\mathrm{PQ}}$	$-(p+q_2)$	$-(p+q_3)$	$-2q_2$	0

Table: The $\widetilde{\mathrm{U}}(1)_T$ and the $\widetilde{\mathrm{U}}(1)_{\mathrm{PQ}}$ charges, $p : q_2 \neq -3 : +2$ and $p : q_3 \neq -3 : +1$.

- One possible (maximally) symmetry breaking pattern [‘74, L.F.Li] of $\mathfrak{su}(8) \to \mathfrak{g}_{441} \to \mathfrak{g}_{341} \to \mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}} \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\mathrm{EM}}$.

Global symmetries in the $\mathfrak{su}(8)$ theory

- The global $\widetilde{\mathrm{U}}(1)_T$ symmetries at different stages

$$
\begin{aligned}
&\mathfrak{su}(8) \to \mathfrak{g}_{441} \; : \; \mathcal{T}' = \mathcal{T} - 4t\mathcal{X}_0 \,, \\
&\mathfrak{g}_{441} \to \mathfrak{g}_{341} \; : \; \mathcal{T}'' = \mathcal{T}' + 8t\mathcal{X}_1 \,, \\
&\mathfrak{g}_{341} \to \mathfrak{g}_{331} \; : \; \mathcal{T}''' = \mathcal{T}'' \,, \quad \mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM} \; : \; \mathcal{B} - \mathcal{L} = \mathcal{T}''' \,.
\end{aligned} \tag{13}
$$

Consistent relations of $(\mathcal{B}-\mathcal{L})(q_L) = \frac{4}{3}t$, $(\mathcal{B}-\mathcal{L})(\ell_L) = -4t$, and etc.

Higgs	$\mathfrak{g}_{441} \to \mathfrak{g}_{341}$	$\mathfrak{g}_{341} \to \mathfrak{g}_{331}$	$\mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM}$	$\mathfrak{g}_{\rm SM} \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\rm EM}$
$\overline{\mathbf{8_H}}_{,\omega}$	✓	✓	✓	✓
$\overline{\mathbf{28_H}}_{,\dot\omega}$	✗	✓	✓	✓
$\mathbf{70_H}$	✗	✗	✗	✓

Symmetry breaking pattern in the $\mathfrak{su}(8)$ theory

- The vectorlike fermions of six $(\mathbf{5_F}, \overline{\mathbf{5_F}})$-pairs and one $(\mathbf{10_F}, \overline{\mathbf{10_F}})$-pair become massive through Eq. (12) as follows

  0 : $\mathfrak{su}(8) \xrightarrow{\mathbf{63_H}} \mathfrak{g}_{441}$, all fermions remain massless.

  1 : $\mathfrak{g}_{441} \xrightarrow{\overline{\mathbf{8_H}}_{,\mathrm{IV}}} \mathfrak{g}_{341}$, one $(\mathbf{5_F}, \overline{\mathbf{5_F}})$-pair.

  2 : $\mathfrak{g}_{341} \xrightarrow{\overline{\mathbf{8_H}}_{,\mathrm{V}}, \overline{\mathbf{28_H}}_{,\dot{1},\mathrm{V}\dot{\mathrm{I}}\mathrm{I}}} \mathfrak{g}_{331}$, two $(\mathbf{5_F}, \overline{\mathbf{5_F}})$-pairs and one $(\mathbf{10_F}, \overline{\mathbf{10_F}})$-pair.

  3 : $\mathfrak{g}_{331} \xrightarrow{\overline{\mathbf{8_H}}_{,3,\mathrm{VI}}, \overline{\mathbf{28_H}}_{,\dot{2},\mathrm{I}\dot{\mathrm{I}}\mathrm{X},\mathrm{I}\dot{\mathrm{X}}}} \mathfrak{g}_{\mathrm{SM}}$, three $(\mathbf{5_F}, \overline{\mathbf{5_F}})$-pairs.

  Each can be precisely counted by anomaly-free conditions.
- 23 out of 27 left-handed sterile neutrinos remain massless by the 't Hooft anomaly matching of $[\widetilde{\mathrm{U}}(1)_T]^3 = ... = [\widetilde{\mathrm{U}}(1)_{B-L}]^3$.

