第20届全国中高能核物理大会
暨第14届全国中高能核物理专题研讨会

RUHR UNIVERSITÄT BOCHUM

Do we still need quark models?

Lu Meng (孟 璐)

Ruhr-Universität Bochum

2025-04-26, 上海

Based on PRD107(2023),054035, PRD109(2024),054034, PRD110 (2024), 094041 …

Together with Yao Ma, Wei-Lin Wu, Yan-Ke Chen, and Shi-Lin Zhu (PKU)

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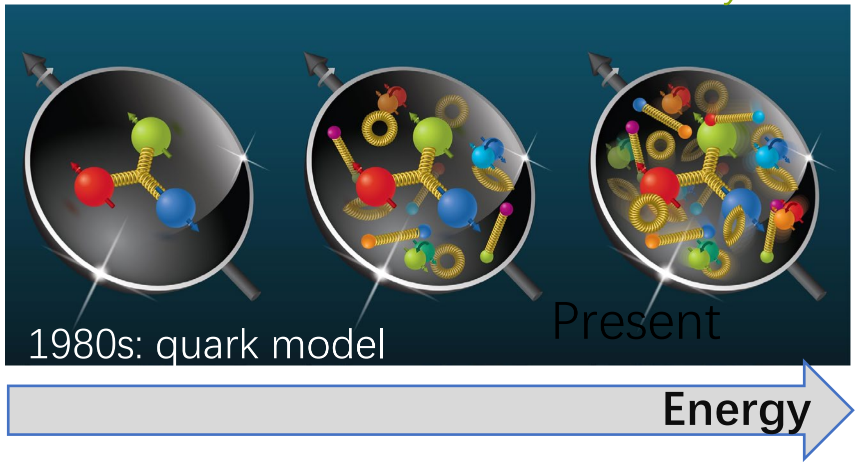
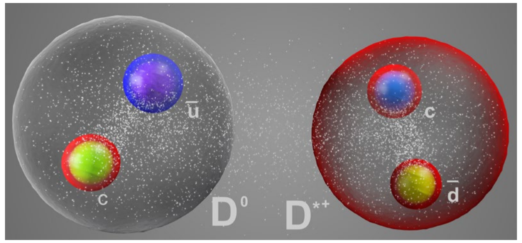
Quark model as a picture

- Evolving view of the proton

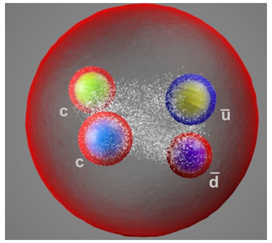
Courtesy of BNL



- Abundance of exotic candidates
- The structure of $T_{cc}(3875)$: tetraquark state



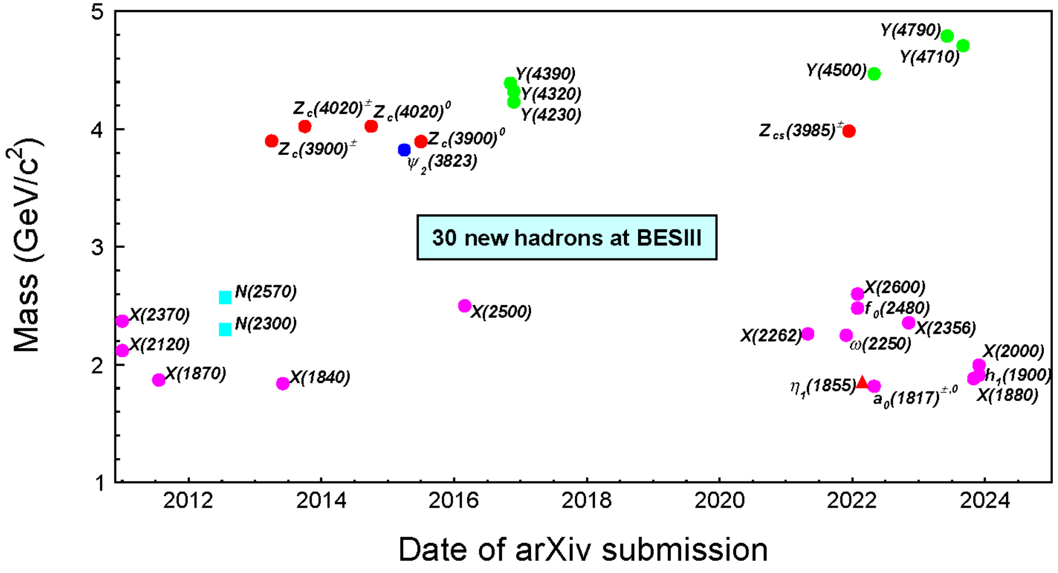
Molecular tetraquark

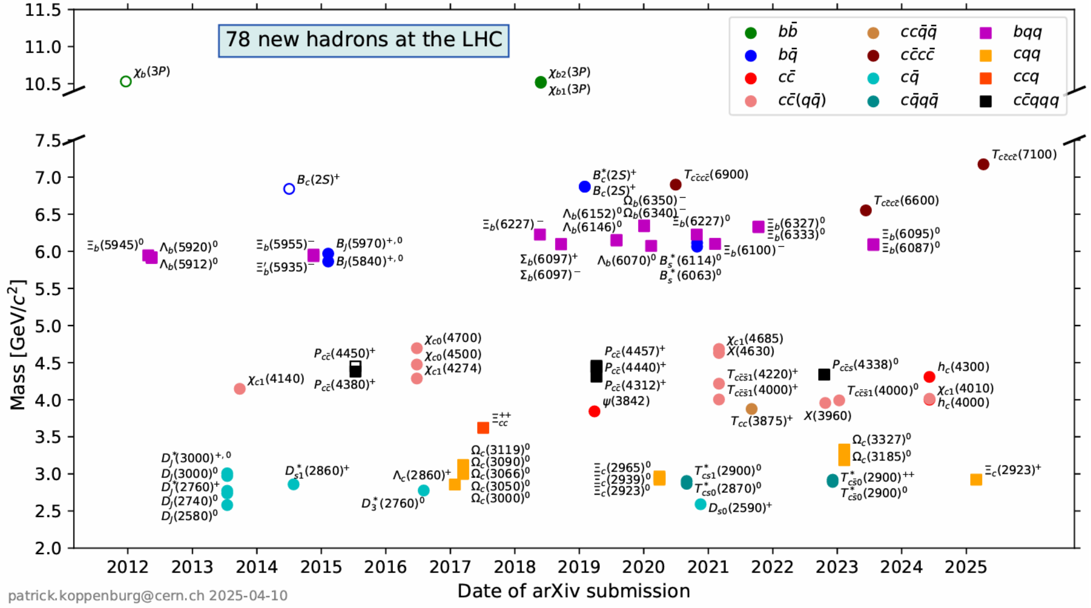


Compact tetraquark

- Given the experimental status, starting from constituent quark models

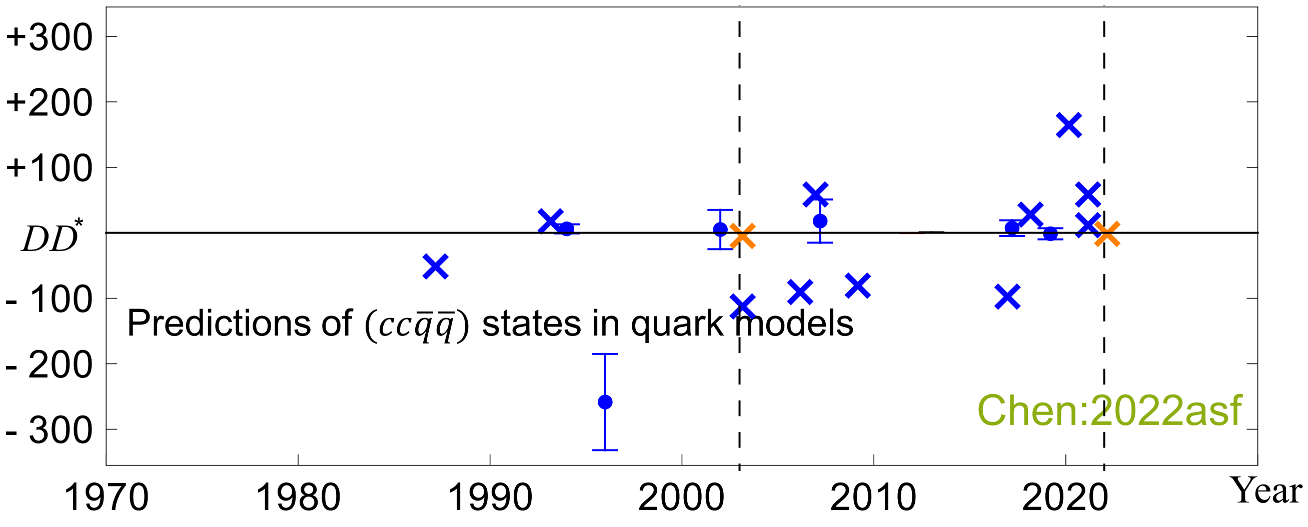
https://en.wikipedia.org/wiki/Proton
https://lhcb-outreach.web.cern.ch/2021/07/29/observation-of-an-exceptionally-charming-tetraquark/



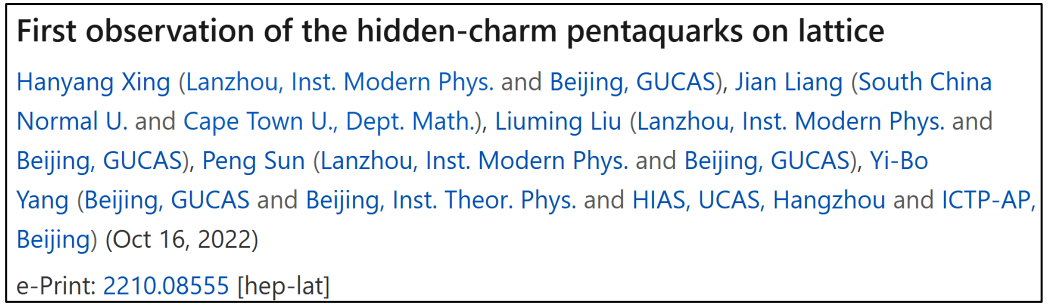


Quark models as calculating methods

- **Quark model**: forward-looking prediction, however large uncertainties
- **Low energy EFT**: order by order improvable, symmetries of QCD, but need experimental inputs
- **Lattice QCD**: from the 1st principles but, need extrapolations and expensive
- …



