Reduction for one-loop Feynman integrals in relativistic quantum field theories at finite temperature and finite density

一个逃兵的回归与流浪 (An interplay among PRD, PRB, and PRC)

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(“夹逼定理”)

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Outline

1. Introduction and Motivation
2. From Passarino-Veltman reduction to Generalized Passarino-Veltman reduction
   - Passarino-Veltman reduction (PVR)
   - Gereralized Passarino-Veltman reduction (GPVR)
   - Two demonstration applications of GPVR
3. Summary and Outlook

Scattering Amplitudes: Bridge the Bap between Experiments and Theories

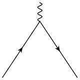
$$d\sigma = \frac{1}{2E_{\mathcal{A}}E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|}\left(\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_f p_f\right)$$

$$d\Gamma = \frac{1}{2m_{\mathcal{A}}}\left(\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}(m_{\mathcal{A}} \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} - \sum_f p_f\right).$$

Typical tree and loop Feynman diagrams

In perturbative QFTs, scattering amplitude $\underset{\sim}{i\mathcal{M}}$ can be expressed in terms of

tree Feynman diagrams



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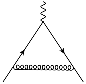
one-loop Feynman diagrams
(承前启后，继往开来)
(重要补充，必要准备)



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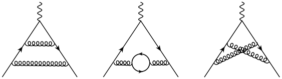
two-loop Feynman diagrams



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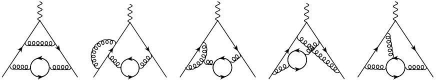
three-loop Feynman diagrams



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Tree Feynman diagrams for $e^+e^- \to \mu^+\mu^-$

Nuclear Physics B160 (1979) 151–207

ONE-LOOP CORRECTIONS FOR e^+e^- ANNIHILATION INTO $\mu^+\mu^-$ IN THE WEINBERG MODEL

G. PASSARINO* and M. VELTMAN
Institute for Theoretical Physics, University of Utrecht, Utrecht, The Netherlands

Received 22 March 1979

G. Passarino, M. Veltman / One-loop corrections for $e^+e^- \to \mu^+\mu^-$

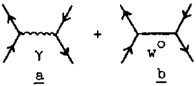


Fig. 2. Lowest-order diagrams for $e^+e^- \to \mu^+\mu^-$.

Feynman diagrams at tree-level

One-loop Feynman diagrams for $e^+e^- \to \mu^+\mu^-$ (1)

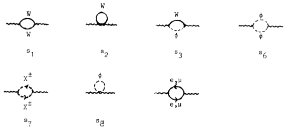
Feynman diagrams at one-loop level

Appendix C

Diagrams

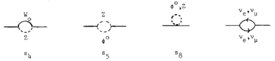
Diagrams which contribute to the amplitude $e^+e^- \to \mu^+\mu^-$, or to the Ward identities are:

Photon self-energy:

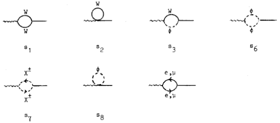


W^0 self-energy:

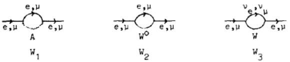
We have the same types of diagrams as in the photon self-energy with external W^0 lines and in addition:



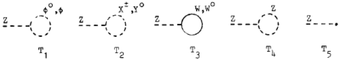
Photon – W^0 transition:



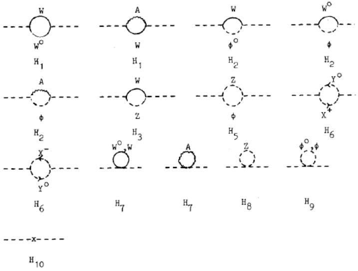
e, μ self-energy:



tadpoles:

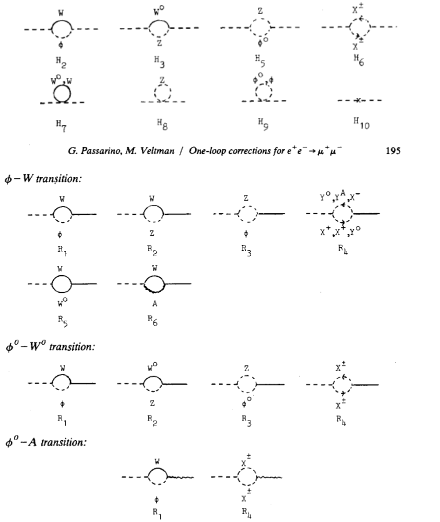


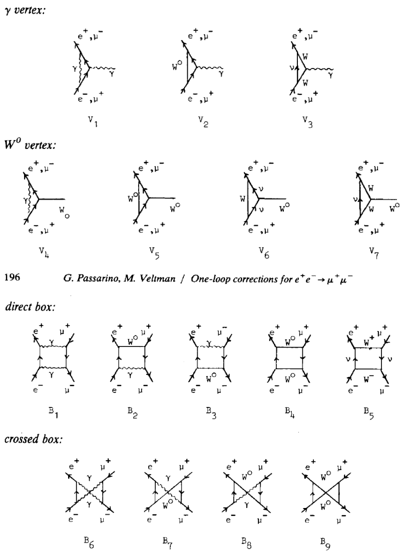
ϕ^+ self-energy:



One-loop Feynman diagrams for $e^+e^- \to \mu^+\mu^-$ (2)

Feynman diagrams at one-loop level (cont.)

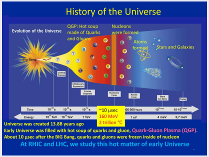


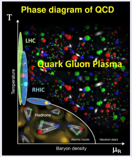


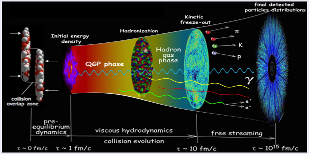
Complexities and Calculation methods

Complexities

- MUCH MORE one-loop Feynman diagrams due to more external legs.
  For $gg \to ggggg$ process in pure Yang-Mills theory, there are 2 485 ($\sim 10^3$) tree Feynman diagrams, and 227 585 ($\sim 10^5$) one-loop Feynman diagrams.
- EXTRA complexity due to finite temperature and finite density.
  Feynman diagrams calculation in relativistic QFTs at finite temperature and finite density. ("火上浇油")







Conventional method: To calculate ONE by ONE (手工抄书)