Symmetry breaking pattern in the $\mathfrak{su}(8)$ theory

- The VEV assignments

$$\mathfrak{g}_{441}\to\mathfrak{g}_{341} \;:\; \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\mathrm{IV}}\rangle\equiv\frac{1}{\sqrt{2}}W_{\overline{\mathbf{4}}\,,\mathrm{IV}}\,, \tag{14a}$$

$$\mathfrak{g}_{341}\to\mathfrak{g}_{331} \;:\; \langle(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\mathrm{V}}\rangle\equiv\frac{1}{\sqrt{2}}w_{\overline{\mathbf{4}}\,,\mathrm{V}}\,,$$
$$\langle(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{1}\,,\dot{\mathrm{V}}\mathrm{II}}\rangle\equiv\frac{1}{\sqrt{2}}w_{\overline{\mathbf{4}}\,,\dot{1}\,,\dot{\mathrm{V}}\mathrm{II}}\,, \tag{14b}$$

$$\mathfrak{g}_{331}\to\mathfrak{g}_{\mathrm{SM}} \;:\; \langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,3\,,\mathrm{VI}\,,\mathrm{I}\dot{\mathrm{X}}}\rangle\equiv\frac{1}{\sqrt{2}}V_{\overline{\mathbf{3}}\,,3\,,\mathrm{VI}\,,\mathrm{I}\dot{\mathrm{X}}}$$
$$\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})'_{\mathbf{H}\,,\dot{2}\,,\dot{\mathrm{V}}\mathrm{III}}\rangle\equiv\frac{1}{\sqrt{2}}V'_{\overline{\mathbf{3}}\,,\dot{2}\,,\dot{\mathrm{V}}\mathrm{III}}\,, \tag{14c}$$

$$\mathrm{EWSB} \;:\; \langle(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})'''_{\mathbf{H}}\rangle\equiv\frac{1}{\sqrt{2}}v_{\mathrm{EW}}\,. \tag{14d}$$

The VEVs in black are the minimal set to integrate out the massive vectorlike fermions. The VEVs in red are necessary for the $(d^i\,,\ell^i)$ masses.

Some remarks on the Higgs VEVs

- If the $\overline{\mathbf{8_H}}_{,\omega=3}$ and the $\overline{\mathbf{28_H}}_{,\dot\omega=\dot 1\,,\dot 2}$ developed the EWSB VEVs, one expects a total of four Higgs doublets at the EW scale together with the $(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{H}}^{\prime\prime\prime}\subset\mathbf{70_H}$, which then lead to thirteen physical Higgs bosons. This is experimentally challenging with the current LHC results.
- If the $\overline{\mathbf{8_H}}_{,\omega=3}$ and the $\overline{\mathbf{28_H}}_{,\dot\omega=\dot 1\,,\dot 2}$ to develop intermediate VEVs, it is possible to assign them so that the inter-generational down-type quark and charge lepton masses become hierarchical.
- Dimensionless parameters

$$
\begin{aligned}
&\zeta_1\equiv\frac{W_{\overline{\mathbf{4}}\,,\mathrm{IV}}}{M_{\mathrm{pl}}}\,,\quad \zeta_2\equiv\frac{w_{\overline{\mathbf{4}}\,,\mathrm{V}}}{M_{\mathrm{pl}}}\,,\quad \dot\zeta_2\equiv\frac{w_{\overline{\mathbf{4}}\,,\dot 1\,,\dot{\mathrm{V}}\mathrm{II}}}{M_{\mathrm{pl}}}\,,\\
&\zeta_3\equiv\frac{V_{\overline{\mathbf{3}}\,,3\,,\mathrm{VI}}}{M_{\mathrm{pl}}}\,,\quad \dot\zeta_3'\equiv\frac{V'_{\overline{\mathbf{3}}\,,\dot 2\,,\mathrm{II}\dot{\mathrm{X}}}}{M_{\mathrm{pl}}}\,,\quad \dot\zeta_3\equiv\frac{V_{\overline{\mathbf{3}}\,,\mathrm{I}\dot{\mathrm{X}}}}{M_{\mathrm{pl}}}\,,\\
&\zeta_1\gg\zeta_2\sim\dot\zeta_2\gg\zeta_3\sim\dot\zeta_3'\sim\dot\zeta_3\,.
\end{aligned}
\tag{15}
$$

The $\mathfrak{su}(8)$ Higgs fields

- Decompositions of $\overline{\mathbf{8_H}}_{,\omega}/\overline{\mathbf{28_H}}_{,\dot\omega}$

$$\begin{aligned}
\overline{\mathbf{8_H}}_{,\omega} &\supset \underline{(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega}} \oplus \underline{(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\omega}} \supset \underline{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}}\,,\\
\overline{\mathbf{28_H}}_{,\dot\omega} &\supset \underline{(\mathbf{1}\,,\mathbf{6}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot\omega}} \oplus \underline{(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot\omega}}\\
&\supset \left[\underline{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})'_{\mathbf{H}\,,\dot\omega}} \oplus (\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega}\right] \oplus \underline{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot\omega}}\,.
\end{aligned} \tag{16}$$