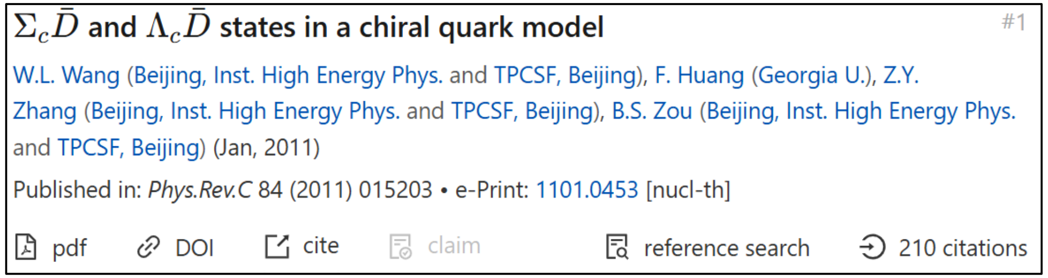
Ader:1981db, Zouzou:1986qh, Carlson:1987hh



First observation of the hidden-charm pentaquarks on lattice

Hanyang Xing (Lanzhou, Inst. Modern Phys. and Beijing, GUCAS), Jian Liang (South China Normal U. and Cape Town U., Dept. Math.), Liuming Liu (Lanzhou, Inst. Modern Phys. and Beijing, GUCAS), Peng Sun (Lanzhou, Inst. Modern Phys. and Beijing, GUCAS), Yi-Bo Yang (Beijing, GUCAS and Beijing, Inst. Theor. Phys. and HIAS, UCAS, Hangzhou and ICTP-AP, Beijing) (Oct 16, 2022)

e-Print: 2210.08555 [hep-lat]



$\Sigma_c\bar{D}$ and $\Lambda_c\bar{D}$ states in a chiral quark model #1

W.L. Wang (Beijing, Inst. High Energy Phys. and TPCSF, Beijing), F. Huang (Georgia U.), Z.Y. Zhang (Beijing, Inst. High Energy Phys. and TPCSF, Beijing), B.S. Zou (Beijing, Inst. High Energy Phys. and TPCSF, Beijing) (Jan, 2011)

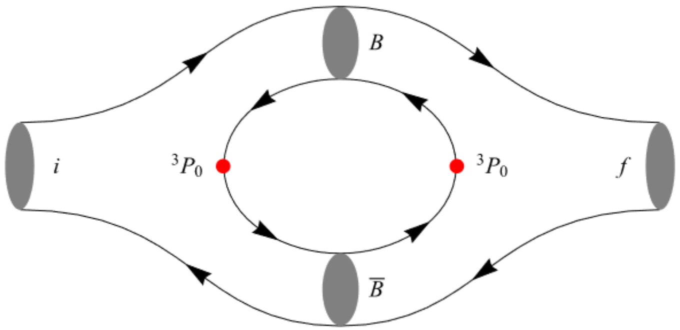
Published in: *Phys.Rev.C* 84 (2011) 015203 • e-Print: 1101.0453 [nucl-th]

pdf DOI cite claim reference search 210 citations

- We still need quark model !
- The right question: How to **adapt the quark model to meet the new demands**?

Quark models in new era

- Heavy-quarkonium-like states: unquenched quark model



- **Multiquark systems**

$[c\bar{c}qqq]$
P_c

$[cc\bar{c}\,\bar{c}]$
$X(6900)$
$X(6600)$
$X(7100)$
2006.16957
2306.07164
2304.08962

$[cs\bar{u}\,\bar{d}]$
$T_{cs1}(2900)$
$T_{cs0}(2900)$
2009.00025
2009.00026

$[cs\bar{u}\,\bar{d}]$
$Z_{cs}(3985)$
$Z_{cs}(4000)$
2011.07855
2103.01803

$[cc\bar{u}\,\bar{d}]$
$T_{cc}(3875)^+$
2109.01038
2109.01056

$[c\bar{s}u\bar{d}][c\bar{s}\bar{u}d]$
$T_{c\bar{s}0}(2900)^{++}$
$T_{c\bar{s}0}(2900)^0$
2212.02716
2212.02717

$[c\bar{c}qqq]$
$P_{cs}(4338)$
$P_{cs}(4459)$
2210.10346
2012.10380

- ▶ Color structures: e.g. $3-\bar{3}$ and $6-\bar{6}$ tetraquark
- ▶ Confinement potential: Y-type or pairwise
- ▶ Resonance above the di-hadron thresholds
- ▶ ….

Challenges
Interactions? few-body (resonance) problem?

Contents

- Background
- **Multiquark package**: resonance, automatic, high efficiency
- **A benchmark test：** interactions and few-body methods
- Outlook

Method 1: Gaussian Expansion Method

- Color functions

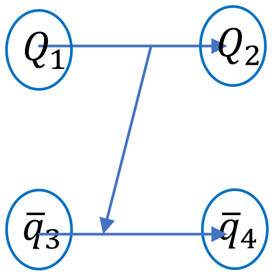
$$\begin{cases}[(Q_1Q_2)_{\bar{3}}(\bar{q}_3\bar{q}_4)_3]_1\\ [(Q_1Q_2)_6(\bar{q}_3\bar{q}_4)_{\bar{6}}]_1\end{cases} \quad \text{Or} \quad \begin{cases}[(Q_1\bar{q}_3)_1(Q_2\bar{q}_4)_1]_1\\ [(Q_1\bar{q}_4)_1(Q_2\bar{q}_3)_1]_1\end{cases} \quad \text{Or} \quad \begin{cases}[(Q_1\bar{q}_3)_1(Q_2\bar{q}_4)_1]_1\\ [(Q_1\bar{q}_4)_8(Q_2\bar{q}_3)_8]_1\end{cases}$$

Eigenvalue projecting method

- Spin wave function

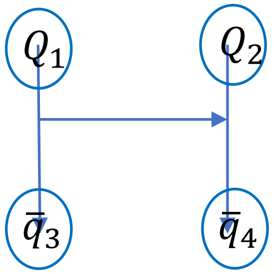
$$S_{12} = 0,1;\ S_{34} = 0,1 \quad S_{12}\otimes S_{34} \to J \qquad \text{Or} \qquad S_{13} = 0,1;\ S_{24} = 0,1 \quad S_{13}\otimes S_{24} \to J \qquad \text{Or} \qquad S_{14} = 0,1;\ S_{23} = 0,1 \quad S_{14}\otimes S_{23} \to J$$

- Spatial wave functions



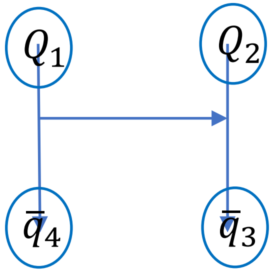
Diquark-antidiquark

And



Di-meson

And



Di-meson

$$\phi_{nlm}(\boldsymbol{r}) = N_{lm} r^l e^{-\frac{r^2}{r_n^2}} Y_{lm}(\hat{r})$$

Geometric progression

$r_n = r_0 a^{n-1}$

Hiyama:2003cu

Embed both long- and short-range correlations

- Antisymmetrization (e.g. $Q_1 = Q_2$ and $q_3 = q_4$) :

$$\psi = \mathcal{A}[\psi_{color} \otimes \psi_{spin} \otimes \psi_{spatial} \otimes \psi_{flavor}], \qquad \mathcal{A} = (1 - P_{12})(1 - P_{34})$$

Complex scaling method

- Original Hamiltonian

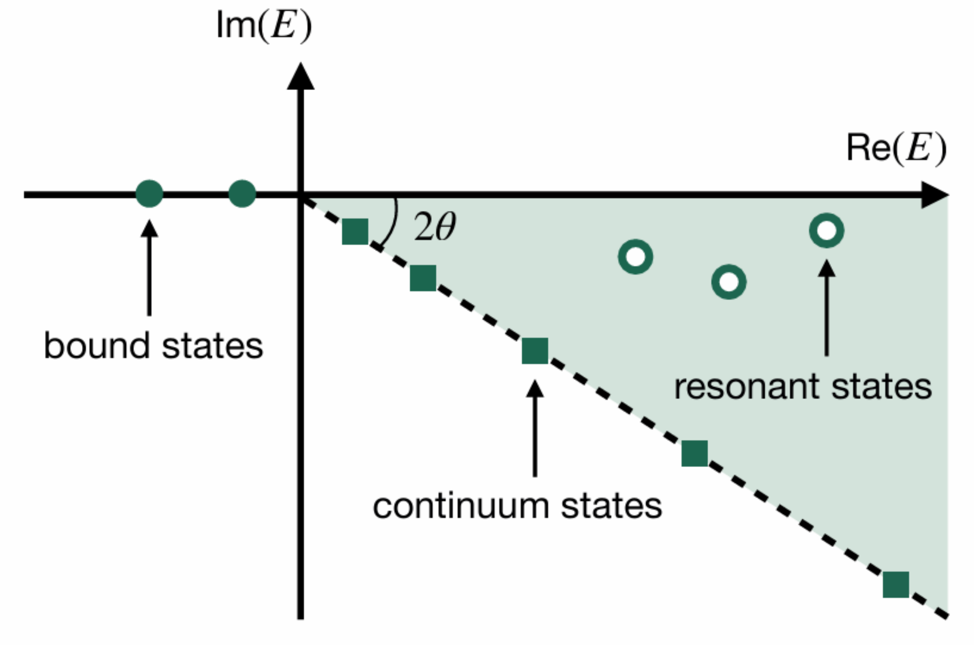
$$H = \sum_i^n \left(m_i + \frac{\boldsymbol{p}_i^2}{2m_i} \right) + \sum_{i<j=1}^n V_{ij} \,,$$

- Complex scaling

$$U(\theta)\boldsymbol{r} = \boldsymbol{r}e^{i\theta}, \quad U(\theta)\boldsymbol{p} = \boldsymbol{p}e^{-i\theta}.$$

