SO(10)-inspired Leptogenesis with flavour coupling effects

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Overview

1. See-Saw mechanism
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See-Saw mechanism

Leptonic mass terms after symmetry breaking

$$-\mathcal{L}_M = \bar{\alpha}_L D_m \alpha_R + \bar{\nu}_{\alpha L} m_{D\alpha i} N_{iR} + \frac{1}{2}\bar{N}^c_{iR} D_M N_{iR} + \text{h.c.}. \tag{1}$$

with $D_{m_l} = \text{diag}(m_e, m_u, m_\tau)$, $D_M = \text{diag}(M_1, M_2, M_3)$. Neutrino mass matrix

$$m_\nu = -m_D \frac{1}{D_M} m_D^T, \tag{2}$$

Diagonalised by the PMNS matrix U,

$$U^\dagger m_\nu U^\star = -D_m. \tag{3}$$

In usual parameterization

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \text{diag}(e^{i\rho}, 1, e^{i\sigma}). \tag{4}$$

Low energy neutrino parameters

Current experiment constraints with (without) SK,

$$
\begin{aligned}
\theta_{13} &= 8.56^\circ \pm 0.11^\circ \in [8.19^\circ, 8.89^\circ], \\
\theta_{12} &= 33.68^{\circ}{}^{+0.73^\circ}_{-0.70^\circ} \in [31.63^\circ, 35.95^\circ], \\
\theta_{23} &= 43.3^{\circ}{}^{+1.0^\circ}_{-0.8^\circ} (48.5^{\circ}{}^{+0.7^\circ}_{-0.9^\circ}) \in [41.3^\circ, 49.9^\circ], \\
\delta &= -148^{\circ}{}^{+26^\circ}_{-41^\circ} (177^{\circ}{}^{+19^\circ}_{-20^\circ}) \in [-236^\circ, 4^\circ].
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
m_{\text{sol}} &\equiv \sqrt{m_2^2 - m_1^2} = (8.6 \pm 0.1)\ \text{meV}, \\
m_{\text{atm}} &\equiv \sqrt{m_3^2 - m_2^2} = (49.9 \pm 0.3)\ \text{meV}
\end{aligned} \tag{6}
$$

Effective neutrino mass,

$$
m_{ee} \equiv |m_{\nu ee}| = \left| U_{e1}^2 m_1 + U_{e2}^2 m_2 + U_{e3}^2 m_3 \right| \tag{7}
$$

SO(10)-inspired Model

Transformation of neutrino Dirac mass matrix between the flavour basis and the Yukawa basis

$$m_D = V_L^\dagger D_{m_D} U_R. \tag{8}$$

Minimal $SO(10)$-inspired model: extend the SM with only additional 3 massive RH neutrinos by see-saw mechanism, and parametrise the three eigenvalues in the Dirac mass matrix in terms of the up quark masses,

$$m_{D1} = \alpha_1 m_{up}, \quad m_{D2} = \alpha_2 m_{charm}, \quad m_{D3} = \alpha_3 m_{top}, \tag{9}$$

under conditions:

$$i) I \leq V_L \leq V_{CKM} (\theta_{ij}^L \leq \theta_{ij}^{CKM}), \tag{10}$$

$$ii) \alpha_i = \mathcal{O}(0.1 - 10). \tag{11}$$

Which implies $M_1 \ll 10^9$ GeV, $\quad 10^9$ GeV $\lesssim M_2 \lesssim 10^{12}$ GeV $\quad$ and $\quad M_3 \gg 10^{12}$ GeV.

Leptogenesis

In the early history of the universe, heavy RH neutrinos decay into leptons $\Gamma(N_i \to L_i + \phi^\dagger)$ and anti-leptons $\Gamma(N_i \to \bar{L}_i + \phi)$ with different decay rate, produce lepton asymmetry given by CP asymmetry

$$\varepsilon_{i\alpha} = -\frac{\Gamma(N_i \to L_\alpha + \phi^\dagger) - \Gamma(N_i \to \bar{L}_\alpha + \phi)}{\Gamma(N_i \to L_\alpha + \phi^\dagger) + \Gamma(N_i \to \bar{L}_\alpha + \phi)}. \tag{12}$$