- The global $\widetilde{\mathrm{U}}(1)_{B-L}$ according to Eq. (13)

$$\begin{aligned}
\mathbf{70_H} &\supset \underline{(\mathbf{4}\,,\overline{\mathbf{4}}\,,+\frac{1}{2})_{\mathbf{H}}} \oplus (\overline{\mathbf{4}}\,,\mathbf{4}\,,-\frac{1}{2})_{\mathbf{H}} \supset \ldots\\
&\supset \underbrace{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})'''_{\mathbf{H}}}_{\mathcal{B}-\mathcal{L}=0} \oplus \underbrace{(\mathbf{1}\,,\mathbf{2}\,,-\frac{1}{2})'''_{\mathbf{H}}}_{\mathcal{B}-\mathcal{L}=-8t}\,.
\end{aligned} \tag{17}$$

Conjecture: the $(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})'''_{\mathbf{H}} \subset \mathbf{70_H}$ is the only SM Higgs doublet.

Top quark mass in the $\mathfrak{su}(8)$ theory

- The natural top quark mass from the tree level

$$
\begin{aligned}
& Y_{\mathcal{T}} \mathbf{28_F} \mathbf{28_F} \mathbf{70_H} \supset Y_{\mathcal{T}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}\,, \mathbf{4}\,, 0)_{\mathbf{F}} \otimes (\mathbf{4}\,, \overline{\mathbf{4}}\,, +\frac{1}{2})_{\mathbf{H}} \\
\supset & \ ... \supset Y_{\mathcal{T}} (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{2}\,, +\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{H}}^{\prime\prime\prime} \\
\Rightarrow & \ \frac{1}{\sqrt{2}} Y_{\mathcal{T}} t_L {t_R}^c v_{\rm EW} \,. \qquad (18)
\end{aligned}
$$

- With $(\mathbf{3}\,, \mathbf{2}\,, +\frac{1}{6})_{\mathbf{F}} \equiv (t_L\,, b_L)^T$ and $(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \equiv {t_R}^c$ coming from the $\mathbf{28_F}$, it is straightforward to infer that $(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}} \equiv {\tau_R}^c$ also lives in the $\mathbf{28_F}$. The third-generational SM $\mathbf{10_F}$ reside in the $\mathbf{28_F}$, while the first- and second-generational SM $\mathbf{10_F}$'s must reside in the $\mathbf{56_F}$.
- Top quark mass conjecture: a rank-2 chiral IRAFFS is necessary so that only the top quark obtains mass with the natural Yukawa coupling at the EW scale. The $\mathfrak{su}(9)$ with $9 \times \overline{\mathbf{9_F}} \oplus \mathbf{84_F}$ is ruled out [2307.07921].

The $\mathfrak{su}(8)$ fermions

SU(8)	$\mathcal{G}_{441}$	$\mathcal{G}_{341}$	$\mathcal{G}_{331}$	$\mathcal{G}_{\rm SM}$
$\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}$	$(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \,:\, \mathcal{D}_R^{\Omega\, c}$
		$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \,:\, \check{\mathcal{N}}_L^{\Omega}$
	$(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \,:\, \mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \,:\, \check{\mathcal{N}}_L^{\Omega'}$
			$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \,:\, \check{\mathcal{N}}_L^{\Omega''}$

SU(8)	$\mathcal{G}_{441}$	$\mathcal{G}_{341}$	$\mathcal{G}_{331}$	$\mathcal{G}_{\rm SM}$
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \,:\, \mathfrak{D}_L$
		$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$\underline{(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \,:\, {t_R}^c}$
	$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}} \,:\, ({\mathfrak{e}_R}^c, {\mathfrak{n}_R}^c)^T$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} \,:\, \check{\mathfrak{n}}_R^c$
			$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}' \,:\, ({\mathfrak{n}_R'}^c, -{\mathfrak{e}_R'}^c)^T$
				$\underline{(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \,:\, {\tau_R}^c}$
	$(\mathbf{4}, \mathbf{4}, 0)_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}$	$\underline{(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \,:\, (t_L, b_L)^T}$
				$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}' \,:\, \mathfrak{D}_L'$
			$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}''$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'' \,:\, \mathfrak{D}_L''$
		$(\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}''$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}'' \,:\, ({\mathfrak{e}_R''}^c, {\mathfrak{n}_R''}^c)^T$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}' \,:\, \check{\mathfrak{n}}_R'^c$
			$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}''$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}'' \,:\, \check{\mathfrak{n}}_R''^c$