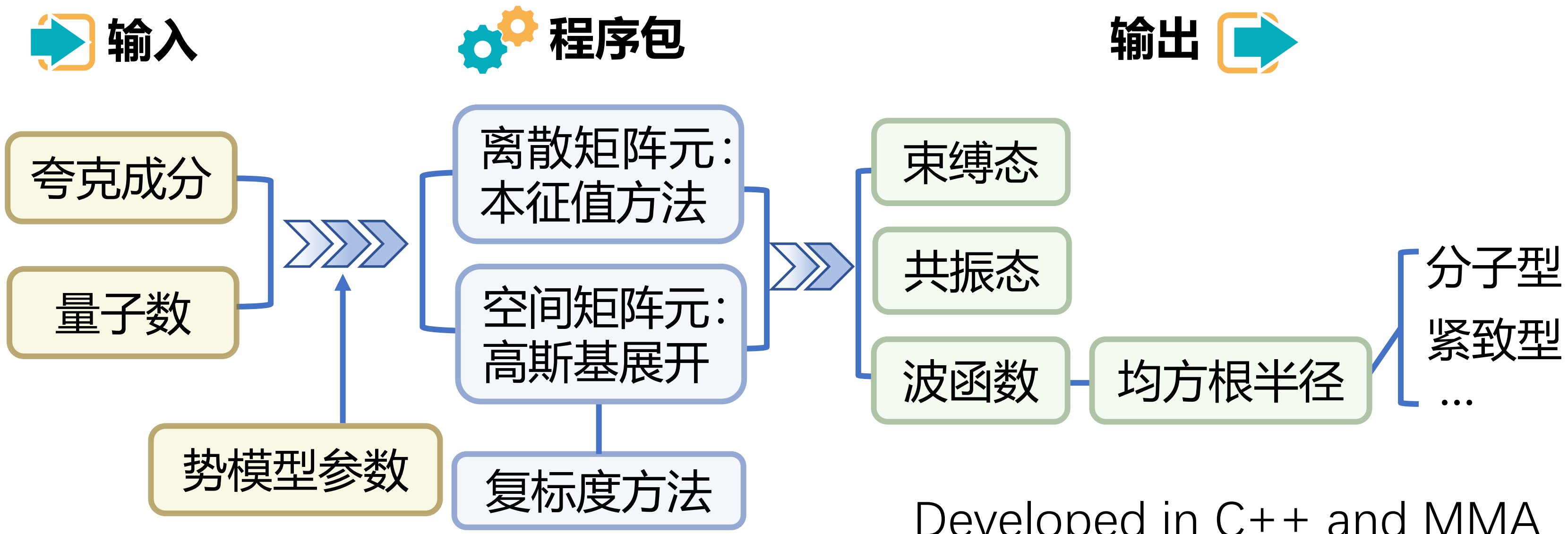
$$H(\theta) = \sum_{i=1}^n \left(m_i + \frac{p_i^2 e^{-2i\theta}}{2m_i} \right) + \sum_{i<j=1}^n V_{ij}\left(r_{ij}e^{i\theta}\right).$$

- Resonance appearing as the eigenvalue of $H(\theta)$



Aguilar:1971ve, Balslev:1971vb, Aoyama:2006hrz…

A multiquark resonance package



Developed in C++ and MMA

- Automatic Features
  - ▶ Automated wave function construction
  - ▶ Automatic matrix element calculation
  - ▶ Unified framework for tetra-, penta-, hexaquark states…(no code refactoring needed)
- High efficiency:
  - ▶ Gaussian expansion method, avoiding numerical integration
  - ▶ Parallelization capability

Tetraquark bound states

- Over 150 tetraquark systems

	$QQ\bar{Q}\bar{Q}$	$QQ\bar{Q}\bar{q}$	$QQ\bar{q}\bar{q}$	$Qs\bar{q}\bar{q}$	$Q\bar{s}q\bar{q}$
$J^P = 0^+$	No bound	No bound	😊	😊	No bound
$J^P = 1^+$	No bound	No bound	😊	😊	No bound
$J^P = 2^+$	No bound	No bound	😊	😊	No bound

$q = u, d, s$; $Q = b, c$; Only S-wave

Semay:1994ht, Silvestre-Brac:1996myf
Vijande:2004he, Gonzalez:2012gka

- Benchmark off three different quark models: AL1, AP1 and SLM

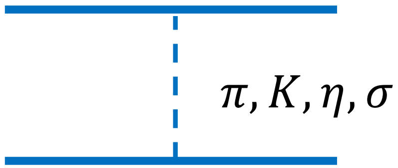
$$V_{ij}(r) = \left[-\frac{\kappa}{r} + \lambda r^p - \Lambda + \frac{2\pi}{3m_i m_j}\kappa' \frac{1}{\pi^{3/2} r_0^3} e^{(-r^2/r_0^2)} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right] \lambda_i \cdot \lambda_j$$

AL1: $p = 1$
AP1: $p = 2/3$

$$V_{ij}(r) = \left[\frac{\alpha_s}{4}\left(\frac{1}{r} - \frac{1}{6m_i m_j}\frac{e^{-r/r_0}}{r_0^2 r}\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right) + \underbrace{\left(-a_c(1 - e^{-\mu_c r}) + \Delta\right)}_{\text{Screened confinement}}\right] \lambda_i \cdot \lambda_j$$

Salamanca model (SLM)

$$+V_\pi + V_K + V_\eta + V_\sigma$$



chiral
quark mode

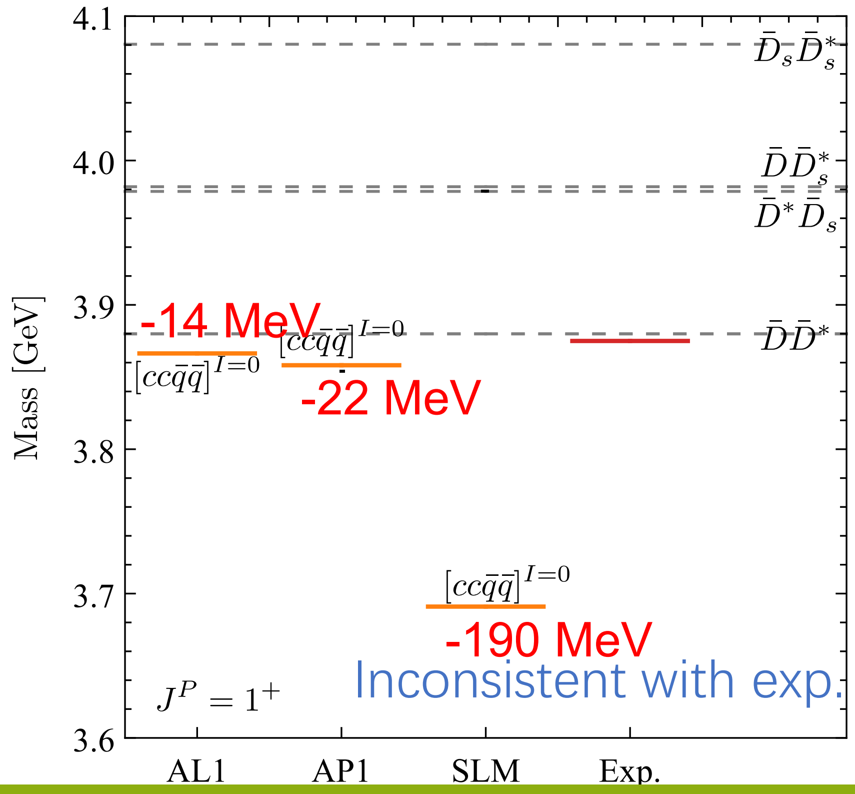
Total runtime:
2.5h on my laptop

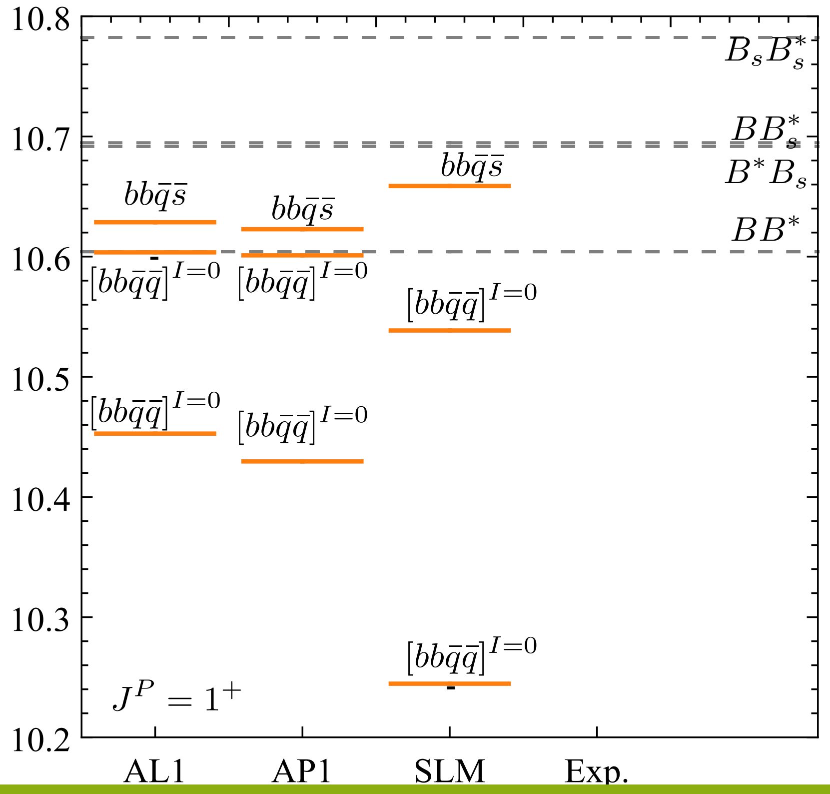
$QQ\bar{q}\,\bar{q}$ with $J^P = 1^+$

"Predictions"

- Points of agreement
  - ▶ $[cc\bar{q}\,\bar{q}]^{I=0}$, $[bb\bar{q}\,\bar{q}]^{I=0}$, $[bb\bar{q}\bar{s}]$ bound states ;
  - ▶ 1st excited state $[bb\bar{q}\,\bar{q}]^{I=0}$ is bound state
  - ▶ No $[bb\bar{q}\,\bar{q}]^{I=1}$ states
- SLM: deeper states

  (1) $[cc\bar{q}\bar{q}]^{I=0}$ and $[bb\bar{q}\bar{q}]^{I=0}$ are deeper than AL1 and AP1





Computational error in literature

- Ref.[Ortega:2022efc]: Same SLM interactions
  - $[bb\bar{q}\bar{q}]^{I=1}$ bound state! Our results: there is no isospin vector states
  - Get a $J^P(I)=0^+(0)$ state dominated by S-wave BB states!