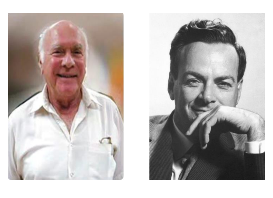
- （数量）少：a few
- （速度）慢：time-consuming
- （质量）差：error-prone
- （成本）费：disposable

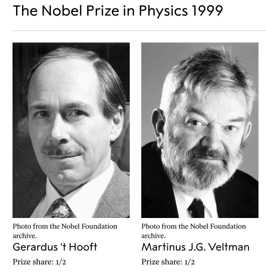
Efficient method: To calculate ONCE for ALL (活字印刷)

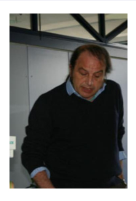
- （数量）多：a great many
- （速度）快：time-saving
- （质量）好：correctness-guaranteed
- （成本）省：reusable

Efficient method at zero temperature and zero density

Based on the Lorentz symmetry (continuous spacetime symmetry)







- L.M. Brown and R.P. Feynman, Radiative Corrections to Compton Scattering, Phys. Rev. **85**, 231 (1952).

- G. 't Hooft and M.J.G. Veltman, Scalar one-loop integrals, Nucl. Phys. B **153**, 365 (1979).
- G. Passarino and M.J.G. Veltman, One-loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model, Nucl. Phys. B **160**, 151 (1979).

Efficient method at finite temperature and finite density

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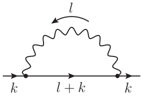
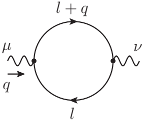
Outline

An observation from typical one-loop Feynman diagrams in QED (1)





$$-i\tilde{\Sigma}(k;0,m) = -e^2\left[\gamma^\alpha(\gamma_\rho k^\rho + m)\gamma_\alpha\right] \times \underbrace{\int \frac{d^D l}{(2\pi)^D} \frac{1}{[l^2 - 0^2][(l+k)^2 - m^2]}}$$

$$- e^2\left[\gamma^\alpha\gamma_\rho\gamma_\alpha\right] \times \underbrace{\int \frac{d^D l}{(2\pi)^D} \frac{l^\rho}{[l^2 - 0^2][(l+k)^2 - m^2]}},$$

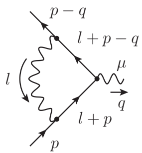
$$i\tilde{\Pi}^{\mu\nu}(q;m;m) = -4e^2\left[g^{\mu\nu}m^2\right] \times \underbrace{\int \frac{d^D l}{(2\pi)^D} \frac{1}{[l^2 - m^2][(l+q)^2 - m^2]}}$$

$$- 4e^2\left[g^\mu{}_\rho g^\nu{}_\sigma - g^{\mu\nu}g_{\rho\sigma} + g^\mu{}_\sigma g^\nu{}_\rho\right] q^\sigma \times \underbrace{\int \frac{d^D l}{(2\pi)^D} \frac{l^\rho}{[l^2 - m^2][(l+q)^2 - m^2]}}$$

$$- 4e^2\left[g^\mu{}_\rho g^\nu{}_\sigma - g^{\mu\nu}g_{\rho\sigma} + g^\mu{}_\sigma g^\nu{}_\rho\right] \times \underbrace{\int \frac{d^D l}{(2\pi)^D} \frac{l^\rho l^\sigma}{[l^2 - m^2][(l+q)^2 - m^2]}},$$

Two-point one-loop Feynman diagram: two square brackets.

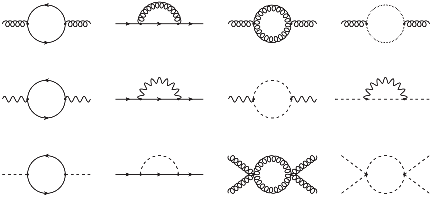
An observation from typical one-loop Feynman diagrams in QED (2)



$$
\begin{aligned}
-ie\delta\tilde{\Gamma}^{\mu}(p,p-q;0;m;m) = & -e^3\left[\gamma^{\alpha}(\not{p}+m)\gamma^{\mu}(\not{p}-\not{q}+m)\gamma_{\alpha}\right] \\
& \times \underbrace{\int \frac{d^D l}{(2\pi)^D}\frac{1}{[l^2-0^2]\left[(l+p)^2-m^2\right]\left[(l+p-q)^2-m^2\right]}} \\
& -e^3\left[\gamma^{\alpha}\gamma_{\rho}\gamma^{\mu}(\not{p}-\not{q}+m)\gamma_{\alpha}+\gamma^{\alpha}(\not{p}+m)\gamma^{\mu}\gamma_{\rho}\gamma_{\alpha}\right] \\
& \times \underbrace{\int \frac{d^D l}{(2\pi)^D}\frac{l^{\rho}}{[l^2-0^2]\left[(l+p)^2-m^2\right]\left[(l+p-q)^2-m^2\right]}} \\
& -e^3\left[\gamma^{\alpha}\gamma_{\rho}\gamma^{\mu}\gamma_{\sigma}\gamma_{\alpha}\right] \\
& \times \underbrace{\int \frac{d^D l}{(2\pi)^D}\frac{l^{\rho}l^{\sigma}}{[l^2-0^2]\left[(l+p)^2-m^2\right]\left[(l+p-q)^2-m^2\right]}}
\end{aligned}
$$

Three-point one-loop Feynman diagram: three square brackets.

A further observation from typical two-point one-loop Feynman diagrams in relativistic QFTs



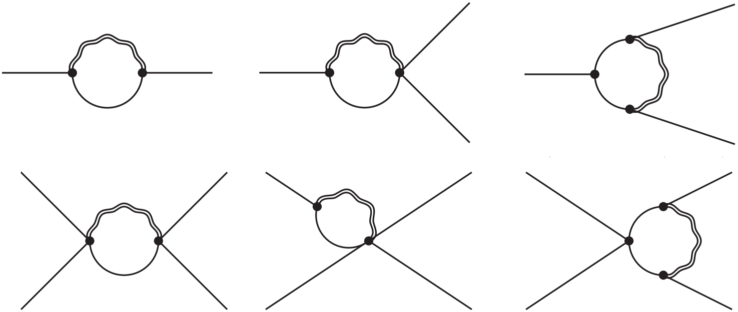
$$\tilde{\mathcal{A}}_0(p;m_1) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left[(l+p)^2 - m_1^2\right]},$$

$$\tilde{\mathcal{B}}_0(p_1;m_1;m_2) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]},$$

$$\tilde{\mathscr{A}}^{\rho}(p;m_1) = \int \frac{d^D l}{i\pi^{D/2}} \frac{l^\rho}{\left[(l+p)^2 - m_1^2\right]},$$