The $\mathfrak{su}(8)$ fermions

SU(8)	$\mathcal{G}_{441}$	$\mathcal{G}_{341}$	$\mathcal{G}_{331}$	$\mathcal{G}_{\rm SM}$
$\mathbf{56_F}$	$(\mathbf{1},\overline{\mathbf{4}},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{4}},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{2}},+\frac{1}{2})'''_{\mathbf{F}}\ :\ (\mathfrak{n}_R^{\prime\prime\prime c},-\mathfrak{e}_R^{\prime\prime\prime c})^T$
				$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}\ :\ {\mu_R}^c$
			$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}\ :\ {\mathfrak{E}_R}^c$
	$(\overline{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}\ :\ {u_R}^c$
		$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}\ :\ \mathfrak{E}_L$
	$(\mathbf{4},\mathbf{6},+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{6},+\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'_{\mathbf{F}}\ :\ (c_L,s_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}\ :\ \mathfrak{D}_L'''$
			$(\mathbf{3},\overline{\mathbf{3}},+\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3},\overline{\mathbf{2}},+\frac{1}{6})''_{\mathbf{F}}\ :\ (\mathfrak{d}_L,-\mathfrak{u}_L)^T$
				$(\mathbf{3},\mathbf{1},+\frac{2}{3})_{\mathbf{F}}\ :\ \mathfrak{U}_L$
		$(\mathbf{1},\mathbf{6},+\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''_{\mathbf{F}}\ :\ (\mathfrak{e}_R^{\prime\prime\prime\prime c},\mathfrak{n}_R^{\prime\prime\prime\prime c})^T$
				$(\mathbf{1},\mathbf{1},0)'''_{\mathbf{F}}\ :\ \check{\mathfrak{n}}_R^{\prime\prime\prime\, c}$
			$(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1},\overline{\mathbf{2}},+\frac{1}{2})'''''_{\mathbf{F}}\ :\ (\mathfrak{n}_R^{\prime\prime\prime\prime\prime c},-\mathfrak{e}_R^{\prime\prime\prime\prime\prime c})^T$
				$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}\ :\ {e_R}^c$
	$(\mathbf{6},\mathbf{4},-\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{4},-\frac{1}{12})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'''_{\mathbf{F}}\ :\ (u_L,d_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}\ :\ \mathfrak{D}_L''''$
			$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''''_{\mathbf{F}}\ :\ \mathfrak{D}_L'''''$
		$(\overline{\mathbf{3}},\mathbf{4},-\frac{5}{12})_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{3},-\frac{1}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6})_{\mathbf{F}}\ :\ ({\mathfrak{d}_R}^c,{\mathfrak{u}_R}^c)^T$
				$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}\ :\ {\mathfrak{U}_R}^c$
			$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}$	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}\ :\ {c_R}^c$

Vectorlike fermions in the $\mathfrak{su}(8)$ theory

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{441} $\{\Omega\}$	$\mathfrak{D}$ IV	-	$(\mathfrak{e}'' , \mathfrak{n}'')$ IV	$\{\check{\mathfrak{n}}' , \check{\mathfrak{n}}''\}$ $\{\mathrm{IV}' , \mathrm{IV}''\}$
v_{341} $\{\Omega\}$	$\mathfrak{d} , \{\mathfrak{D}'' , \mathfrak{D}'''''\}$ $\{\mathrm{V} , \dot{\mathrm{V}}\mathrm{II}\}$	$\mathfrak{u} , \mathfrak{U}$	$\mathfrak{E} , (\mathfrak{e} , \mathfrak{n}) , (\mathfrak{e}'''' , \mathfrak{n}'''')$ $\{\mathrm{V} , \dot{\mathrm{V}}\mathrm{II}\}$	$\{\check{\mathfrak{n}} , \check{\mathfrak{n}}'''\}$ $\{\mathrm{V}' , \dot{\mathrm{V}}\mathrm{II}'\}$
v_{331} $\{\Omega\}$	$\{\mathfrak{D}' , \mathfrak{D}''' , \mathfrak{D}''''\}$ $\{\mathrm{VI} , \mathrm{I}\dot{\mathrm{I}}\mathrm{X} , \mathrm{I}\dot{\mathrm{X}}\}$	-	$(\mathfrak{e}' , \mathfrak{n}') , (\mathfrak{e}''' , \mathfrak{n}''') , (\mathfrak{e}''''' , \mathfrak{n}''''')$ $\{\mathrm{VI} , \mathrm{I}\dot{\mathrm{I}}\mathrm{X} , \mathrm{I}\dot{\mathrm{X}}\}$	-

Table: The vectorlike fermions at different intermediate symmetry breaking scales in the $\mathfrak{su}(8)$ theory.