Thanks to $B - L$ conserving sphaleron processes, this generates a baryon asymmetry, which can be expressed by the baryon-to-photon ratio as

$$\eta_{B0}^{\text{lep}} \simeq 0.96 \times 10^{-2} N_{B-L}^{\text{lep}} = 0.96 \times 10^{-2} \sum_\alpha N_{\Delta\alpha}. \tag{13}$$

A successfully Leptogenesis should produce the correct value of asymmetry observed by CMB

$$\eta_B^{\text{CMB}} = (6.1 \pm 0.1) \times 10^{-10} \tag{14}$$

Analytic solutions

The $B-L$ asymmetry abundance correspond to different lepton species can be evaluate by Boltzmann equation

$$\frac{dN_{N_i}}{dz_i} = -D_i \left(N_{N_i} - N_{N_i}^{\mathrm{eq}} \right), \qquad i = 1, 2, 3 \tag{15}$$

$$\frac{dN_{\Delta\alpha}}{dz_i} = \varepsilon_{i\alpha} D_{N_i} (N_{N_i} - N_{N_i}^{\mathrm{eq}}) - P_{i\alpha}^0 W_{N_i} N_{\Delta\alpha}, \tag{16}$$

Here $z_i = M_i/T$, and we normalize the abundance of heavy neutrinos in ultra-relativistic thermal equilibrium as $N_{N_i}^{\mathrm{eq}}(T \gg M_i) = 1$. The decay factor and wash-out term are defined as

$$D_i(z_i) \equiv \frac{\tilde{\Gamma}_i}{Hz_i}, \qquad W_{N_i}(z_i) = \frac{1}{4} K_i \, \mathcal{K}_1(z_i) \, z_i^3. \tag{17}$$

Analytic solutions

and the decay parameters

$$K_i = \sum_\alpha K_{i\alpha} \text{ with } K_{i\alpha} \equiv \frac{\Gamma_{i\alpha} + \bar{\Gamma}_{i\alpha}}{H(T = M_i)} \text{ and } P^0_{i\alpha} = \frac{K_{i\alpha}}{K_i}. \tag{18}$$

The analytic expression of $B - L$ asymmetry can be related to decay parameters as

$$\begin{aligned} N^{\text{lep,f}}_{B-L} \simeq & \frac{K_{2e}}{K_{2\tau^\perp}} \varepsilon_{2\tau^\perp} \kappa(K_{2\tau^\perp}) e^{-\frac{3\pi}{8} K_{1e}} + \frac{K_{2\mu}}{K_{2\tau^\perp}} \varepsilon_{2\tau^\perp} \kappa(K_{2\tau^\perp}) e^{-\frac{3\pi}{8} K_{1\mu}} \\ & + \varepsilon_{2\tau} \kappa(K_{2\tau}) e^{-\frac{3\pi}{8} K_{1\tau}}, \end{aligned} \tag{19}$$

with the efficiency factors calculated by

$$\kappa(K_{2\alpha}) = \frac{2}{z_B(K_{2\alpha}) K_{2\alpha}} \left(1 - e^{-\frac{K_{2\alpha} z_B(K_{2\alpha})}{2}}\right), \quad z_B(K_{2\alpha}) \simeq 2 + 4 K_{2\alpha}^{0.13} e^{-\frac{2.5}{K_{2\alpha}}}. \tag{20}$$

Flavour coupling effects

More precisely, the flavour effects in Leptogenesis arise from a direct way and an indirect way. In direct form, leptons with different flavors with different values of CP violation rate $\varepsilon_\alpha(\alpha = e, \mu, \tau)$ give a different value contribution to the total asymmetry $B - L$

$$N^{\mathrm{lep}}_{B-L} = \sum_\alpha N_{\Delta\alpha} \tag{21}$$

In the indirect way, the Higgs bosons produced in the RH neutrino decay is unflavoured. This effects the inverse decay of different flavoured leptons $\bar{\Gamma}(L_i + \phi^\dagger \to N_i)$ and anti-leptons in the so-called wash out process, leading the change of the Boltzmann equation

$$W_{N_i} N_{\Delta\alpha} \Rightarrow W_{N_i} \sum_{\alpha,\beta} C^{(n)}_{\beta\alpha} N_{\Delta\alpha}, \tag{22}$$

where $C^{(n)}$ are flavour coupling matrices.