Violating Boson principle

Mass	E_B	$\mathcal{P}_{B^0B^{*+}}$	$\mathcal{P}_{B^+B^{*0}}$	$\mathcal{P}_{B^{*+}B^{*0}}$	$\mathcal{P}_{I=0}$	$\mathcal{P}_{I=1}$
10582.2	21.9	47.8	50.0	2.2	99.99	0.01
10593.5	10.5	51.0	48.6	0.4	0.02	99.98

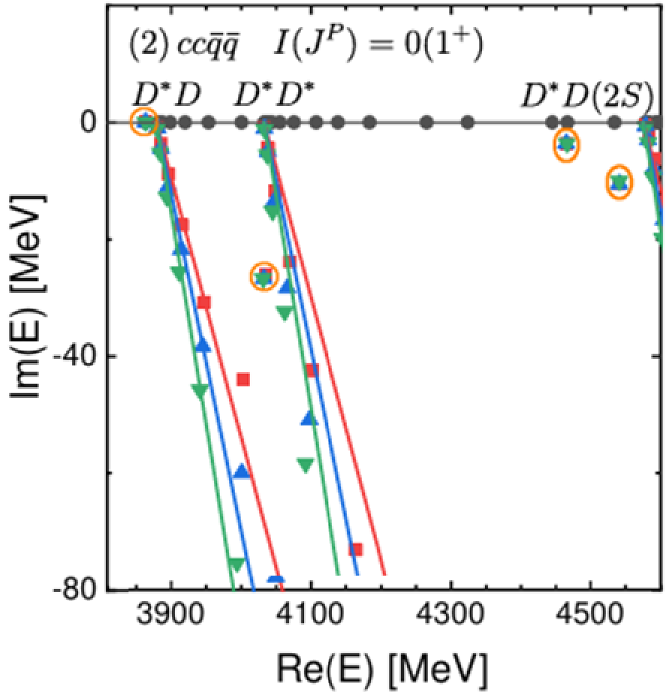
J^P	I	Mass	Width	E_B	$\mathcal{P}_{BB}$	$\mathcal{P}_{B^*B^*}$	Γ_{BB}	$\Gamma_{B^*B^*}$
0^+	0	10553.0	0	6.0	92%	8%	0	0
		10640.7	2.8	8.7	76%	24%	2.8	0
	1	10545.9	0	13.1	93%	7%	0	0
		10672.6	72.0	−23.2	39%	61%	30.7	41.3
2^+	1	10642.3	0	7.1	-	100%	-	0

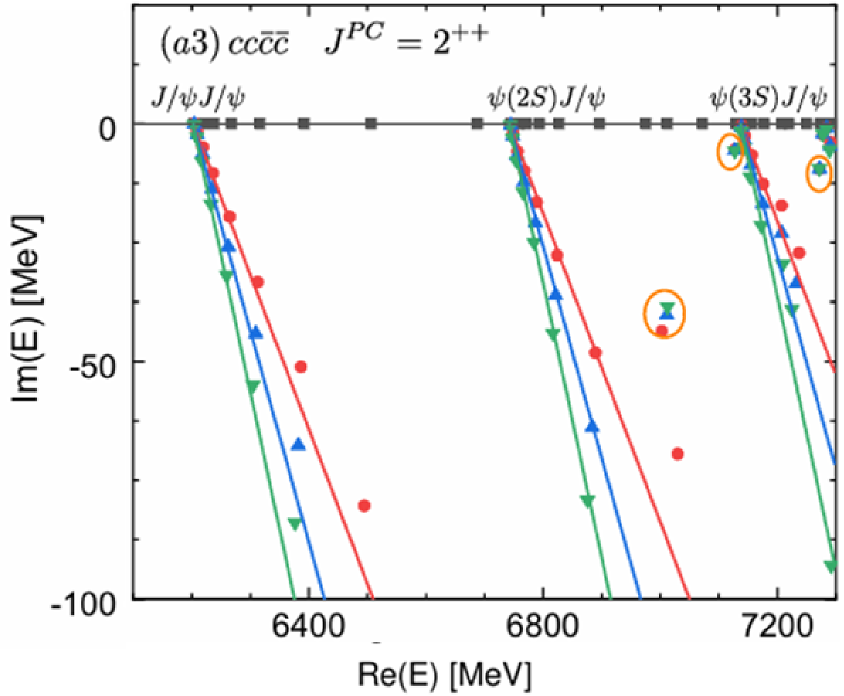
Bound and resonant results

- Recommended tetraquark states below di-meson thresholds (consistent predictions of AP1 and AL1 models)

$J^P = 1^+$	$[cc\bar{q}\bar{q}]^{I=0}$	$[bb\bar{q}\bar{q}]^{I=0}$	$[bc\bar{q}\bar{q}]^{I=0}$	$bb\bar{q}\bar{s}$	$[bs\bar{q}\bar{q}]^{I=0}$
$J^P = 0^+$	$[cb\bar{q}\bar{q}]^{I=0}$	$[cs\bar{q}\bar{q}]^{I=0}$	$[bs\bar{q}\bar{q}]^{I=0}$		
$J^P = 2^+$	$[cb\bar{q}\bar{q}]^{I=0}$				

- Wave functions: T_{cc} molecular state, T_{bb} compact tetraquark state
- Resonances from the complex scaling method
  - ▶ $QQ\bar{Q}\bar{Q}$
  - ▶ $\bar{Q}\bar{Q}qq$
  - ▶ $Qs\bar{q}\bar{q}$
  - ▶ $ss\bar{s}\bar{s}$
  - ▶ …





Benchmark test calculation of a four-nucleon bound states

Benchmark Test Calculation of a Four-Nucleon Bound State

H. Kamada [*], A. Nogga and W. Glöckle

Institut für Theoretische Physik II, Ruhr-Universität Bochum, 44780 Bochum, Germany

E. Hiyama[§] and M. Kamimura[§§]

[§] High Energy Accelerator Research Organization, Institute of Particle and Nuclear Studies, Japan

[§§] Department of Physics, Kyushu University, Fukuoka 812-8581, Japan

K. Varga[**] and Y. Suzuki[***]

[**] Solid State Division, Oak Ridge National Laboratory, Oak Ridge, TN 37380

[**] Institute of Nuclear Research of the Hungarian Academy of Sciences (ATOMKI), Debrecen, 4000, PO Box 51. Hungary

[***] Department of Physics, Niigata University, Niigata 950-2181, Japan

M. Viviani[♯], A. Kievsky[♯], and S. Rosati[♯,♭]

[♯] INFN, Sezione di Pisa, I-56100 Pisa, Italy

[♭] Department of Physics, University of Pisa, I-56100 Pisa, Italy

J. Carlson[%], Steven C. Pieper[%%] and R. B. Wiringa[%%]

[%] Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

[%%] Physics Division, Argonne National Laboratory, Argonne, Illinois 60439

P. Navrátil[a] and B. R. Barrett[+]

[a] Lawrence Livermore National Laboratory, P. O. Box 808, Livermore, CA 94551

[a] Nuclear Physics Institute, Academy of Sciences of the Czech Republic, 250 68 Řež near Prague, Czech Republic

[+] Department of Physics, P.O. Box 210081, University of Arizona, Tucson, AZ, 85721 USA

N. Barnea[++], W. Leidemann[+++] and G. Orlandini[+++]

[++] The Racah Institute of Physics, The Hebrew University, 91904, Jerusalem, Israel

[+++] Dipartimento di Fisica and INFN (Gruppo Collegato di Trento), Università di Trento,

Kamada:2001tv

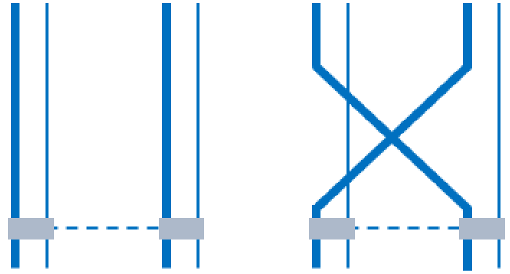
	E_b
Faddeev-Yakubovsky Eq.	-25.94(5)
Gaussian basis expansion	-25.90
stochastic variational method	-25.92
Hyperspherical variational	-25.90(1)
Green's function MC/Diffusion MC	-25.93(2)
No-core shell model	-25.80(20)
Hyperspherical harmonic methods	-25.944(10)
GEM, our results	-25.9

Method 2: Resonating Group Method

- Dimeson-wave function

$$\psi_{AB}(\boldsymbol{P}) = \mathcal{A}[\phi_A(\boldsymbol{p}_A)\phi_B(\boldsymbol{p}_B)\chi(\boldsymbol{P})\chi_{AB}^{CST}]$$

  - ▶ ϕ_A and ϕ_B are meson wave functions: from GEM



Entem:2000mq, Ortega:2022efc

RGM is valid only when clustering behavior is assumed.

- Schrödinger equation of RGM

$$\left(\frac{\boldsymbol{P}'^2}{2\mu} - E\right)\chi(\boldsymbol{P}') + \int d^3\boldsymbol{P}\left(V_D(\boldsymbol{P}',\boldsymbol{P}) + K_{Ex}(\boldsymbol{P}',\boldsymbol{P})\right)\chi(\boldsymbol{P}) = 0$$

V_D direct interaction, K_{Ex} the exchange kernel

- Only the di-meson-type spatial wave are included, not as general as GEM

$$E_{RGM} \gtrsim E_{GEM}$$

  - ▶ Valid only when clustering behavior is assumed

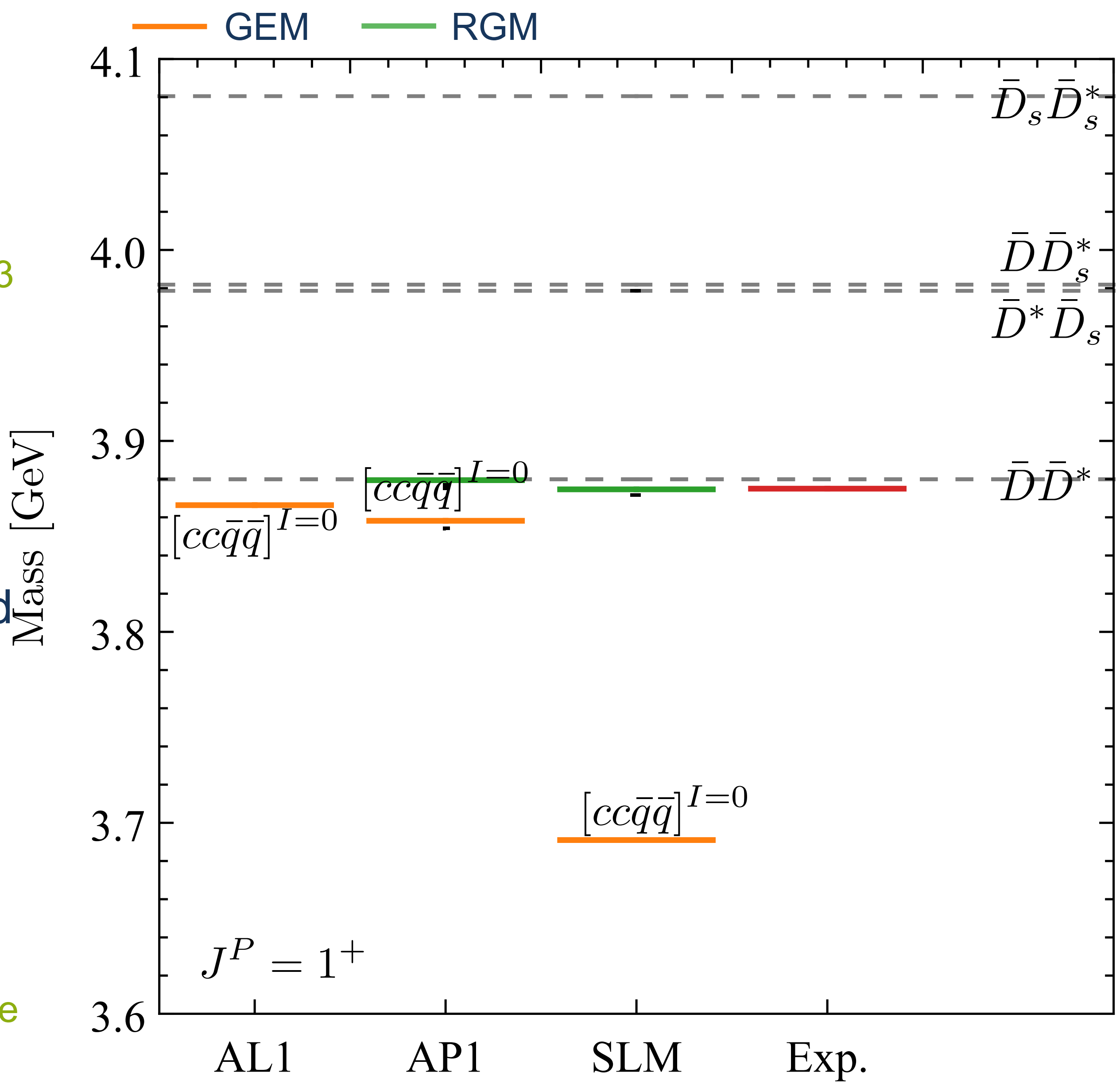
RGM results

- The RGM results agree with the GEM neglecting diquark-antidiquark correlation
  - ▶ Similar conclusion in literature