SM fermion masses in the $\mathfrak{su}(8)$ theory

- To generate other lighter SM fermion masses: the gravitational effects through $d = 5$ operators, which break the global symmetries in Eq. (11) explicitly.
- The direct Yukawa couplings of $\mathcal{O}_{\mathcal{F}}^{d=5}$:

$$
\begin{aligned}
& c_3 \mathcal{O}_{\mathcal{F}}^{(3\,,2)} \equiv c_3 \overline{\mathbf{8_F}}^{\dot\omega} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}^{\dagger}_{,\dot\kappa} \cdot \mathbf{70_H}^{\dagger} \\
\Rightarrow\ & c_3 \left[\dot\zeta_3 (s_L \mathcal{D}_R^{\dot\omega\,c} - \mathcal{E}_L^{\dot\omega} \mu_R{}^c) + \dot\zeta_3' (d_L \mathcal{D}_R^{\dot\omega\,c} - \mathcal{E}_L^{\dot\omega} e_R{}^c) \right] v_{\rm EW} \,, \\
& c_4 \mathcal{O}_{\mathcal{F}}^{(4\,,1)} \equiv c_4 \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{\,,\dot\omega} \cdot \mathbf{70_H} \Rightarrow c_4 \dot\zeta_2 (c_L u_R{}^c + \cancel{u_L c_R{}^c}) v_{\rm EW} \,, \\
& c_5 \mathcal{O}_{\mathcal{F}}^{(5\,,1)} \equiv c_5 \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} \\
\Rightarrow\ & c_5 \left[\zeta_1 (u_L t_R{}^c + t_L u_R{}^c) + \zeta_2 (c_L t_R{}^c + t_L c_R{}^c) \right] v_{\rm EW} \,. \qquad (19)
\end{aligned}
$$

- All $(u\,, c\,, t)$ obtain hierarchical masses, while all $(d^i\,, \ell^i)$ are massless.

SM fermion masses in the $\mathfrak{su}(8)$ theory

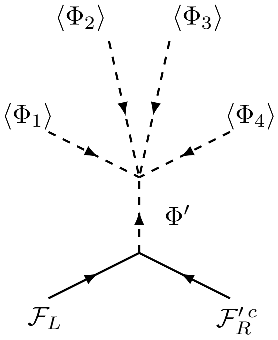


Figure: The indirect Yukawa couplings.

$(d^i\,,\ell^i)$ obtain masses through the EWSB components in renormalizable Yukawa couplings of $\overline{\mathbf{8_F}}^{\omega}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\omega}$ and $\overline{\mathbf{8_F}}^{\dot\omega}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega}$ with the $d=5$ Higgs mixing operators.

SM fermion masses in the $\mathfrak{su}(8)$ theory

- There are two indirect Yukawa couplings with the irreducible Higgs mixing operators of $\mathcal{O}_{\mathcal{H}}^{d=5}$:

$$
\begin{aligned}
\mathcal{O}_{\mathscr{A}}^{d=5} &\equiv \epsilon_{\omega_1\omega_2\omega_3\omega_4} \overline{\mathbf{8_H}}^{\dagger}{}_{,\omega_1} \overline{\mathbf{8_H}}^{\dagger}{}_{,\omega_2} \overline{\mathbf{8_H}}^{\dagger}{}_{,\omega_3} \overline{\mathbf{8_H}}^{\dagger}{}_{,\omega_4} \mathbf{70_H}^{\dagger}\,, \\
&\quad \mathcal{P}\mathcal{Q} = 2(2p+3q_2) \neq 0\,,
\end{aligned} \tag{20a}
$$