Two flavour regime

Defining transformation matrix

$$U \equiv \begin{pmatrix} U_{\gamma'\gamma} & U_{\gamma'\tau} \\ U_{\tau'\gamma} & U_{\tau'\tau} \end{pmatrix}, \quad \begin{pmatrix} N_{\Delta_{\gamma'}} \\ N_{\Delta_{\tau'}} \end{pmatrix} = U \begin{pmatrix} N_{\Delta_\gamma} \\ N_{\Delta_\tau} \end{pmatrix}, \tag{23}$$

where γ denotes the $e+\mu$ flavour composition, that diagonalizes

$$UP_2^0 U^{-1} = \mathrm{diag}(P_{2\gamma'}^0, P_{2\tau'}^0), \quad P_2^0 \equiv \begin{pmatrix} P_{2\gamma}^0 C_{\gamma\gamma}^{(2)} & P_{2\gamma}^0 C_{\gamma\tau}^{(2)} \\ P_{2\tau}^0 C_{\tau\gamma}^{(2)} & P_{2\tau}^0 C_{\tau\tau}^{(2)} \end{pmatrix} \tag{24}$$

This decouples the flavour mixing for Boltzmann equations

$$\frac{dN_{\Delta_{\gamma'}}}{dz_2} = \varepsilon_{2\gamma'} D_2 \left(N_{N_2} - N_{N_2}^{\mathrm{eq}} \right) - P_{2\gamma'}^0 W_2 N_{\Delta_{\gamma'}}, \tag{25}$$

$$\frac{dN_{\Delta_{\tau'}}}{dz_2} = \varepsilon_{2\tau'} D_2 \left(N_{N_2} - N_{N_2}^{\mathrm{eq}} \right) - P_{2\tau'}^0 W_2 N_{\Delta_{\tau'}}. \tag{26}$$

Three flavour regime

During the N_1 wash-out, the evolution of the flavour asymmetries is governed by the equation

$$\frac{dN_{\Delta_\alpha}}{dz_1} = -P^0_{1\alpha} \sum_\beta C^{(3)}_{\alpha\beta} W^{\rm ID}_1 N_{\Delta_\beta}, \qquad (\alpha, \beta = e, \mu, \tau) \tag{27}$$

To simplify the system, we perform a basis transformation using the matrix V that diagonalizes P^0_1:

$$\vec{N}_{\Delta''} = V \vec{N}_\Delta, \quad V = \begin{pmatrix} V_{ee''} & V_{e\mu''} & V_{e\tau''} \\ V_{\mu e''} & V_{\mu\mu''} & V_{\mu\tau''} \\ V_{\tau e''} & V_{\tau\mu''} & V_{\tau\tau''} \end{pmatrix} \tag{28}$$

such that $V P^0_1 V^{-1} = P^{0''}_1 = \mathrm{diag}(P^0_{1e''}, P^0_{1\mu''}, P^0_{1\tau''})$.
In this diagonal basis, the kinetic equations decouple and take the simple form

$$\frac{d\vec{N}_{\Delta''}}{dz_1} = -W_1 P^{0''}_1 \vec{N}_{\Delta''} \tag{29}$$

Three flavour regime

The solution in this basis is then given by

$$\vec{N}^f_{\Delta''} = \left(N^{T\sim T_L}_{\Delta_{e''}} e^{-\frac{3\pi}{8} K_{1e''}}, \quad N^{T\sim T_L}_{\Delta_{\mu''}} e^{-\frac{3\pi}{8} K_{1\mu''}}, \quad N^{T\sim T_L}_{\Delta_{\tau''}} e^{-\frac{3\pi}{8} K_{1\tau''}} \right) \tag{30}$$