Y. Yang, C. Deng, J. Ping and T. Goldman, RRD (2009), 114023

- SLM group: SLM interaction+ RGM method
  - ▶ Loosely bound T_{cc} states
  - ▶ A correct result coincidentally from two wrong premises

Entem:2000mq, Vijande:2004he



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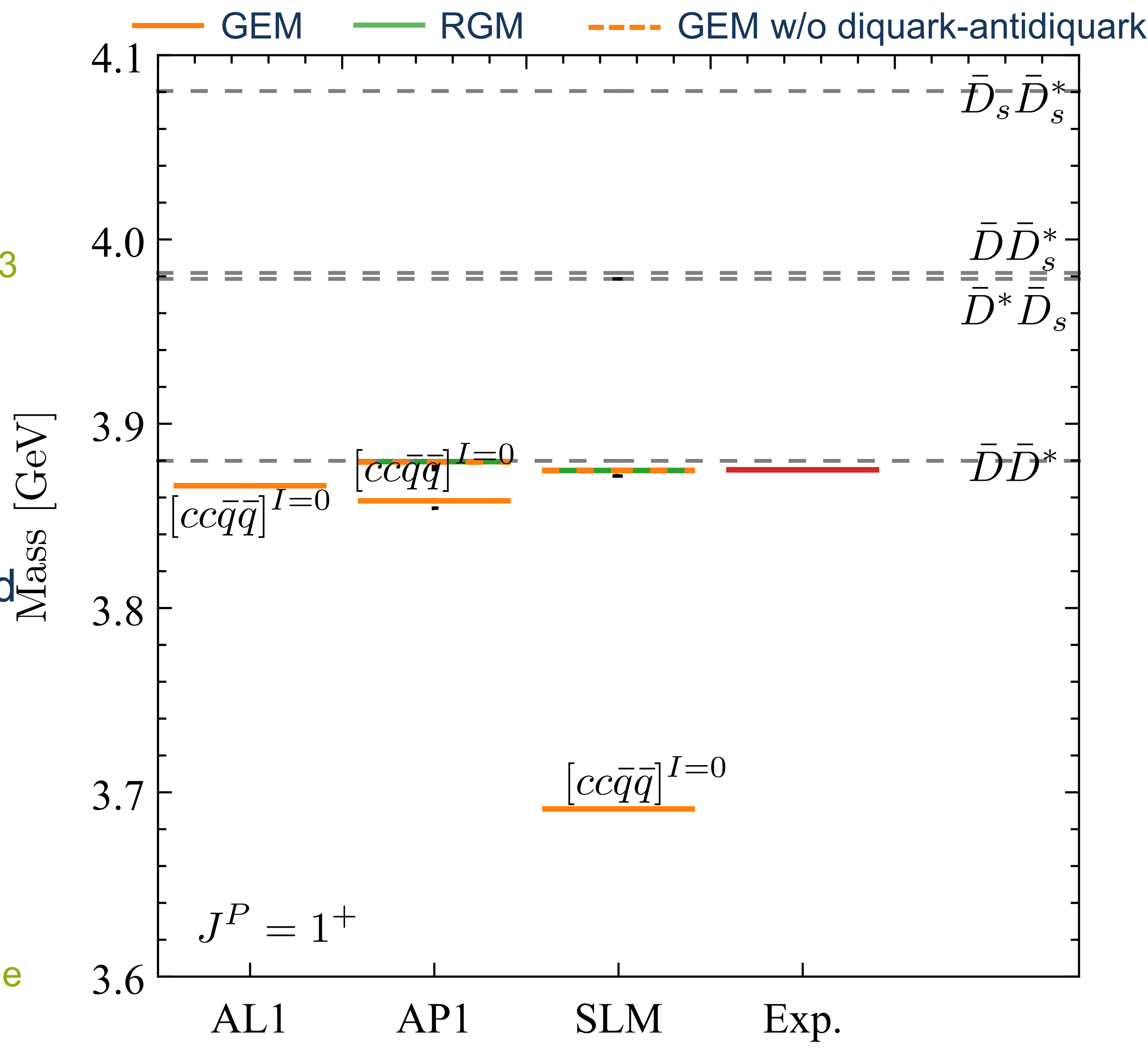


Method 3: Diffusion Monte Carlo

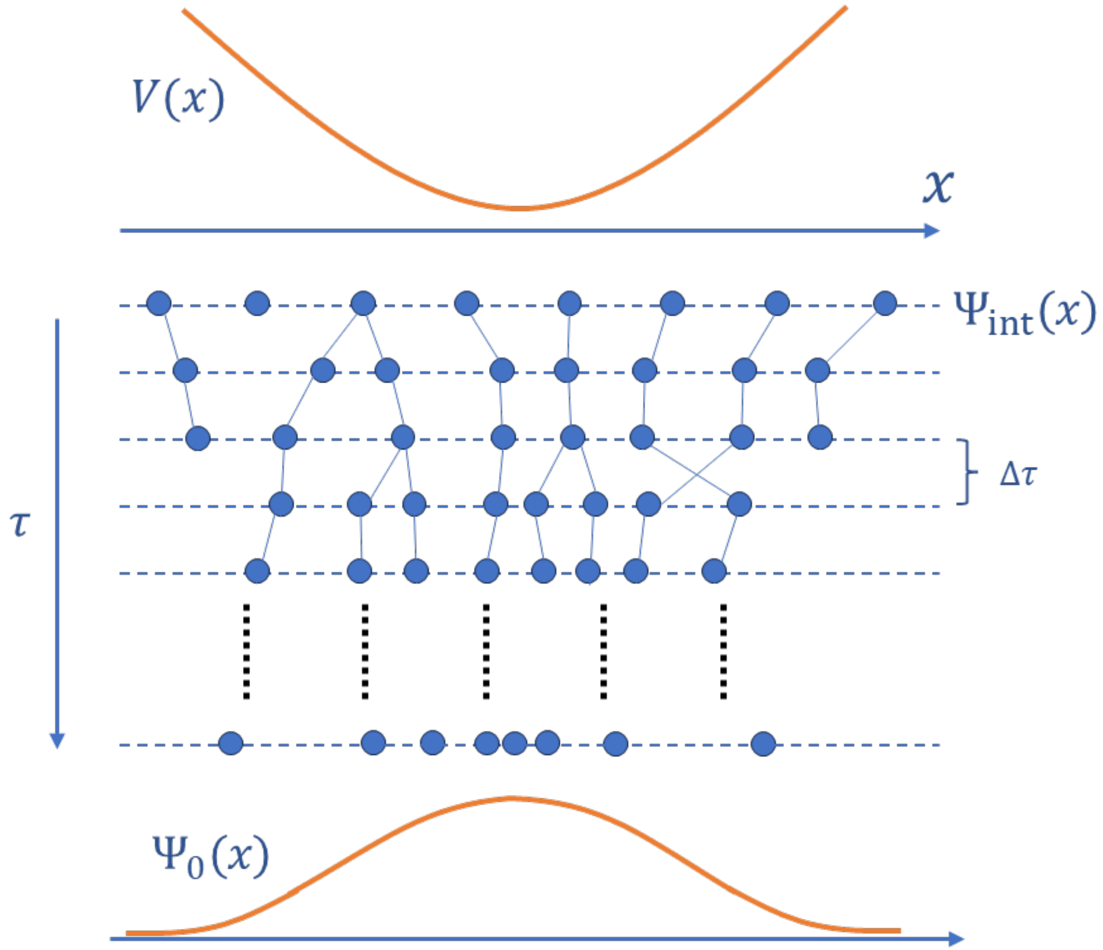
- $t \to i\tau$; Schrödinger equation→Diffusion equation

$$-\frac{\partial\Psi(\boldsymbol{R},\tau)}{\partial\tau} = \left[\underbrace{-\frac{\nabla^2}{2m}}_{\text{Diffusion}} + \underbrace{V(\boldsymbol{R}) - E_R}_{\text{Source or Sink}}\right]\Psi(\boldsymbol{R},\tau),$$

- $E_R \to E_0$, the $\Psi(\boldsymbol{R},t)$ →ground state when $t \to \infty$

$$\Psi(\boldsymbol{R},\tau) = \sum_i c_i\Phi_i(\boldsymbol{R})e^{-[E_i-E_R]\tau},$$