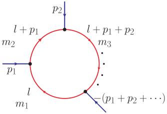
$$\tilde{\mathscr{B}}^{\{\rho,\rho\sigma\}}(p_1;m_1;m_2) = \int \frac{d^D l}{i\pi^{D/2}} \frac{\{l^\rho, l^\rho l^\sigma\}}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]}.$$

An even further observation from typical one-loop Feynman diagrams in relativistic QFTs inlcuding gravitons



Generic one-loop Feynman integrals at zero temperature and zero density

$$\Big\{ -i\tilde{\Sigma}, i\tilde{\Pi}^{\mu\nu}, -ie\delta\tilde{\Gamma}^{\mu}, \cdots \Big\}$$
$$\sim \Big\{ \tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0, \tilde{\mathcal{C}}_0, \cdots, \tilde{\mathscr{A}}^{\rho}, \tilde{\mathscr{B}}^{\rho}, \tilde{\mathscr{B}}^{\rho\sigma}, \tilde{\mathscr{C}}^{\rho}, \tilde{\mathscr{C}}^{\rho\sigma}, \tilde{\mathscr{C}}^{\rho\sigma\tau}, \cdots \Big\},$$



where the one-loop scalar Feynman integrals (OLSFIs) at zero temperature and zero density

$$\tilde{\mathcal{A}}_0(p; m_1) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left[(l+p)^2 - m_1^2\right]},$$

$$\tilde{\mathcal{B}}_0(p_1; m_1; m_2) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]},$$

$$\tilde{\mathcal{C}}_0(p_1, p_2; m_1; m_2; m_3) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]\left[(l+p_1+p_2)^2 - m_3^2\right]},$$

are Lorentz invariant, and the one-loop tensor Feynman integrals (OLTFIs) at zero temperature and zero density

$$\tilde{\mathscr{A}}^{\rho}(p; m_1) = \int \frac{d^D l}{i\pi^{D/2}} \frac{l^{\rho}}{\left[(l+p)^2 - m_1^2\right]},$$

$$\tilde{\mathscr{B}}^{\{\rho,\rho\sigma\}}(p_1; m_1; m_2) = \int \frac{d^D l}{i\pi^{D/2}} \frac{\{l^{\rho}, l^{\rho} l^{\sigma}\}}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]},$$

$$\tilde{\mathscr{C}}^{\{\rho,\rho\sigma,\rho\sigma\tau\}}(p_1, p_2; m_1; m_2; m_3) = \int \frac{d^D l}{i\pi^{D/2}} \frac{\{l^{\rho}, l^{\rho} l^{\sigma}, l^{\rho} l^{\sigma} l^{\tau}\}}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]\left[(l+p_1+p_2)^2 - m_3^2\right]}.$$

are Lorentz covariant and symmetric for ranks $r \geq 2$. After integrating over internal momentum l, the tensor indices must be inherited by external momenta and metric tensor.

Passarino-Veltman reduction in a nutshell (1)

Step-1: Reduce OLTFIs to master integrals at zero temperature and zero density

Based on the Lorentz covariance (continuous spacetime symmetry), the Lorentz-covariant OLTFIs

$$\left\{\tilde{\mathscr{A}}^{\rho},\tilde{\mathscr{B}}^{\rho},\tilde{\mathscr{B}}^{\rho\sigma},\tilde{\mathscr{C}}^{\rho},\tilde{\mathscr{C}}^{\rho\sigma},\tilde{\mathscr{C}}^{\rho\sigma\tau},\cdots\right\}$$

$$\sim\left\{p^{\rho},p_1^{\rho},p_2^{\rho},\cdots,g^{\rho\sigma},p_1^{\rho}p_1^{\sigma},(p_1^{\rho}p_2^{\sigma}+p_2^{\rho}p_1^{\sigma}),p_2^{\rho}p_2^{\sigma},\cdots,\right.$$
$$\left(p_1^{\rho}g^{\sigma\tau}+p_1^{\sigma}g^{\tau\rho}+p_1^{\tau}g^{\rho\sigma}\right),\left(p_2^{\rho}g^{\sigma\tau}+p_2^{\sigma}g^{\tau\rho}+p_2^{\tau}g^{\rho\sigma}\right),$$
$$\left.p_1^{\rho}p_1^{\sigma}p_1^{\tau},\left(p_1^{\rho}p_1^{\sigma}p_2^{\tau}+p_1^{\sigma}p_1^{\tau}p_2^{\rho}+p_1^{\tau}p_1^{\rho}p_2^{\sigma}\right),\left(p_1^{\rho}p_2^{\sigma}p_2^{\tau}+p_1^{\sigma}p_2^{\tau}p_2^{\rho}+p_1^{\tau}p_2^{\rho}p_2^{\sigma}\right),p_2^{\rho}p_2^{\sigma}p_2^{\tau},\cdots\right\}$$
$$\times\left\{\tilde{\mathcal{A}}_1,\tilde{\mathcal{B}}_1,\tilde{\mathcal{C}}_1,\tilde{\mathcal{C}}_2,\cdots,\tilde{\mathcal{B}}_{00},\tilde{\mathcal{B}}_{11},\tilde{\mathcal{C}}_{00},\tilde{\mathcal{C}}_{11},\tilde{\mathcal{C}}_{12},\tilde{\mathcal{C}}_{22},\cdots,\tilde{\mathcal{C}}_{001},\tilde{\mathcal{C}}_{002},\tilde{\mathcal{C}}_{111},\tilde{\mathcal{C}}_{112},\tilde{\mathcal{C}}_{122},\tilde{\mathcal{C}}_{222},\cdots\right\}.$$

Lorentz-invariant form factors can be expressed as a linear combination of master integrals

$$\left\{\tilde{\mathcal{A}}_1,\tilde{\mathcal{B}}_1,\tilde{\mathcal{C}}_1,\tilde{\mathcal{C}}_2,\cdots,\tilde{\mathcal{B}}_{00},\tilde{\mathcal{B}}_{11},\tilde{\mathcal{C}}_{00},\tilde{\mathcal{C}}_{11},\tilde{\mathcal{C}}_{12},\tilde{\mathcal{C}}_{22},\cdots,\tilde{\mathcal{C}}_{001},\tilde{\mathcal{C}}_{002},\tilde{\mathcal{C}}_{111},\tilde{\mathcal{C}}_{112},\cdots\right\}\sim\left\{\tilde{\mathcal{A}}_0,\tilde{\mathcal{B}}_0,\tilde{\mathcal{C}}_0,\cdots\right\}.$$