Applying the inverse transformation, we obtain the final flavour asymmetries $\vec{N}^f_\Delta = V^{-1} \vec{N}^f_{\Delta''}$. Each final asymmetry in flavour α is then given by

$$N^f_{\Delta_\alpha} = \sum_{\alpha''} V^{-1}_{\alpha\alpha''} \left[N^{T\sim T_L}_{\Delta_{\alpha''}} e^{-\frac{3\pi}{8} K_{1\alpha''}} \right] \tag{31}$$

$$= \sum_{\alpha''} V^{-1}_{\alpha\alpha''} e^{-\frac{3\pi}{8} K_{1\alpha''}} \sum_{\beta} V_{\alpha''\beta} N^{T\sim T_L}_{\Delta_\beta}, \tag{32}$$

with

$$N^{T\sim T_L}_{\Delta_\alpha} \simeq \frac{K_{2\alpha}}{K_{2\gamma}} \left[\varepsilon_{2\gamma}\, \kappa(K_{2\gamma}) - C^{(2)}_{\gamma\tau}\, \varepsilon_{2\tau}\, \kappa(K_{2\tau}) \right], \qquad \alpha = e, \mu \tag{33}$$

$$N^{T\sim T_L}_{\Delta_\tau} \simeq \varepsilon_{2\tau}\, \kappa(K_{2\tau}) \tag{34}$$

Strong thermal leptogenesis

- Pre-existing lepton asymmetry N_{B-L}^{pre} may arise from unknown early mechanisms.
- To ensure predictivity, it must be washed out by RH neutrino inverse decays.
- This requirement defines the **strong thermal condition**.
- The heaviest RH neutrino N_3 is assumed too heavy to contribute.

If a pre-existing asymmetry is generated at $T \geq T_B^{\text{ext}}$, then for temperatures $T \gg M_2$, the total $B - L$ asymmetry remains constant:

$$N_{B-L}^{\mathrm{p}}(T \gg M_2) = N_{B-L}^{\mathrm{p,i}}$$

Pre-existing Asymmetry and Washout Stages

For $10^{12}\,\mathrm{GeV} \gg T \gg M_2$, fast tauon interactions break coherence between flavour states. The pre-existing asymmetry can be written as

$$N^{\mathrm{p}}_{B-L} = N^{\mathrm{p,i}}_{\Delta_\tau} + N^{\mathrm{p,i}}_{\Delta_{\tau_\perp}}\,,$$

with

$$N^{\mathrm{p,i}}_{\Delta_\tau} = p^0_{\mathrm{p}\tau}\, N^{\mathrm{p,i}}_{B-L}, \qquad N^{\mathrm{p,i}}_{\Delta_{\tau_\perp}} = (1 - p^0_{\mathrm{p}\tau})\, N^{\mathrm{p,i}}_{B-L}.$$

Here $p^0_{\mathrm{p}\tau}$ is the probability that the pre-existing lepton lies in the tau flavour. Differences in flavour composition are neglected here (phantom terms).

As temperature drops to $T \simeq T_{B2} \simeq M_2/z_{B2}$, the N_2 washout affects only certain components. The residual asymmetries are:

$$N^{\mathrm{p}}_{\Delta_\tau}(T_{B2}) = p^0_{\mathrm{p}\tau}\, e^{-\frac{3\pi}{8}K_{2\tau}}\, N^{\mathrm{p,i}}_{B-L}$$

$$N^{\mathrm{p}}_{\Delta_{\tau_2^\perp}}(T_{B2}) = (1 - p^0_{\mathrm{p}\tau})\, p^0_{\tau_2^\perp}\, e^{-\frac{3\pi}{8}(K_{2e}+K_{2\mu})}\, N^{\mathrm{p,i}}_{B-L}$$

$$N^{\mathrm{p}}_{\Delta_{\tau_2^\perp}}(T_{B2}) = (1 - p^0_{\mathrm{p}\tau})(1 - p^0_{\tau_2^\perp})\, N^{\mathrm{p,i}}_{B-L}$$

Wash-out pre-existing asymmetry

At temperatures $T \simeq 10^9\,\mathrm{GeV}$, also muon lepton interactions become effective, breaking the residual coherence of the $e-\mu$ lepton components in such a way that in the range $10^9\,\mathrm{GeV} \gg T \gg M_1$, the total asymmetry can be regarded as the sum of three charged lepton flavour components:

$$N_{B-L}^{\mathrm{p}}(10^9\,\mathrm{GeV} \gg T \gg M_1) = \sum_{\alpha=e,\mu,\tau} N_{\Delta_\alpha}^{\mathrm{p}}(T_{B2}), \tag{35}$$