$$
\begin{aligned}
\mathcal{O}_{\mathscr{B}}^{d=5} &\supset (\overline{\mathbf{28_H}}^{\dagger}{}_{,\dot{1}} \overline{\mathbf{28_H}}_{,\dot{\mathrm{VII}}}) \cdot \overline{\mathbf{28_H}}^{\dagger}{}_{,\mathrm{II}\dot{\mathrm{X}}} \overline{\mathbf{28_H}}^{\dagger}{}_{,\dot{1}} \mathbf{70_H}^{\dagger}\,, \\
&\quad (\overline{\mathbf{28_H}}^{\dagger}{}_{,\dot{1}} \overline{\mathbf{28_H}}_{,\dot{\mathrm{VII}}}) \cdot \overline{\mathbf{28_H}}^{\dagger}{}_{,\mathrm{I}\dot{\mathrm{X}}} \overline{\mathbf{28_H}}^{\dagger}{}_{,\dot{2}} \mathbf{70_H}^{\dagger}\,, \\
&\quad \mathcal{P}\mathcal{Q} = 2(p+q_2+q_3)\,.
\end{aligned} \tag{20b}
$$

- Each operator of $\mathcal{O}_{\mathcal{H}}^{d=5}$
  1. *breaks the global symmetries explicitly*;
  2. *can not be further partitioned into subset of renormalizable operators, among which any of them can be allowed by both the gauge and the global symmetries. This relies on the VEV assignments in Eqs.* (14).

SM fermion masses in the $\mathfrak{su}(8)$ theory

- The $(u\,,c\,,t)$ masses

$$\mathcal{M}_u = \frac{v_{\rm EW}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & c_5\zeta_1/\sqrt{2} \\ c_4\dot{\zeta}_2/\sqrt{2} & 0 & c_5\zeta_2/\sqrt{2} \\ c_5\zeta_1/\sqrt{2} & c_5\zeta_2/\sqrt{2} & Y_\mathcal{T} \end{pmatrix} . \quad (21)$$

- The $(d\,,s\,,b)$ masses

$$\mathcal{M}_d \approx \frac{v_{\rm EW}}{4} \begin{pmatrix} (2c_3 + Y_\mathcal{D} d_\mathscr{B})\dot{\zeta}_3' & (2c_3 + Y_\mathcal{D} d_\mathscr{B}\Delta_{\dot{2}})\dot{\zeta}_3' & 0 \\ (2c_3 + Y_\mathcal{D} d_\mathscr{B}\Delta_{\dot{1}}')\dot{\zeta}_3 & (2c_3 + Y_\mathcal{D} d_\mathscr{B}\zeta_{23}^{-2})\dot{\zeta}_3 & 0 \\ 0 & 0 & Y_\mathcal{B} d_\mathscr{A}\zeta_{23}^{-1}\zeta_1 \end{pmatrix} . \quad (22)$$

  The charged lepton masses are $\mathcal{M}_\ell = (\mathcal{M}_d)^T$.

- Natural fermion mass textures by following the gravity-induced $d = 5$ operators.

SM fermion masses in the $\mathfrak{su}(8)$ theory

- The $(u\,,c\,,t)$ masses

$$m_u \approx c_4 \frac{\zeta_2 \dot{\zeta}_2}{2\zeta_1} v_{\rm EW}\,, \quad m_c \approx c_5^2 \frac{\zeta_1^2}{2\sqrt{2} Y_{\mathcal{T}}} v_{\rm EW}\,, \quad m_t \approx \frac{Y_{\mathcal{T}}}{\sqrt{2}} v_{\rm EW}\,. \tag{23}$$

- The $(d\,,s\,,b)$ and $(e\,,\mu\,,\tau)$ masses

$$\begin{aligned} m_d = m_e &\approx \frac{c_3 \dot{\zeta}_3}{2} |\tan\lambda| v_{\rm EW}\,, \quad m_s = m_\mu \approx \frac{1}{4}(2c_3 + Y_{\mathcal{D}} d_{\mathscr{B}} \zeta_{23}^{-2}) \dot{\zeta}_3 v_{\rm EW}\,, \\ m_b = m_\tau &\approx Y_{\mathcal{B}} \frac{d_{\mathscr{A}} \zeta_1 \zeta_2}{4\zeta_3} v_{\rm EW}\,. \end{aligned} \tag{24}$$

- The CKM mixing:

$$\hat{V}_{\rm CKM}\Big|_{\mathfrak{su}(8)} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & \frac{c_5}{Y_{\mathcal{T}}}\zeta_2 \\ -\lambda & 1-\lambda^2/2 & -\frac{c_5}{Y_{\mathcal{T}}}\zeta_1 \\ -\frac{c_5}{Y_{\mathcal{T}}}(\lambda\zeta_1+\zeta_2) & -\frac{c_5}{Y_{\mathcal{T}}}\zeta_1 & 1 \end{pmatrix}\,. \tag{25}$$