Step-2: Calculate master integrals at zero temperature and zero density

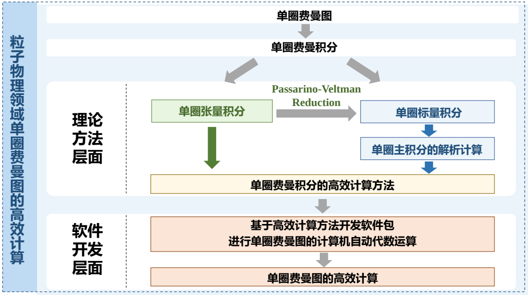
The master integrals $\tilde{\mathcal{A}}_0,\tilde{\mathcal{B}}_0,\tilde{\mathcal{C}}_0,\cdots$ are nothing but the Lorentz-invariant OLSFIs and had been analytically calculated, say, in "G. 't Hooft and M.J.G. Veltman, Scalar one-loop integrals, Nucl. Phys. B **153**, 365 (1979)".

As a consequence, the one-loop Feynman diagrams in relativistic QFTs at zero temperature and zero density

$$\left\{-i\tilde{\Sigma},i\tilde{\Pi}^{\mu\nu},-ie\delta\tilde{\Gamma}^{\mu},\cdots\right\}\sim\left\{\tilde{\mathcal{A}}_0,\tilde{\mathcal{B}}_0,\tilde{\mathcal{C}}_0,\cdots,\tilde{\mathscr{A}}^{\rho},\tilde{\mathscr{B}}^{\rho},\tilde{\mathscr{B}}^{\rho\sigma},\tilde{\mathscr{C}}^{\rho},\tilde{\mathscr{C}}^{\rho\sigma},\tilde{\mathscr{C}}^{\rho\sigma\tau},\cdots\right\}$$
$$\sim\left\{\tilde{\mathcal{A}}_0,\tilde{\mathcal{B}}_0,\tilde{\mathcal{C}}_0,\cdots\right\}.$$

Passarino-Veltman reduction in a nutshell (2)

零温度场论的“活字印刷”



A brief summary of PVR

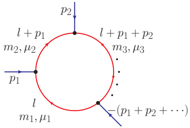
RQFTs	At zero temperature and zero density	At finite temperature and finite density
EMs	PVR	?
LS	with	?
OLSFIs	LI: $\tilde{\mathscr{A}}_0$, $\tilde{\mathscr{B}}_0$, $\tilde{\mathscr{C}}_0$, $\cdots$	?
OLTFIs	LC: $\tilde{\mathscr{A}}^{\rho}$, $\tilde{\mathscr{B}}^{\rho}$, $\tilde{\mathscr{B}}^{\rho\sigma}$, $\tilde{\mathscr{C}}^{\rho}$, $\tilde{\mathscr{C}}^{\rho\sigma}$, $\tilde{\mathscr{C}}^{\rho\sigma\tau}$, $\cdots$	?
TSs	LC	?
MIs	LI: $\tilde{\mathscr{A}}_0$, $\tilde{\mathscr{B}}_0$, $\tilde{\mathscr{C}}_0$, $\cdots$	?

Relativistic QFTs (RQFTs), Efficient Methods (EMs), Lorentz symmetry (LS),
One-loop scalar Feynman integrals (OLSFIs), One-loop tensor Feynman integrals (OLTFIs),
Lorentz-invariant (LI), Lorentz-covariant (LC), Tensor structures (TSs), Master integrals (MIs).

Outline

Generic OLSFIs at finite temperature and finite density

$$\Big\{-i\Sigma, i\Pi^{\mu\nu}, -ie\delta\Gamma^{\mu}, \cdots\Big\}$$
$$\sim \Big\{\mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \cdots, \mathscr{A}^{\rho}, \mathscr{B}^{\rho}, \mathscr{B}^{\rho\sigma}, \mathscr{C}^{\rho}, \mathscr{C}^{\rho\sigma}, \mathscr{C}^{\rho\sigma\tau}, \cdots\Big\}.$$



Generic OLSFIs in relativistic QFTs at finite temperature and finite density

$$\mathcal{A}_0(p; m_1, \mu_1; \beta) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^0 + p^0 + \mu_1)^2 - \left[(\boldsymbol{l} + \boldsymbol{p})^2 + m_1^2\right]},$$

$$\mathcal{B}_0(p_1; m_1, \mu_1; m_2, \mu_2; \beta) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left\{(l^0 + \mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0 + p_1^0 + \mu_2)^2 - \left[(\boldsymbol{l} + \boldsymbol{p}_1)^2 + m_2^2\right]\right\}},$$

$$\mathcal{C}_0(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left\{(l^0 + \mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0 + p_1^0 + \mu_2)^2 - \left[(\boldsymbol{l} + \boldsymbol{p}_1)^2 + m_2^2\right]\right\}} \times \frac{1}{\left\{(l^0 + p_1^0 + p_2^0 + \mu_3)^2 - \left[(\boldsymbol{l} + \boldsymbol{p}_1 + \boldsymbol{p}_2)^2 + m_3^2\right]\right\}},$$

$\cdots\cdots$

Generic OLTFIs at finite temperature and finite density

Generic OLTFIs in relativistic QFTs at finite temperature and finite density

$$\mathscr{A}^{\rho}(p;m_1,\mu_1;\beta) = \int \frac{d^D l}{i\pi^{D/2}} \frac{l^{\rho}}{(l^0+p^0+\mu_1)^2 - \left[(\boldsymbol{l}+\boldsymbol{p})^2+m_1^2\right]},$$

$$\mathscr{B}^{\{\rho,\rho\sigma\}}(p_1;m_1,\mu_1;m_2,\mu_2;\beta) = \int \frac{d^D l}{i\pi^{D/2}} \frac{\{l^{\rho}, l^{\rho}l^{\sigma}\}}{\left\{(l^0+\mu_1)^2-\left[\boldsymbol{l}^2+m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1)^2+m_2^2\right]\right\}},$$