where

$$\begin{aligned}
N_{\Delta_\tau}^{\mathrm{p}}(T \gg M_1) &= p_{\mathrm{p}\tau}^0\, e^{-\frac{3\pi}{8}K_{2\tau}}\, N_{B-L}^{\mathrm{p,i}}, \\
N_{\Delta_\mu}^{\mathrm{p}}(T \gg M_1) &= (1-p_{\mathrm{p}\tau}^0)\left[p_{\mu\tau_2^\perp}^0 p_{p\tau_2^\perp}^0\, e^{-\frac{3\pi}{8}(K_{2e}+K_{2\mu})} + (1-p_{\mu\tau_2^\perp}^0)(1-p_{p\tau_2^\perp}^0)\right] N_{B-L}^{\mathrm{p,i}}, \\
N_{\Delta_e}^{\mathrm{p}}(T \gg M_1) &= (1-p_{\mathrm{p}\tau}^0)\left[p_{e\tau_2^\perp}^0 p_{p\tau_2^\perp}^0\, e^{-\frac{3\pi}{8}(K_{2e}+K_{2\mu})} + (1-p_{e\tau_2^\perp}^0)(1-p_{p\tau_2^\perp}^0)\right] N_{B-L}^{\mathrm{p,i}}
\end{aligned} \tag{36}$$

Wash-out pre-existing asymmetry

The probabilities $p^0_{\beta\tau^\perp}$ are related to the decay parameters as

$$p^0_{\beta\tau^\perp} = \frac{p^0_{2\beta}}{p^0_{2e} + p^0_{2\mu}} = \frac{K_{2\beta}}{K_{2e} + K_{2\mu}}, \tag{37}$$

and analogously for $p^0_{2\mu^\perp}$. These expressions show that the tauon component is the only component of the pre-existing asymmetry that can be completely washed out by the N_2 wash-out processes.

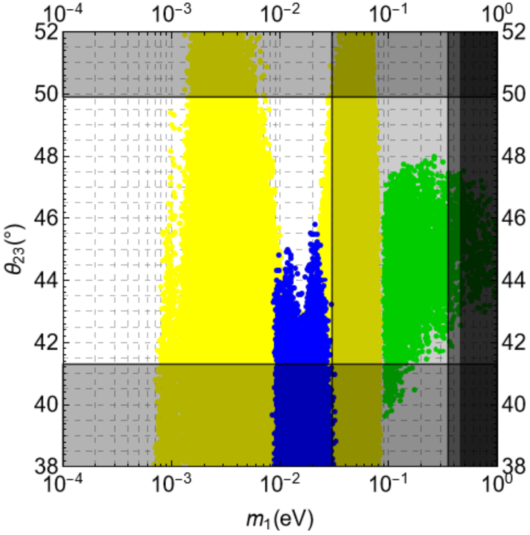
Finally, at temperatures $T \sim M_1$, the lightest RH neutrino wash-out processes act on the flavour asymmetries in a way that the relic values of the pre-existing asymmetries are given by:

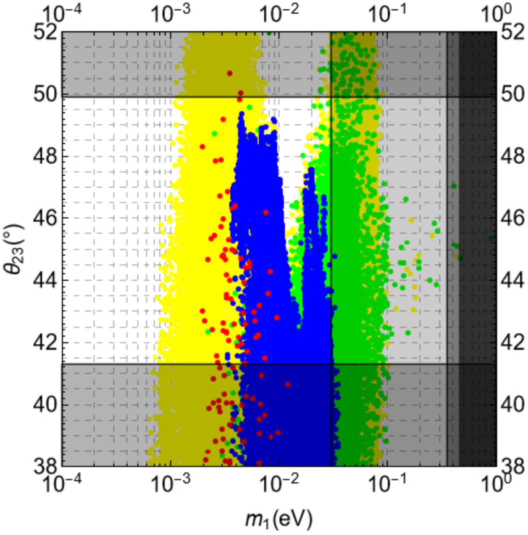
$$N^{\mathrm{p,f}}_{\Delta_\tau} = p^0_{\mathrm{p}\tau}\, e^{-\frac{3\pi}{8}(K_{1\tau}+K_{2\tau})}\, N^{\mathrm{p,i}}_{B-L}, \tag{38}$$