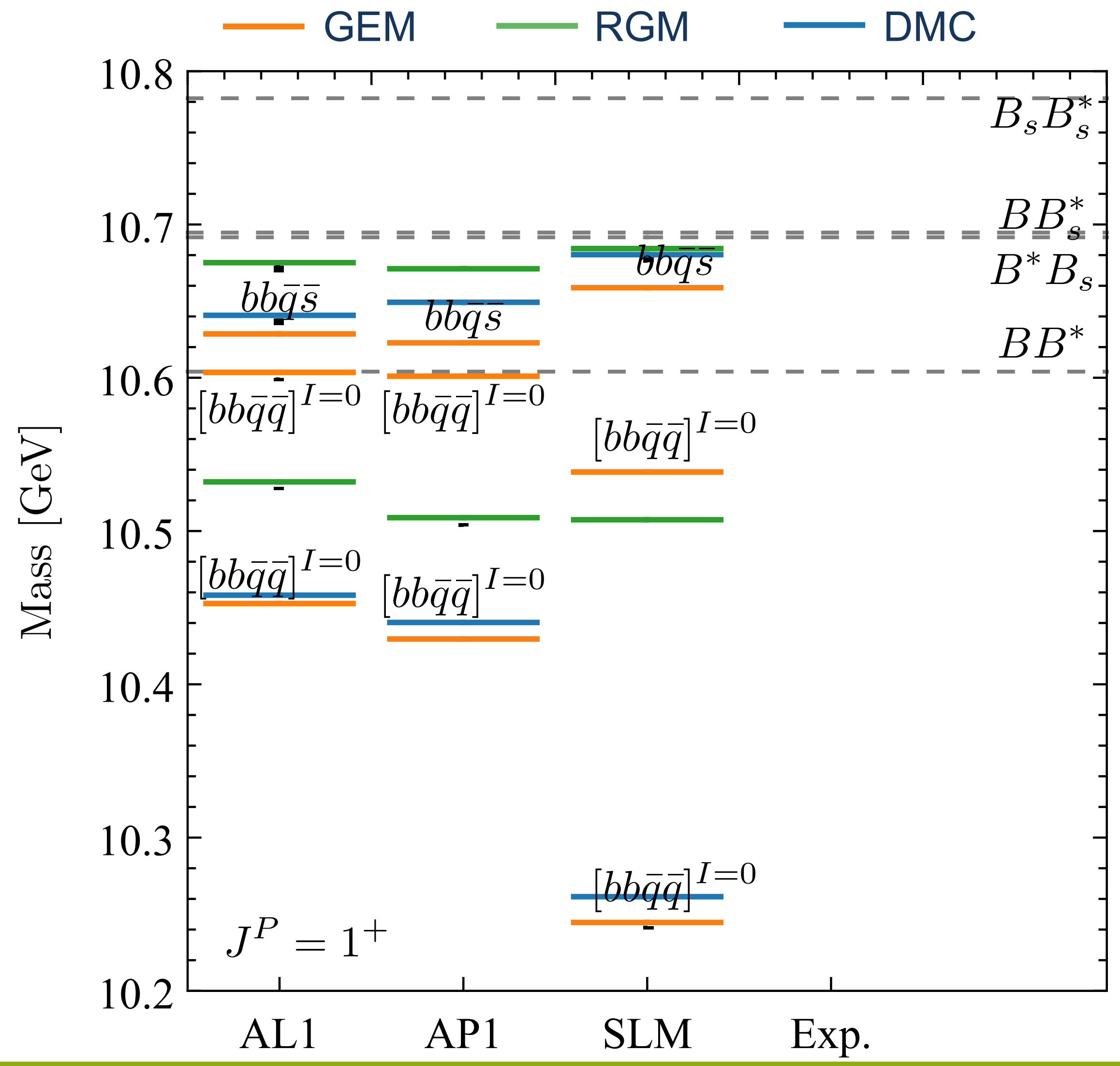
- In practices: importance sampling
- Milder increase computational cost as particles numbers



Results from DMC

- The DMC with the present coupled-channel strategy give the higher energy than GEM
- The coupled-channel strategy need to be improved in the future

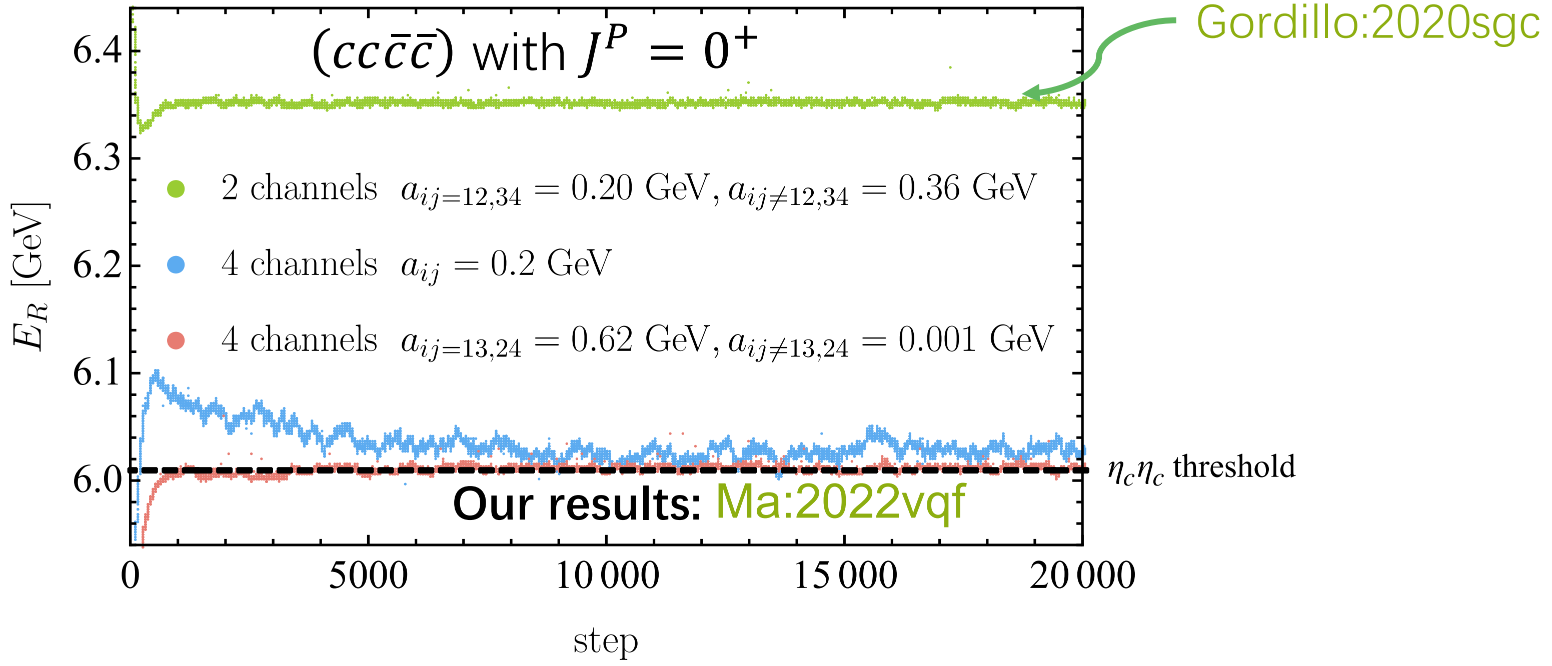
Y. L. Yang and P. W. Zhao, PhysRevC.107.034320



DMC in quark models

- Ref [Gordillo:2020sgc]:

  got a wrong ground state



- Ref [Assi:2023dlu]:

  claimed the existence of fully heavy tetraquark bund state in version 1

FERMILAB-PUB-23-660-T

Tetraquarks made of sufficiently heavy quarks are bound in QCD

Benoît Assi[1] and Michael L. Wagman[1]

[1] *Fermi National Accelerator Laboratory, Batavia, IL, 60510*

version 1

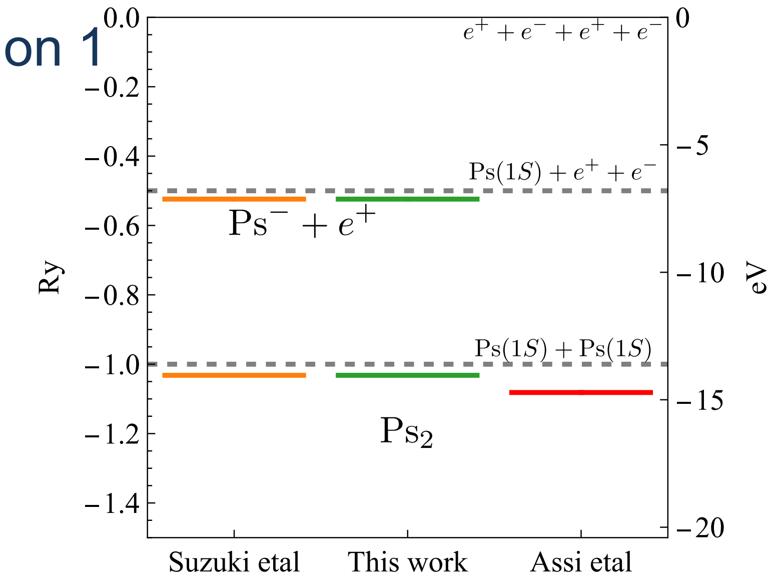
FERMILAB-PUB-23-660-T

Tetraquarks made of sufficiently unequal-mass heavy quarks are bound in QCD

Benoît Assi[1] and Michael L. Wagman[1]

[1] *Fermi National Accelerator Laboratory, Batavia, IL, 60510*

version 2



Summary and outlook

- Do we still need quark model? Yes！ The quark model should be adapted to the era of multiquarks.
- An efficient automatic multiquark resonance package
  - ▶ Gaussian expansion method + Complex scaling method
- Benchmark test (AL1,AP1,SLM)⊗(GEM,RGM,DMC)
  - ▶ $(QQ\bar{Q}\bar{Q})$, $(QQ\bar{Q}\bar{q})$, $(QQ\bar{q}\bar{q})$, $(Qs\bar{q}\bar{q})$, $(Q\bar{s}q\bar{q})$
  - ▶ Recommended tetraquark states below di-meson thresholds

$J^P=1^+$	$[cc\bar{q}\bar{q}]^{I=0}$	$[bb\bar{q}\bar{q}]^{I=0}$	$[bc\bar{q}\bar{q}]^{I=0}$	$bb\bar{q}\bar{s}$	$[bs\bar{q}\bar{q}]^{I=0}$
$J^P=0^+$	$[cb\bar{q}\bar{q}]^{I=0}$	$[cs\bar{q}\bar{q}]^{I=0}$	$[bs\bar{q}\bar{q}]^{I=0}$		
$J^P=2^+$	$[cb\bar{q}\bar{q}]^{I=0}$				

  - ▶ SLM interaction: too attractive Improve it: + vector meson exchange?
  - ▶ RGM: valid only when the clustering behavior is assumed
  - ▶ Point out several computational errors in literature
  - ▶ Predicting resonant states
- Outlook
  - ▶ Flux-tube confinement potentials
  - ▶ Decays
  - ▶ Relativistic effect
  - ▶ Release the package as open source

Thanks for your attention!!