$$\begin{aligned}\mathscr{C}^{\{\rho,\rho\sigma,\rho\sigma\tau\}}&(p_1,p_2;m_1,\mu_1;m_2,\mu_2;m_3,\mu_3;\beta)\\ = \int \frac{d^D l}{i\pi^{D/2}}& \frac{\{l^{\rho}, l^{\rho}l^{\sigma}, l^{\rho}l^{\sigma}l^{\tau}\}}{\left\{(l^0+\mu_1)^2-\left[\boldsymbol{l}^2+m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1)^2+m_2^2\right]\right\}}\\ &\times \frac{1}{\left\{(l^0+p_1^0+p_2^0+\mu_3)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1+\boldsymbol{p}_2)^2+m_3^2\right]\right\}},\end{aligned}$$

$$\cdots\cdots$$

- Finite densities μ_a are introduced via $i\partial_0 \to i\partial_0 + \mu_a$.
- Finite temperature T is introduced via $l^0 \to i\omega_n = \begin{cases} i\frac{(2n+1)\pi}{\beta}, & \text{fermion}, \\ i\frac{2n\pi}{\beta}, & \text{boson}, \end{cases}$ with $\beta = 1/(k_B T)$.

Challenges and Countermeasures

In relativistic QFTs at finite temperature and finite density,

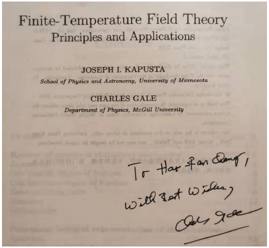
- No Lorentz invariance due to a rest reference frame of the many-body system in which the temperature and density are measured.
- The continuous spacetime symmetry is spatial SO(3) symmetry rather than spacetime SO$(1,3)$ symmetry.

- In line with the habits of high-energy theorists, we prefer to work with Lorentz-covariant tensors.
- The Lorentz-covariant tensors are incomplete to expand the one-loop tensor Feynman integrals.
- Introduce a constant vector $u^\rho = (1,0,0,0)$ in momentum space, whose spatial momentum is zero, to denote the rest reference frame.
- Construct a complete set of (symmetric) tensor structures by combining Lorentz-covariant tensors with this constant vector.

Hao-Ran Chang, Phys. Rev. D **110**, 016022 (2024)

The propagator, its inverse, and the self-energy, are all symmetric second-rank tensors. Assuming rotational invariance (which would not be correct for a solid) the most general tensor of this type is a linear combination of $g_{\mu\nu}$, $k_\mu k_\nu$, $u_\mu u_\nu$, and $k_\mu u_\nu + k_\nu u_\mu$. Here $u_\mu = (1, 0, 0, 0)$ specifies the rest frame of the many-body system. Taking into account

In the vacuum there is no preferred rest frame, so the vector u_μ cannot play any role (it is not defined). Also, in the vacuum $\Pi^{\mu\nu}$ must be proportional to $g^{\mu\nu} - k^\mu k^\nu / k^2$; hence $F = G$. Furthermore, G can only depend on k^2. At finite temperature and density, however, F and G can depend on $k^0 = u \cdot k$ and $|\mathbf{k}| = \sqrt{(u\cdot k)^2 - k^2}$ separately, owing to the lack of Lorentz invariance.



Step-1 in the GPVR (1)

To reduce tensor integrals to master integrals at zero temperature and finite density:

$$\mathscr{A}^{\rho}(p;m_1,\mu_1;\beta) = p^{\rho}\mathcal{A}_1 + u^{\rho}\mathcal{A}_2,$$

$$\mathscr{B}^{\rho}(p_1;m_1,\mu_1;m_2,\mu_2;\beta) = p_1^{\rho}\mathcal{B}_1 + u^{\rho}\mathcal{B}_2,$$

$$\mathscr{B}^{\rho\sigma}(p_1;m_1,\mu_1;m_2,\mu_2;\beta) = g^{\rho\sigma}\mathcal{B}_{00} + p_1^{\rho}p_1^{\sigma}\mathcal{B}_{11} + \left(p_1^{\rho}u^{\sigma} + p_1^{\sigma}u^{\rho}\right)\mathcal{B}_{12} + u^{\rho}u^{\sigma}\mathcal{B}_{22},$$

$$\mathscr{C}^{\rho}(p_1,p_2;m_1,\mu_1;m_2,\mu_2;m_3,\mu_3;\beta) = p_1^{\rho}\mathcal{C}_1 + p_2^{\rho}\mathcal{C}_2 + u^{\rho}\mathcal{C}_3,$$

$$\mathscr{C}^{\rho\sigma}(p_1,p_2;m_1,\mu_1;m_2,\mu_2;m_3,\mu_3;\beta) = g^{\rho\sigma}\mathcal{C}_{00} + p_1^{\rho}p_1^{\sigma}\mathcal{C}_{11} + \left(p_1^{\rho}p_2^{\sigma} + p_1^{\sigma}p_2^{\rho}\right)\mathcal{C}_{12} + p_2^{\rho}p_2^{\sigma}\mathcal{C}_{22} + \left(p_1^{\rho}u^{\sigma} + p_1^{\sigma}u^{\rho}\right)\mathcal{C}_{13} + \left(p_2^{\rho}u^{\sigma} + p_2^{\sigma}u^{\rho}\right)\mathcal{C}_{23} + u^{\rho}u^{\sigma}\mathcal{C}_{33},$$

Step-1 in the GPVR (2)