$$N^{\mathrm{p,f}}_{\Delta_\mu} = (1 - p^0_{\mathrm{p}\tau})\, e^{-\frac{3\pi}{8}K_{1\mu}} \left[p^0_{\mu\tau_2^\perp} p^0_{p\tau_2^\perp}\, e^{-\frac{3\pi}{8}(K_{2e}+K_{2\mu})} + (1 - p^0_{\mu\tau_2^\perp})(1 - p^0_{p\tau_2^\perp}) \right] N^{\mathrm{p,i}}_{B-L}, \tag{39}$$

$$N^{\mathrm{p,f}}_{\Delta_e} = (1 - p^0_{\mathrm{p}\tau})\, e^{-\frac{3\pi}{8}K_{1e}} \left[p^0_{e\tau_2^\perp} p^0_{p\tau_2^\perp}\, e^{-\frac{3\pi}{8}(K_{2e}+K_{2\mu})} + (1 - p^0_{e\tau_2^\perp})(1 - p^0_{p\tau_2^\perp}) \right] N^{\mathrm{p,i}}_{B-L}. \tag{40}$$

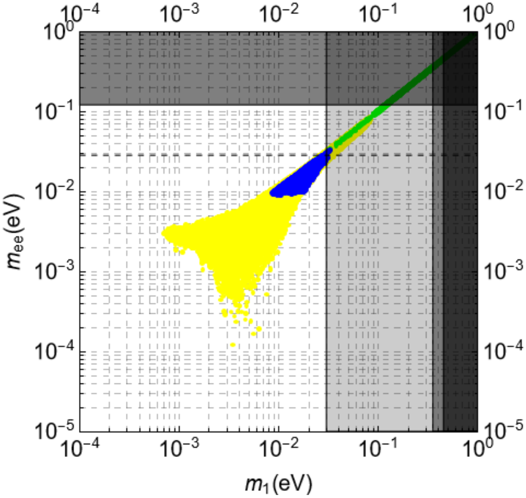
θ_{23} *vs* m_1



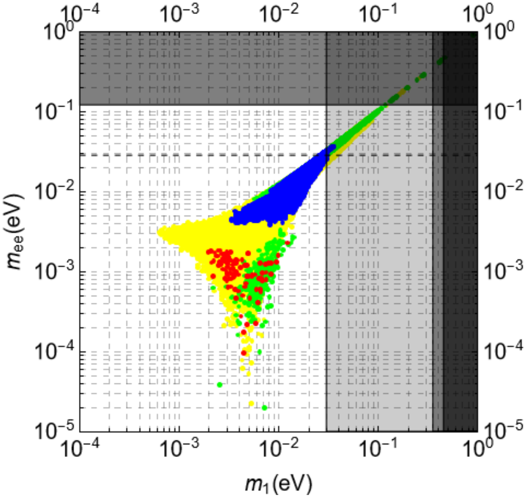


A: Without flavour coupling effect **B: With flavour coupling effect**

m_{ee} vs m_1

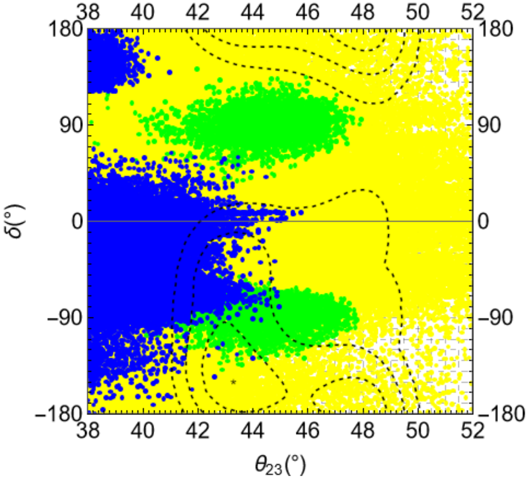


C: Without flavour coupling effect

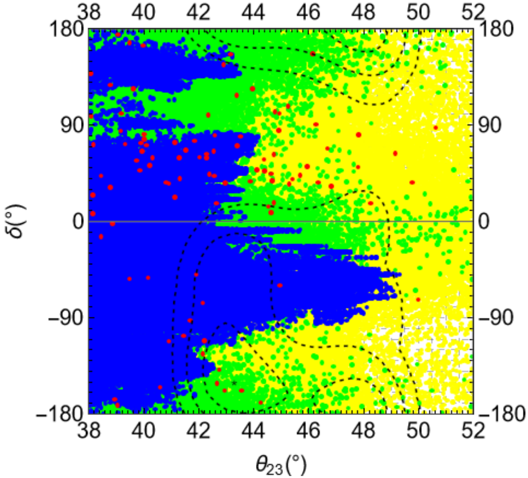


D: With flavour coupling effect

δ vs θ_{23}



E: Without flavour coupling effect



F: With flavour coupling effect

Summary

- The flavour coupling effect originates mainly from the emission of flavourless Higgs bosons in the decay processes of right-handed (RH) neutrinos.
- The Boltzmann equations should be modified by inserting flavour coupling matrices in the washout terms. By applying a transpose transformation, the system can be rotated into a diagonal basis, where analytical solutions can then be derived.
- The flavour coupling effect enlarges the allowed parameter space, both in scenarios without and with strong thermal conditions.
- Absolute neutrino mass scale experiments are beginning to probe a crucial regime: that of partially hierarchical neutrino masses.
- The current best-fit results, which favour the first octant for the atmospheric mixing angle, are quite encouraging for model building.