SM fermion masses in the $\mathfrak{su}(8)$ theory: benchmark

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	2.0×10^{-5}	0.5	0.5	0.8
c_3	c_4	c_5	$d_{\mathscr{A}}$	$d_{\mathscr{B}}$	λ
1.0	0.2	1.0	0.01	0.01	0.22
m_u	m_c	m_t	$m_d = m_e$	$m_s = m_\mu$	$m_b = m_\tau$
1.6×10^{-3}	0.6	139.2	0.5×10^{-3}	6.4×10^{-2}	1.5
$\lvert V_{ud}\rvert$	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$			
0.98	0.22	3.0×10^{-3}			
$\lvert V_{cd}\rvert$	$\lvert V_{cs}\rvert$	$\lvert V_{cb}\rvert$			
0.22	0.98	7.5×10^{-2}			
$\lvert V_{td}\rvert$	$\lvert V_{ts}\rvert$	$\lvert V_{tb}\rvert$			
0.019	7.5×10^{-2}	1			

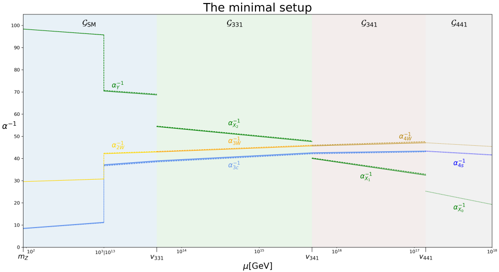
Table: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

Flavor non-universality in the $\mathfrak{su}(8)$

	u	c	t	
$g_f^{V\,\mathcal{M}}$	$-\frac{1}{24}+\frac{7}{24}s_{\vartheta_C}^2$	$-\frac{1}{24}+\frac{7}{24}s_{\vartheta_C}^2$	$-\frac{1}{24}+\frac{7}{24}s_{\vartheta_C}^2$	
$g_f^{A\,\mathcal{M}}$	$\frac{1}{8}+\frac{3}{8}s_{\vartheta_C}^2$	$-\frac{1}{8}+\frac{1}{8}s_{\vartheta_C}^2$	$-\frac{1}{8}+\frac{3}{8}s_{\vartheta_C}^2$	
	d	s	b	
$g_f^{V\,\mathcal{M}}$	$-\frac{1}{24}-\frac{5}{24}s_{\vartheta_C}^2$	$\frac{1}{12}-\frac{1}{12}s_{\vartheta_C}^2$	$\frac{1}{12}-\frac{5}{24}s_{\vartheta_C}^2$	
$g_f^{A\,\mathcal{M}}$	$\frac{1}{8}-\frac{1}{8}s_{\vartheta_C}^2$	$-\frac{1}{4}s_{\vartheta_C}^2$	$-\frac{1}{8}s_{\vartheta_C}^2$	
	e	μ	τ	$\nu_{e\,L}/\nu_{\mu\,L}/\nu_{\tau\,L}$
$g_f^{V\,\mathcal{M}}$	$\frac{1}{8}-\frac{3}{8}s_{\vartheta_C}^2$	$-\frac{1}{2}s_{\vartheta_C}^2$	$-\frac{3}{8}s_{\vartheta_C}^2$	$-\frac{1}{8}s_{\vartheta_C}^2$
$g_f^{A\,\mathcal{M}}$	$\frac{1}{8}-\frac{1}{8}s_{\vartheta_C}^2$	$-\frac{1}{4}s_{\vartheta_C}^2$	$-\frac{1}{8}s_{\vartheta_C}^2$	$\frac{1}{8}s_{\vartheta_C}^2$

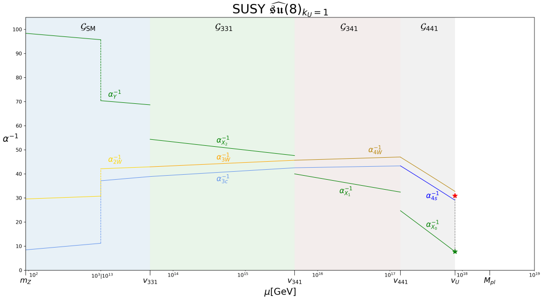
Table: The couplings of the neutral currents mediated by the neutral $\mathcal{G}_{441}$ gauge boson M_μ in the $V-A$ basis, $\tan\vartheta_C \equiv g_{X_0}/(\sqrt{6}g_{4s})$. [2406.09970].