$$
\begin{aligned}
&\mathscr{C}^{\rho\sigma\tau}(p_1,p_2;m_1,\mu_1;m_2,\mu_2;m_3,\mu_3;\beta)\\
&= \left(p_1^\rho g^{\sigma\tau} + p_1^\sigma g^{\tau\rho} + p_1^\tau g^{\rho\sigma}\right)\mathcal{C}_{001}\\
&+ \left(p_2^\rho g^{\sigma\tau} + p_2^\sigma g^{\tau\rho} + p_2^\tau g^{\rho\sigma}\right)\mathcal{C}_{002}\\
&+ p_1^\rho p_1^\sigma p_1^\tau \mathcal{C}_{111} + \left(p_1^\rho p_1^\sigma p_2^\tau + p_1^\sigma p_1^\tau p_2^\rho + p_1^\tau p_1^\rho p_2^\sigma\right)\mathcal{C}_{112}\\
&+ \left(p_1^\rho p_2^\sigma p_2^\tau + p_1^\sigma p_2^\tau p_2^\rho + p_1^\tau p_2^\rho p_2^\sigma\right)\mathcal{C}_{122} + p_2^\rho p_2^\sigma p_2^\tau \mathcal{C}_{222}\\
&+ \left(u^\rho g^{\sigma\tau} + u^\sigma g^{\tau\rho} + u^\tau g^{\rho\sigma}\right)\mathcal{C}_{003}\\
&+ \left(p_1^\rho p_1^\sigma u^\tau + p_1^\sigma p_1^\tau u^\rho + p_1^\tau p_1^\rho u^\sigma\right)\mathcal{C}_{113}\\
&+ \left(p_2^\rho p_2^\sigma u^\tau + p_2^\sigma p_2^\tau u^\rho + p_2^\tau p_2^\rho u^\sigma\right)\mathcal{C}_{223}\\
&+ \left[p_1^\rho\left(p_2^\sigma u^\tau + p_2^\tau u^\sigma\right) + p_2^\rho\left(p_1^\sigma u^\tau + p_1^\tau u^\sigma\right) + u^\rho\left(p_1^\sigma p_2^\tau + p_1^\tau p_2^\sigma\right)\right]\mathcal{C}_{123}\\
&+ \left(p_1^\rho u^\sigma u^\tau + p_1^\sigma u^\tau u^\rho + p_1^\tau u^\rho u^\sigma\right)\mathcal{C}_{133}\\
&+ \left(p_2^\rho u^\sigma u^\tau + p_2^\sigma u^\rho u^\tau + p_2^\tau u^\rho u^\sigma\right)\mathcal{C}_{233}\\
&+ u^\rho u^\sigma u^\tau \mathcal{C}_{333},
\end{aligned}
$$

$$\cdots\cdots$$

The form factors can be expressed as a linear combination of master integrals as

$$
\begin{aligned}
&\Big\{\mathcal{A}_1,\mathcal{B}_1,\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{B}_{00},\mathcal{B}_{11},\mathcal{C}_{00},\mathcal{C}_{11},\mathcal{C}_{12},\mathcal{C}_{22},\cdots,\mathcal{C}_{001},\mathcal{C}_{002},\mathcal{C}_{111},\mathcal{C}_{112},\mathcal{C}_{122},\mathcal{C}_{222},\cdots,\\
&\mathcal{A}_2,\mathcal{B}_2,\mathcal{C}_3,\mathcal{B}_{12},\mathcal{B}_{22},\mathcal{C}_{13},\mathcal{C}_{23},\mathcal{C}_{33},\mathcal{C}_{003},\mathcal{C}_{113},\mathcal{C}_{223},\mathcal{C}_{123},\mathcal{C}_{133},\mathcal{C}_{233},\mathcal{C}_{333},\cdots\Big\}\\
&\sim\Big\{\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0,\cdots,\mathscr{B}^0,\mathscr{B}^{00},\mathscr{C}^0,\mathscr{C}^{00},\mathscr{C}^{000},\cdots\Big\}.
\end{aligned}
$$

Step-2 in the GPVR (1)

To calculate master integrals at finite temperature and finite density:

$$\mathcal{A}_0(p;m_1,\mu_1;\beta)=\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{(l^0+p^0+\mu_1)^2-\left[(\boldsymbol{l}+\boldsymbol{p})^2+m_1^2\right]},$$

$$\mathcal{B}_0(p_1;m_1,\mu_1;m_2,\mu_2;\beta)=\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{\left\{(l^0+\mu_1)^2-\left[\boldsymbol{l}^2+m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1)^2+m_2^2\right]\right\}},$$

$$\mathcal{C}_0(p_1,p_2;m_1,\mu_1;m_2,\mu_2;m_3,\mu_3;\beta)=\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{\left\{(l^0+\mu_1)^2-\left[\boldsymbol{l}^2+m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1)^2+m_2^2\right]\right\}}\times\frac{1}{\left\{(l^0+p_1^0+p_2^0+\mu_3)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1+\boldsymbol{p}_2)^2+m_3^2\right]\right\}},$$

$$\mathscr{B}^{\{0,00\}}(p_1;m_1,\mu_1;m_2,\mu_2;\beta)=\int\frac{d^Dl}{i\pi^{D/2}}\frac{\{l^0,\,l^0l^0\}}{\left\{(l^0+\mu_1)^2-\left[\boldsymbol{l}^2+m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1)^2+m_2^2\right]\right\}},$$

$$\mathscr{C}^{\{0,00,000\}}(p_1,p_2;m_1,\mu_1;m_2,\mu_2;m_3,\mu_3;\beta)=\int\frac{d^Dl}{i\pi^{D/2}}\frac{\{l^0,\,l^0l^0,\,l^0l^0l^0\}}{\left\{(l^0+\mu_1)^2-\left[\boldsymbol{l}^2+m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1)^2+m_2^2\right]\right\}}\times\frac{1}{\left\{(l^0+p_1^0+p_2^0+\mu_3)^2-\left[(\boldsymbol{l}+\boldsymbol{p}_1+\boldsymbol{p}_2)^2+m_3^2\right]\right\}}.$$

Step-2 in the GPVR (2)

To calculate these master integrals, one can work in the imaginary time formalism, where

$$l^0 \to i\omega_n = \begin{cases} i\frac{(2n+1)\pi}{\beta}, & \text{fermion}, \\ i\frac{2n\pi}{\beta}, & \text{boson}, \end{cases} \qquad \int_{-\infty}^{+\infty} \frac{dl^0}{2\pi} f(l^0) \to \frac{1}{\beta}\sum_{i\omega_n} f(i\omega_n),$$

for $n = 0, \pm 1, \pm, 2, \cdots, \pm\infty$ and $\beta = 1/(k_B T)$.

In relativistic QFTs at finite temperature and finite density,

Calculation of scalar master integrals

- P. Rehberg and S.P. Klevansky, and J. Hüfner, Phys. Rev. C **53**, 410 (1996), Hadronization in the SU(3) Nambu-Jona-Lasinio model. $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{C}_0$
- P. Rehberg and S.P. Klevansky, Annals of Phys, **252**, 422 (1996), One Loop Integrals at Finite Temperature and Density. $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{C}_0$
- A.S. Khvorostukhin, Acta Physica Polonica B, **52**, 1303 (2021), arXiv:2011.14596, Calculation of the one loop box integral at Finite Temperature and Density. $\mathcal{D}_0$

for fermionic internal lines.

Calculation of tensor master integrals

- Tensor master integrals ($\mathscr{B}^0, \mathscr{B}^{00}, \mathscr{C}^0, \mathscr{C}^{00}, \mathscr{C}^{000}, \cdots$) have not yet been analytically calculated

for fermionic and bosonic internal lines.