Thank you!

Initially,

$$
\begin{aligned}
N_{\Delta\tau}^{\mathtt{p},\mathtt{i}} &= p_{\mathtt{p}\tau}^{0} N_{B-L}^{\mathtt{p},\mathtt{i}}, \\
N_{\Delta\tau^{\perp}}^{\mathtt{p},\mathtt{i}} &= p_{\mathtt{p}\tau^{\perp}}^{0} N_{B-L}^{\mathtt{p},\mathtt{i}}
\end{aligned}
\tag{41}
$$

When $T \ll M_2$,

$$
\begin{aligned}
N_{\Delta\tau'}^{\mathtt{p},T\ll M_2} &= e^{-\frac{3\pi}{8}K'_{2\tau}} N_{\Delta\tau'}^{\mathtt{p}} = e^{-\frac{3\pi}{8}K'_{2\tau}} \left(U_{\tau'\tau} N_{\Delta\tau}^{\mathtt{p},i} + U_{\tau'\tau^{\perp}} N_{\Delta\tau^{\perp}}^{\mathtt{p},i}\right), \\
N_{\Delta\tau'_{\perp}}^{\mathtt{p},T\ll M_2} &= e^{-\frac{3\pi}{8}K'_{2\tau^{\perp}}} N_{\Delta\tau'_{\perp}}^{\mathtt{p}} = e^{-\frac{3\pi}{8}K'_{2\tau^{\perp}}} \left(U_{\tau'_{\perp}\tau} N_{\Delta\tau}^{\mathtt{p},i} + U_{\tau'_{\perp}\tau^{\perp}} p_{\mathtt{p}\tau_2^{\perp}}^{0} N_{\Delta\tau^{\perp}}^{\mathtt{p},i}\right) \\
N_{\Delta\tau'_{\perp_{2\perp}}}^{\mathtt{p},T\ll M_2} &= U_{\tau'_{\perp}\tau_{\perp}} (1 - p_{\mathtt{p}\tau_2^{\perp}}^{0}) N_{\Delta\tau_{\perp}}^{\mathtt{p},\mathtt{i}}
\end{aligned}
\tag{42}
$$

Wash-out factor

Rotating back, we have

$$
\begin{aligned}
N_{\Delta\tau}^{\mathrm{p},T\ll M_2} &= U^{-1}_{\tau\tau'} N_{\Delta\tau'}^{\mathrm{p},T\ll M_2} + U^{-1}_{\tau\tau'_\perp} N_{\Delta\tau'_\perp}^{\mathrm{p},T\ll M_2}, \\
N_{\Delta\tau^\perp}^{\mathrm{p},T\ll M_2} &= U^{-1}_{\tau_\perp\tau'_\perp} N_{\Delta\tau'}^{\mathrm{p},T\ll M_2} + U^{-1}_{\tau_\perp\tau'_\perp} N_{\Delta\tau'_\perp}^{\mathrm{p},T\ll M_2}, \\
N_{\Delta\tau_{\perp_{2\perp}}}^{\mathrm{p},T\ll M_2} &= U^{-1}_{\tau_\perp\tau'_\perp} N_{\Delta\tau'_{\perp_{2\perp}}}^{\mathrm{p},T\ll M_2}.
\end{aligned}
\tag{43}
$$