Backup

DMC results from [Gordillo:2020sgc]

- Results from [Gordillo:2020sgc]

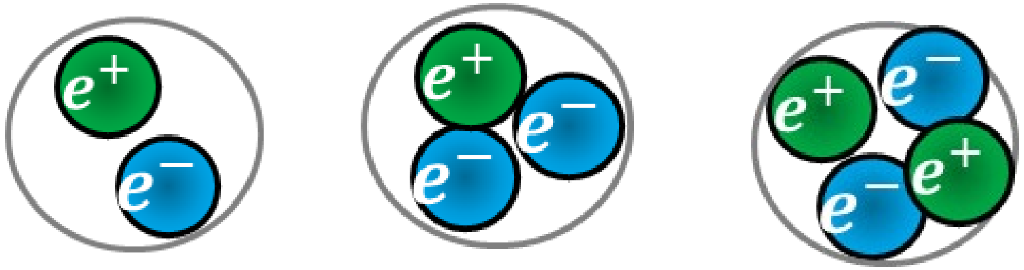
	$n^{2S+1}L_J$	J^{PC}	DMC
η_c	1^1S_0	0^{-+}	3005
J/ψ	1^3S_1	1^{--}	3101
B_c	1^1S_0	0^{-+}	6292
B_c^*	1^3S_1	1^{--}	6343
η_b	1^1S_0	0^{-+}	9424
$\Upsilon(1S)$	1^3S_1	1^{--}	9462

$cc\bar{c}\bar{c}$

J^{PC}	DMC
0^{++}	6351
1^{+-}	6441
2^{++}	6471

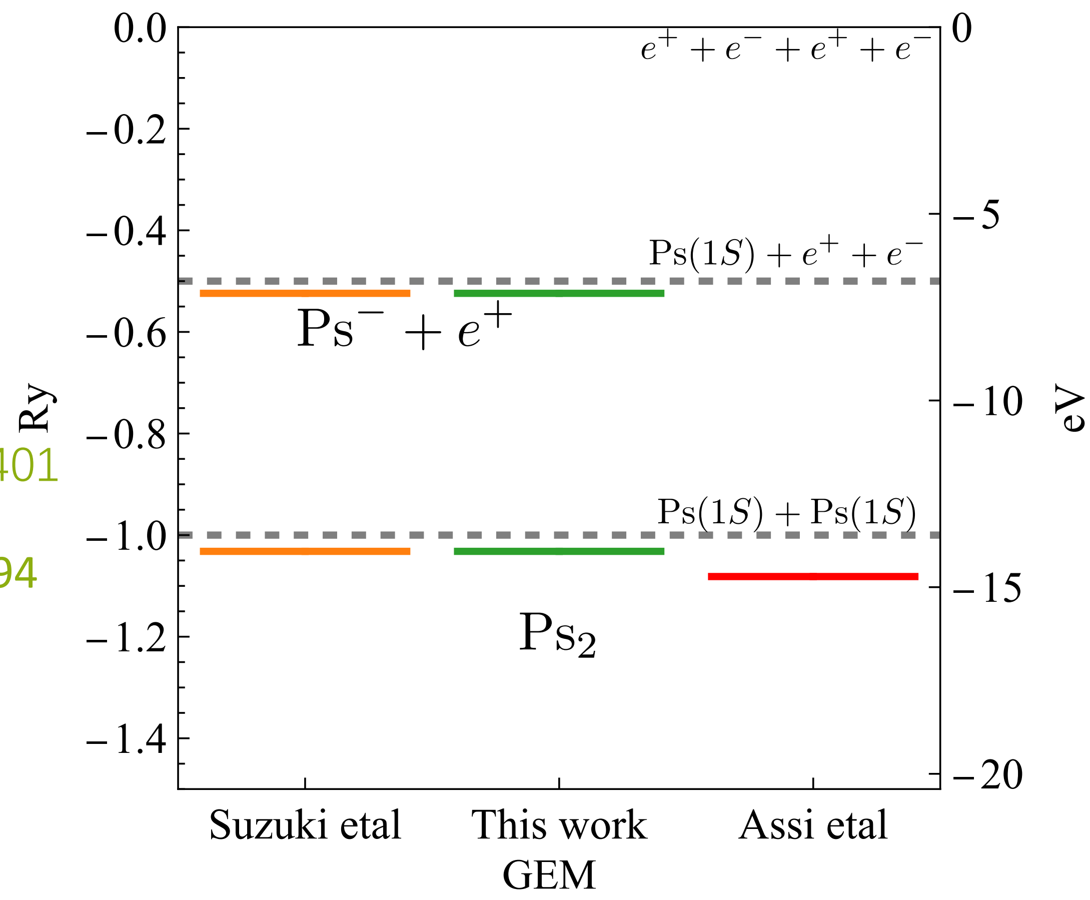
- The mass of $T_{cc\bar{c}\bar{c}}$ is about several hundreds MeV above the related di-meson thresholds

Di-positronium



Ps $\quad$ Ps^- $\quad$ Ps_2

- It was predicted to exist in 1946 by J.A Wheeler
- The binding energy is 0.435 485 eV PhysRevLett.92.043401
- It was not observed until 2007 in an experiment 10.1038/nature06094
- Accuracy of our results: 0.435 eV
- However, Assi etal get a result 1.1 eV



Tetraquarks made of sufficiently heavy quarks are bound in QCD

Benoît Assi[1] and Michael L. Wagman[1]

[1] *Fermi National Accelerator Laboratory, Batavia, IL, 60510*

(Dated: November 6, 2023)

Tetraquarks, bound states composed of two quarks and two antiquarks, have been the subject of intense study but have yet to be understood from first principles. Previous studies of fully-heavy tetraquarks in nonrelativistic effective field theories of quantum chromodynamics (QCD) suggest different conclusions for their existence. We apply variational and Green's function Monte Carlo methods to compute tetraquarks' ground- and excited-state energies in potential nonrelativistic QCD. We robustly demonstrate that fully-heavy tetraquarks are bound in QCD for sufficiently heavy quark masses. We also predict the masses of tetraquark bound states comprised of b and c quarks, which are experimentally accessible, and suggest possible resolutions for previous theoretical discrepancies.

arXiv:2311.01498

Di-positronium - Wikipedia

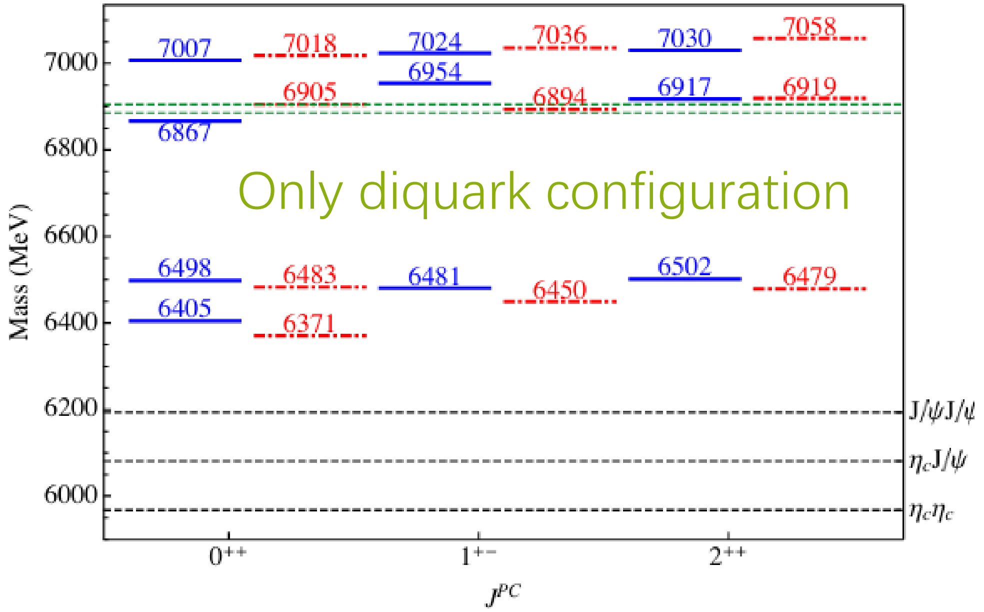
Define molecular radius

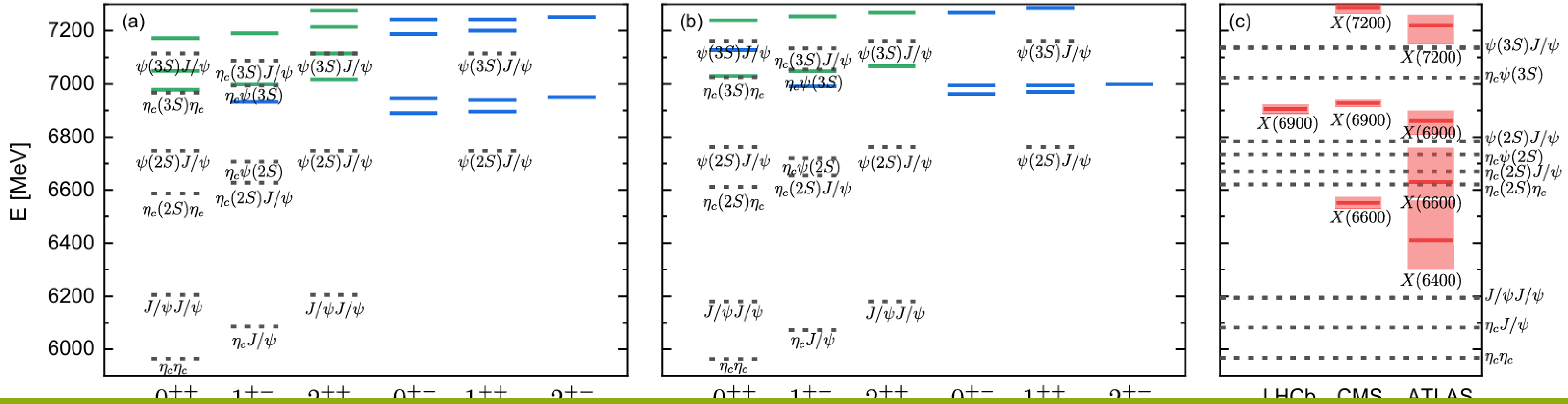
$$\Psi_{I,J}(\theta) = |\{Q\bar{q}_1\}_{1_c}\{s\bar{q}_2\}_{1_c}\rangle \otimes |\psi_1^\theta) + |\{Q\bar{q}_2\}_{1_c}\{s\bar{q}_1\}_{1_c}\rangle \otimes |\psi_2^\theta)$$

$$r_{ij}^{\mathrm{rms}} \equiv \mathrm{Re}\left[\sqrt{\frac{(\psi_1^\theta|r_{ij}^2 e^{2i\theta}|\psi_1^\theta)}{(\psi_1^\theta|\psi_1^\theta)}}\right]$$