Take-home message of GPVR (1)

In relativistic QFTs at finite temperature and finite density, the one-loop Feynman diagrams

$$\Big\{-i\Sigma, i\Pi^{\mu\nu}, -ie\delta\Gamma^{\mu}, \cdots\Big\} \sim \Big\{\mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \cdots, \mathscr{A}^{\rho}, \mathscr{B}^{\rho}, \mathscr{B}^{\rho\sigma}, \mathscr{C}^{\rho}, \mathscr{C}^{\rho\sigma}, \mathscr{C}^{\rho\sigma\tau}, \cdots\Big\}.$$

Based on the residual symmetry [SO(3)], the non-Lorentz-covariant OLTFIs

$$\begin{aligned}
&\Big\{\mathscr{A}^{\rho}, \mathscr{B}^{\rho}, \mathscr{B}^{\rho\sigma}, \mathscr{C}^{\rho}, \mathscr{C}^{\rho\sigma}, \mathscr{C}^{\rho\sigma\tau}, \cdots\Big\} \\
&\sim \Big\{p^{\rho}, p_1^{\rho}, p_2^{\rho}, \cdots, g^{\rho\sigma}, p_1^{\rho}p_1^{\sigma}, (p_1^{\rho}p_2^{\sigma} + p_2^{\rho}p_1^{\sigma}), p_2^{\rho}p_2^{\sigma}, \cdots, \\
&\quad (p_1^{\rho}g^{\sigma\tau} + p_1^{\sigma}g^{\tau\rho} + p_1^{\tau}g^{\rho\sigma}), (p_2^{\rho}g^{\sigma\tau} + p_2^{\sigma}g^{\tau\rho} + p_2^{\tau}g^{\rho\sigma}), \\
&\quad p_1^{\rho}p_1^{\sigma}p_1^{\tau}, (p_1^{\rho}p_1^{\sigma}p_2^{\tau} + p_1^{\sigma}p_1^{\tau}p_2^{\rho} + p_1^{\tau}p_1^{\rho}p_2^{\sigma}), (p_1^{\rho}p_2^{\sigma}p_2^{\tau} + p_1^{\sigma}p_2^{\tau}p_2^{\rho} + p_1^{\tau}p_2^{\rho}p_2^{\sigma}), p_2^{\rho}p_2^{\sigma}p_2^{\tau}, \cdots, \\
&\quad u^{\rho}, (p_1^{\rho}u^{\sigma} + p_1^{\sigma}u^{\rho}), (p_2^{\rho}u^{\sigma} + p_2^{\sigma}u^{\rho}), \cdots, u^{\rho}u^{\sigma}, \cdots, \\
&\quad (u^{\rho}g^{\sigma\tau} + u^{\sigma}g^{\tau\rho} + u^{\tau}g^{\rho\sigma}), (p_1^{\rho}p_1^{\sigma}u^{\tau} + p_1^{\sigma}p_1^{\tau}u^{\rho} + p_1^{\tau}p_1^{\rho}u^{\sigma}), (p_2^{\rho}p_2^{\sigma}u^{\tau} + p_2^{\sigma}p_2^{\tau}u^{\rho} + p_2^{\tau}p_2^{\rho}u^{\sigma}), \\
&\quad (p_1^{\rho}u^{\sigma}u^{\tau} + p_1^{\sigma}u^{\tau}u^{\rho} + p_1^{\tau}u^{\rho}u^{\sigma})\,\big[p_1^{\rho}(p_2^{\sigma}u^{\tau} + p_2^{\tau}u^{\sigma}) + p_2^{\rho}(p_1^{\sigma}u^{\tau} + p_1^{\tau}u^{\sigma}) + u^{\rho}(p_1^{\sigma}p_2^{\tau} + p_1^{\tau}p_2^{\sigma})\big], \\
&\quad (p_1^{\rho}u^{\sigma}u^{\tau} + p_1^{\sigma}u^{\tau}u^{\rho} + p_1^{\tau}u^{\rho}u^{\sigma}), (p_2^{\rho}u^{\sigma}u^{\tau} + p_2^{\sigma}u^{\rho}u^{\tau} + p_2^{\tau}u^{\rho}u^{\sigma}), u^{\rho}u^{\sigma}u^{\tau}, \cdots\Big\} \\
&\times \Big\{\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{B}_{00}, \mathcal{B}_{11}, \mathcal{C}_{00}, \mathcal{C}_{11}, \mathcal{C}_{12}, \mathcal{C}_{22}, \cdots, \mathcal{C}_{001}, \mathcal{C}_{002}, \mathcal{C}_{111}, \mathcal{C}_{112}, \mathcal{C}_{122}, \mathcal{C}_{222}, \cdots, \\
&\quad \mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_3, \mathcal{B}_{12}, \mathcal{B}_{22}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{33}, \mathcal{C}_{003}, \mathcal{C}_{113}, \mathcal{C}_{223}, \mathcal{C}_{123}, \mathcal{C}_{133}, \mathcal{C}_{233}, \mathcal{C}_{333}, \cdots\Big\}.
\end{aligned}$$

Take-home message of GPVR (2)

The form factors are non-Lorentz-invariant, and can be expressed as a linear combination of OLSFIs and one-loop purely-temporal tensor Feynman integrals as

$$\Big\{\mathcal{A}_1,\mathcal{B}_1,\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{B}_{00},\mathcal{B}_{11},\mathcal{C}_{00},\mathcal{C}_{11},\mathcal{C}_{12},\mathcal{C}_{22},\cdots,\mathcal{C}_{001},\mathcal{C}_{002},\mathcal{C}_{111},\mathcal{C}_{112},\mathcal{C}_{122},\mathcal{C}_{222},\cdots,$$

$$\mathcal{A}_2,\mathcal{B}_2,\mathcal{C}_3,\mathcal{B}_{12},\mathcal{B}_{22},\mathcal{C}_{13},\mathcal{C}_{23},\mathcal{C}_{33},\mathcal{C}_{003},\mathcal{C}_{113},\mathcal{C}_{223},\mathcal{C}_{123},\mathcal{C}_{133},\mathcal{C}_{233},\mathcal{C}_{333},\cdots\Big\}$$

$$\sim\Big\{\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0,\cdots,\mathscr{B}^0,\mathscr{B}^{00},\mathscr{C}^0,\mathscr{C}^{00},\mathscr{C}^{000},\cdots\Big\}.$$