For 3 flavour regime, splitting $N_{\Delta\tau_\perp}^{\mathrm{p},T\ll M_2}$ and $N_{\Delta\tau_{\perp_{2\perp}}}^{\mathrm{p},T\ll M_2}$ into,

$$
\begin{aligned}
N_{\Delta\mu}^{\mathrm{p},T\ll M_2} &= p_{\mu\tau_2^\perp} N_{\Delta\tau^\perp}^{\mathrm{p},T\ll M_2} + (1 - p_{\mu\tau_2^\perp}) N_{\Delta\tau_{\perp_{2\perp}}}^{\mathrm{p},T\ll M_2}, \\
N_{\Delta e}^{\mathrm{p},T\ll M_2} &= p_{e\tau_2^\perp} N_{\Delta\tau^\perp}^{\mathrm{p},T\ll M_2} + (1 - p_{e\tau_2^\perp}) N_{\Delta\tau_{\perp_{2\perp}}}^{\mathrm{p},T\ll M_2}
\end{aligned}
\tag{44}
$$

Wash-out factor

After the K_1 wash out process in the temperature $T \ll M_1$,

$$\begin{aligned}
N^{\mathrm{p},T\ll M_1}_{\Delta\tau''} &= e^{-\frac{3\pi}{8}K''_{1\tau}}(V_{\tau''\tau}N^{\mathrm{p},T\ll M_2}_{\Delta\tau} + V_{\tau''\mu}N^{\mathrm{p},T\ll M_2}_{\Delta\mu} + V_{\tau''e}N^{\mathrm{p},T\ll M_2}_{\Delta e}),\\
N^{\mathrm{p},T\ll M_1}_{\Delta\mu''} &= e^{-\frac{3\pi}{8}K''_{1\mu}}(V_{\mu''\tau}N^{\mathrm{p},T\ll M_2}_{\Delta\tau} + V_{\mu''\mu}N^{\mathrm{p},T\ll M_2}_{\Delta\mu} + V_{\mu''e}N^{\mathrm{p},T\ll M_2}_{\Delta e}),\\
N^{\mathrm{p},T\ll M_1}_{\Delta e''} &= e^{-\frac{3\pi}{8}K''_{1e}}(V_{e''\tau}N^{\mathrm{p},T\ll M_2}_{\Delta\tau} + V_{e''\mu}N^{\mathrm{p},T\ll M_2}_{\Delta\mu} + V_{e''e}N^{\mathrm{p},T\ll M_2}_{\Delta e}).
\end{aligned} \tag{45}$$

Rotating back again, finally

$$\begin{aligned}
N^{\mathrm{p,f}}_{\Delta\tau} &= V^{-1}_{\tau\tau''}N^{\mathrm{p},T\ll M_1}_{\Delta\tau''} + V^{-1}_{\tau\mu''}N^{\mathrm{p},T\ll M_1}_{\Delta\mu''} + V^{-1}_{\tau e''}N^{\mathrm{p},T\ll M_1}_{\Delta e''},\\
N^{\mathrm{p,f}}_{\Delta\mu} &= V^{-1}_{\mu\tau''}N^{\mathrm{p},T\ll M_1}_{\Delta\tau''} + V^{-1}_{\mu\mu''}N^{\mathrm{p},T\ll M_1}_{\Delta\mu''} + V^{-1}_{\mu e''}N^{\mathrm{p},T\ll M_1}_{\Delta e''},\\
N^{\mathrm{p,f}}_{\Delta e} &= V^{-1}_{e\tau''}N^{\mathrm{p},T\ll M_1}_{\Delta\tau''} + V^{-1}_{e\mu''}N^{\mathrm{p},T\ll M_1}_{\Delta\mu''} + V^{-1}_{ee''}N^{\mathrm{p},T\ll M_1}_{\Delta e''}.
\end{aligned} \tag{46}$$

where

$$K''_{1\alpha} = P^0_{1\alpha''}K_1, \text{and } K'_{2\alpha} = P^0_{2\alpha'}K_2 \tag{47}$$