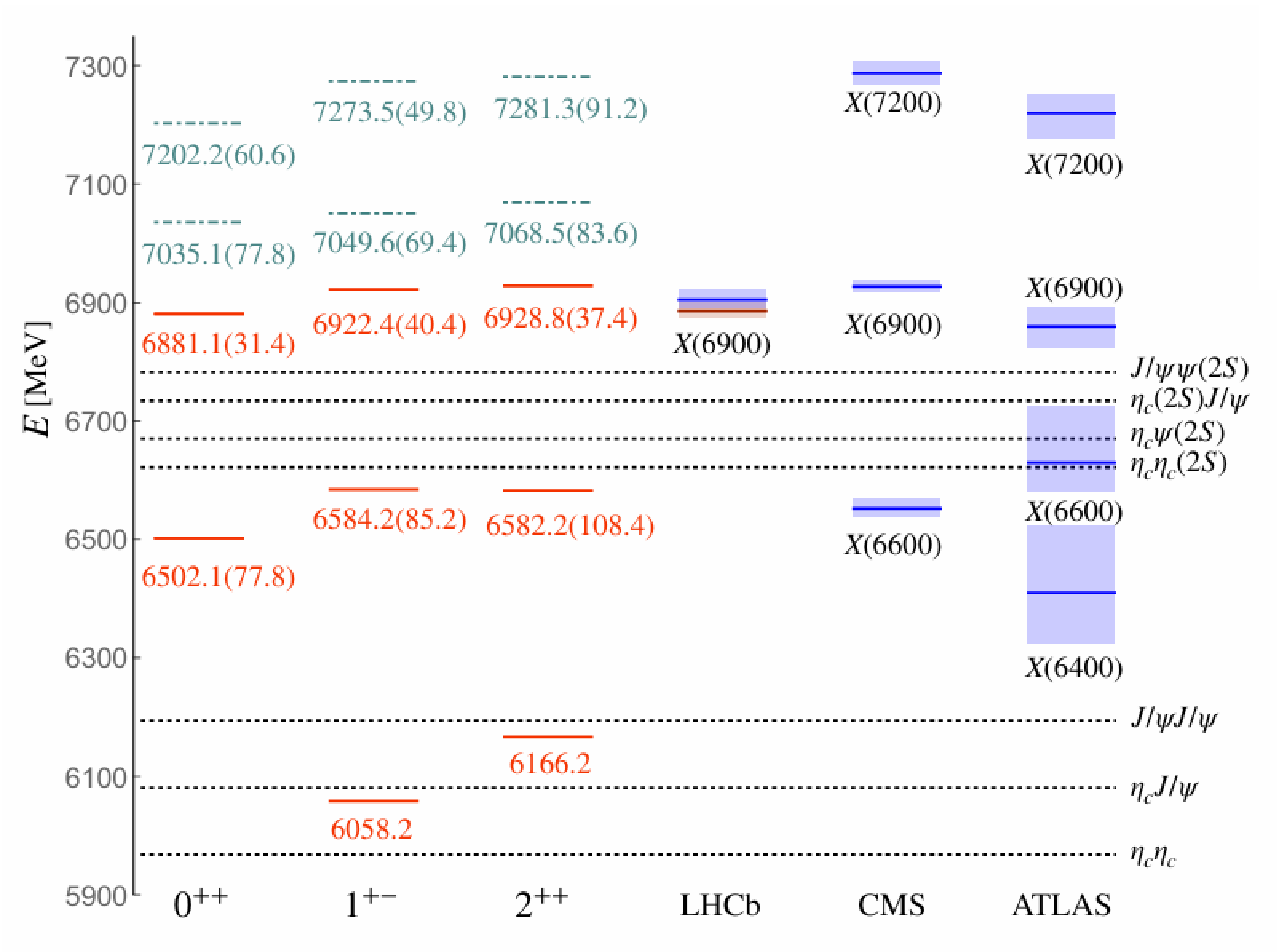
Status of fully heavy tetraquark

- Including di-meson configuration introduce large discrepancy with Ex. results





Fully tetraquark: confinement



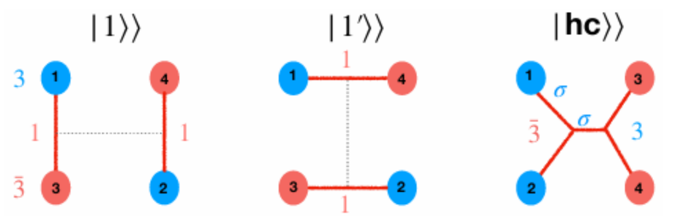


FIG. 1. Three color basis states: $|\mathbf{1}\rangle\rangle$, $|\mathbf{1}'\rangle\rangle$, and $|\mathbf{hc}\rangle\rangle$ in the novel string-type color confinement model.

$$\Psi(1,2,3,4) = \psi_{\mathbf{1}}|\mathbf{1}\rangle\rangle + \psi_{\mathbf{1}'}|\mathbf{1}'\rangle\rangle + \psi_{\mathbf{hc}}|\mathbf{hc}\rangle\rangle, \quad (5)$$

2307.04310

Color-electric interaction for multiquark states

- For color-singlet multiquark states $\{Q_1, Q_2, ..., Q_n\}$, $Q_i = Q$ or $\bar{Q}$, if two-body interaction $V_{Q_iQ_j} = V_8(r_{ij})\lambda_i \cdot \lambda_j$, then

$$V_{\{Q_1,Q_2,...Q_n\}} = \sum_{i<j} a_{ij} V_8(r_{ij}), \quad \sum_{i<j} a_{ij} = -\frac{8}{3}n$$

Proof: $2\langle\sum_{i<j} \lambda_i \cdot \lambda_j\rangle = \langle\sum_i \lambda_i\rangle^2 - \sum_i \langle\lambda_i\rangle^2$

- A general problem: For fixed $\sum_{i<j} a_{ij}$, what distribution of a_{ij} give the lowest mass? Walter Thirring, E.M. Harrell, Quantum mathematical physics: atoms, molecules and large systems

$\Rightarrow$ the symmetric case gives the worse result: $M_n(a_{ij}) \leq M_n^{(S)}$

Proof:

$$H = H^S + \Delta V = H^S + aV_8(r_{12}) - aV_8(r_{34}), \quad \langle\psi^S|\Delta V|\psi^S\rangle = 0, \quad (32)$$

$$\langle\psi^S|H^S|\psi^S\rangle = \langle\psi^S|H|\psi^S\rangle \geq \min_{\psi\in\mathcal{H}}\langle\psi|H|\psi\rangle \quad (33)$$

- Intuitively, less symmetric a_{ij}, more deeply bound ground states

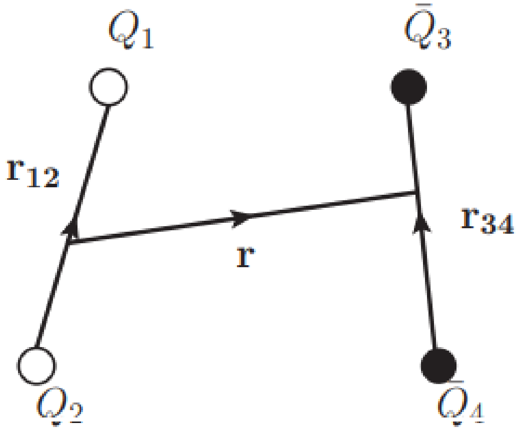
Phys.Rev. D25 (1982) 2370

Color-electric interaction for $QQ\bar{Q}\bar{Q}$

- $\{a_{ij}\}$ distribution for $QQ\bar{Q}\bar{Q}$

a_{ij}	$a_{12} = a_{34}$	$a_{13} = a_{24}$	$a_{14} = a_{24}$
Di-meson	0	$-\frac{16}{3}$	0
$\bar{3}_c - 3_c$	$-\frac{8}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$
$\bar{6}_c - 6_c$	$\frac{4}{3}$	$-\frac{10}{3}$	$-\frac{10}{3}$

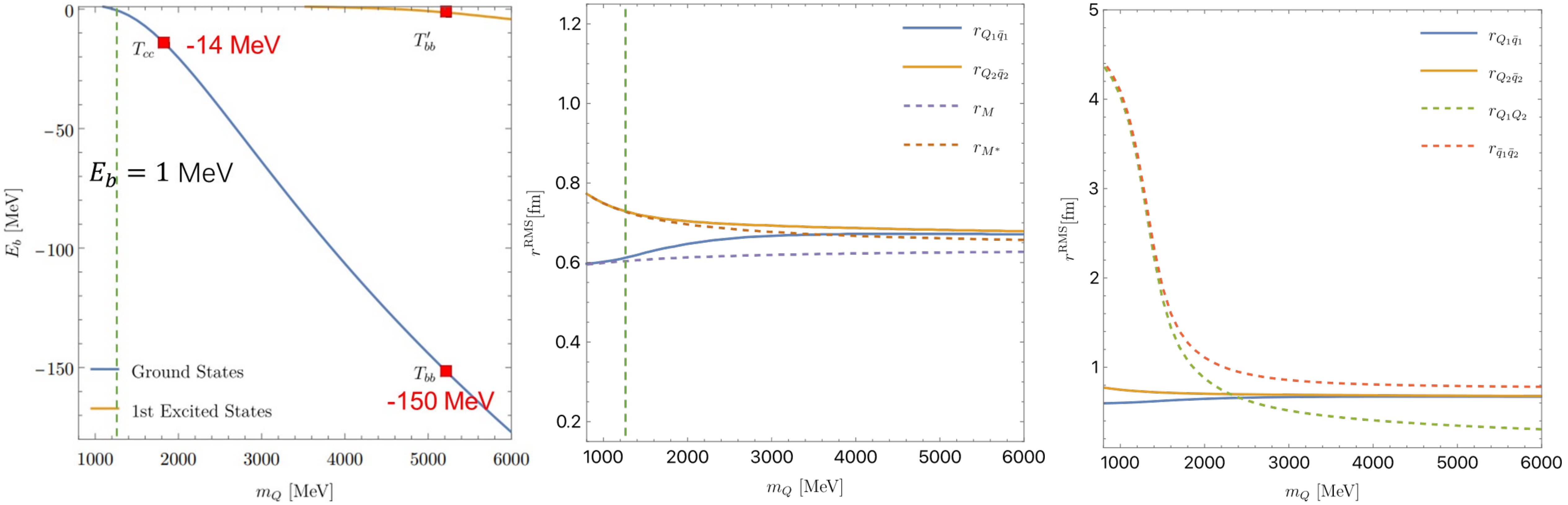
$$2M(Q\bar{Q}) < M(6-\bar{6}) < M(3-\bar{3})$$



- Things become different, when
  - ⇒ unequal quark masses e.g. $QQ\bar{q}\bar{q}$ PRL118 142001; PRL119 202001; PRL119 202002
  - ⇒ hyperfine correction, e.g. color-magnetic interaction $S_i \cdot S_j \lambda_i \cdot \lambda_j$
  - ⇒ multibody interaction, e.g. doubly-Y interaction
  - ⇒...

Phys.Rev. D25 (1982) 2370

Molecular or compact ?



- Tuning the m_Q to m_b: (bb) compact diquark
- Tuning the m_Q to make $E_b < 1$ MeV: molecular states