The RGE of the minimal $\mathfrak{su}(8)$ theory



- No gauge coupling unification within the minimal $\mathfrak{su}(8)$ theory [2406.09970], and the large discrepancy cannot be compensated by the threshold effects.

The RGE of the $\mathcal{N}=1$ $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory



- In the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory [2411.12979], we find the conformal embedding of $\widehat{\mathfrak{su}}(4)_{k_s=1}\oplus\widehat{\mathfrak{su}}(4)_{k_W=1}\oplus\widehat{\mathfrak{u}}(1)_{k_1=1/4}$. The gauge coupling unification is achieved through the relation of

$$g_{4s}^2=g_{4W}^2=\frac{1}{4}g_{X_0}^2=\frac{8\pi G_N}{\alpha'}\,. \qquad (26)$$

The $\mathcal{N}=1$ $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory

- To ensure $[\mathfrak{su}(8)]^3=0$ and $[\mathfrak{su}(8)]^2\cdot\widetilde{\mathrm{U}}(1)_T=0$, the SUSY extension with chiral superfields of $\{H\}=\underbrace{\mathbf{8_H}^{\omega}}_{-t}\oplus\underbrace{\mathbf{28_H}^{\dot\omega}}_{-2t}\oplus\underbrace{\overline{\mathbf{70_H}}}_{+4t}$.
- Constraints to the affine Lie algebra: $c(\hat{\mathfrak{g}}_{k_U})\equiv\frac{k_U\dim(\mathfrak{g})}{k_U+g^\vee}\leq 22$ and $h(\mathcal{R})\equiv\frac{C_2(\mathcal{R})/(\alpha_h,\alpha_h)}{k_U+g^\vee}\leq 1$.

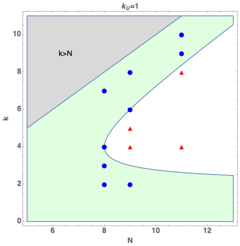


Figure: The unitarity allowed region (green shaded) to the rank-k anti-symmetric irreps of the $\widehat{\mathfrak{su}}(N)_{k_U=1}$ theories. The blue circles and red triangles represent the allowed and the excluded irreps, respectively.

Summary

- We propose an $\mathfrak{su}(8)$ theory to address the SM flavor puzzle. Different generations transform differently in the UV-complete theory and their repetitive structure only emerge in the IR, which lead to the flavor non-universality of SM quarks/leptons.
- Our construction relaxes Georgi's "third law" in 1979, and we avoid the repetitions of one IRAFFS. The global symmetries based on the chiral IRAFFSs are vital to: (i) determine one single SM Higgs doublet through the non-anomalous $\widetilde{\mathrm{U}}(1)_{B-L}$ symmetry, (ii) count the massless left-handed sterile neutrinos precisely through the 't Hooft anomaly matching, (iii) organize the $d=5$ operators for the SM fermion mass (mixing) terms.
- The symmetry-breaking pattern of the $\mathfrak{su}(8)$ theory is described, and all light SM fermion masses besides of the top quark are due to the inevitable gravitational effects that break the emergent global symmetries explicitly.

Summary

- Crucial assumptions: (i) the VEV assignments of three intermediate symmetry-breaking scales in Eq. (15), (ii) the SM flavor IDs in the $\mathbf{28_F}$ and $\mathbf{56_F}$, and (iii) the $d = 5$ operators of direct and indirect Yukawa couplings containing the SM Higgs doublet.
- Main results: all SM quark/lepton masses, as well as the CKM mixing pattern can be quantitatively recovered with $\mathcal{O}(0.1) - \mathcal{O}(1)$ direct Yukawa couplings and $\mathcal{O}(0.01)$ Higgs mixing coefficients.
- All SM quark/leptons are flavor non-universal under the extended strong/weak symmetries, while the SM neutrinos $\nu_L \in \overline{\mathbf{8_F}}^\Omega$ are flavor universal. Neutrino masses and mixings are to be determined.
- The degenerate $m_{d^i} = m_{\ell^i}$ will be further probed based on the RGEs of $\frac{d m_f(\mu)}{d \log \mu} \equiv \gamma_{m_f} m_f(\mu)$, $\gamma_{m_f}(\alpha^\Upsilon) = \frac{\alpha^\Upsilon}{4\pi} \gamma_0(\mathcal{R}_f^\Upsilon)$.