As a consequence, the one-loop Feynman diagrams

$$\Big\{-i\Sigma,i\Pi^{\mu\nu},-ie\delta\Gamma^{\mu},\cdots\Big\}\sim\Big\{\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0,\cdots,\mathscr{A}^{\rho},\mathscr{B}^{\rho},\mathscr{B}^{\rho\sigma},\mathscr{C}^{\rho},\mathscr{C}^{\rho\sigma},\mathscr{C}^{\rho\sigma\tau},\cdots\Big\}$$

$$\sim\Big\{\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0,\cdots\mathscr{B}^0,\mathscr{B}^{00},\mathscr{C}^0,\mathscr{C}^{00},\mathscr{C}^{000},\cdots\Big\}.$$

有限温度场论的“活字印刷”

A brief comparison between PVR and GPVR

RQFTs	At zero temperature and zero density	At finite temperature and finite density
EMs	PVR	GPVR
LS	with	without
OLSFIs	LI: $\tilde{\mathcal{A}}_0,\tilde{\mathcal{B}}_0,\tilde{\mathcal{C}}_0,\cdots$	Non-LI: $\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0,\cdots$
OLTFIs	LC: $\tilde{\mathscr{A}}^{\rho},\tilde{\mathscr{B}}^{\rho},\tilde{\mathscr{B}}^{\rho\sigma},\tilde{\mathscr{C}}^{\rho},\tilde{\mathscr{C}}^{\rho\sigma},\tilde{\mathscr{C}}^{\rho\sigma\tau},\cdots$	Non-LC: $\mathscr{A}^{\rho},\mathscr{B}^{\rho},\mathscr{B}^{\rho\sigma},\mathscr{C}^{\rho},\mathscr{C}^{\rho\sigma},\mathscr{C}^{\rho\sigma\tau},\cdots$
TSs	LC	Non-LC (u^{ρ}) and LC
MIs	LI: $\tilde{\mathcal{A}}_0,\tilde{\mathcal{B}}_0,\tilde{\mathcal{C}}_0,\cdots$	Non-LI: $\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0,\cdots,\mathscr{B}^0,\mathscr{B}^{00},\mathscr{C}^0,\mathscr{C}^{00},\mathscr{C}^{000},\cdots$

Relativistic QFTs (RQFTs), Efficient Methods (EMs), Lorentz symmetry (LS),
One-loop scalar Feynman integrals (OLSFIs), One-loop tensor Feynman integrals (OLTFIs),
Lorentz-invariant (LI), Lorentz-covariant (LC), Tensor structures (TSs), Master integrals (MIs).

Outline

1. Introduction and Motivation
2. From Passarino-Veltman reduction to Generalized Passarino-Veltman reduction
   - Passarino-Veltman reduction (PVR)
   - Gereralized Passarino-Veltman reduction (GPVR)
   - Two demonstration applications of GPVR
3. Summary and Outlook

The first demonstration application of GPVR

After decomposition and combination by hand, the one-loop pseudoscalar polarization function in the Nambu-Jona-Lasinio model can be expressed as

$$\begin{aligned}
&-i\Pi^{\mathrm{PS}}_{f\bar{f}'}(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta)\\
&=-N_c\int\frac{d^4l}{(2\pi)^4}\frac{\mathrm{tr}\left\{\gamma^5\left[l^\rho\gamma_\rho+\mu_f\gamma_0+m_f\right]\gamma^5\left[(l^\sigma-k^\sigma)\gamma_\sigma+\mu_{f'}\gamma_0+m_{f'}\right]\right\}}{\left[(l^0+\mu_f)^2-\boldsymbol{l}^2-m_f^2\right]\left[(l^0+k^0+\mu_{f'})^2-(\boldsymbol{l}-\boldsymbol{k})^2-m_{f'}^2\right]}\\
&=\frac{iN_c}{8\pi^2}\left\{\mathcal{A}_0\left(0;m_f,\mu_f;\beta\right)+\mathcal{A}_0\left(0;m_{f'},\mu_{f'};\beta\right)\right.\\
&\qquad\left.+\left[\left(m_f-m_{f'}\right)^2-\left(k^0+\mu_f-\mu_{f'}\right)^2-\boldsymbol{k}^2\right]\mathcal{B}_0\left(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta\right)\right\}. \qquad (1)
\end{aligned}$$

P. Rehberg and S.P. Klevansky, and J. Hüfner, Phys. Rev. C **53**, 410 (1996),
P. Rehberg and S.P. Klevansky, Annals of Phys, **252**, 422 (1996).

In the spirit of tensor reduction, the one-loop pseudoscalar polarization function can also be expressed as

$$\begin{aligned}
&-i\Pi^{\mathrm{PS}}_{f\bar{f}'}(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta)\\
&=\frac{4iN_c}{(4\pi)^2}\left\{g_{\rho\sigma}\left[\mathscr{B}^{\rho\sigma}\left(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta\right)-\mathscr{B}^{\rho}\left(k;m_f,\mu_f;m_{f'},\mu_{f'}\right)k^\sigma\right]\right.\\
&\qquad+g_{\rho0}\mathscr{B}^{\rho}\left(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta\right)\left(\mu_f+\mu_{f'}\right)-g_{\rho0}k^\rho\mathcal{B}_0\left(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta\right)\\
&\qquad\left.+\left(\mu_f\mu_{f'}-m_fm_{f'}\right)\mathcal{B}_0\left(k;m_f,\mu_f;m_{f'},\mu_{f'};\beta\right)\right\}. \qquad (2)
\end{aligned}$$

In terms of two one-loop tensor Feynman integrals $\mathscr{B}^{\rho}(q;m,\mu;m,\mu;\beta)$ and $\mathscr{B}^{\rho\sigma}(q;m,\mu;m,\mu;\beta)$, and a one-loop scalar Feynman integral $\mathcal{B}_0(q;m,\mu;m,\mu;\beta)$, the one-loop vacuum polarization in the D-dimensional QED can be recast as

$$\begin{aligned} i\Pi^{\lambda\tau}(q;m,\mu;m,\mu;\beta) &= -e^2\int\frac{d^Dl}{(2\pi)^D}\frac{\mathrm{tr}\left\{\gamma^\lambda\left[l^\rho\gamma_\rho+\mu\gamma_0+m\right]\gamma^\tau\left[(l^\sigma+q^\sigma)\gamma_\sigma+\mu\gamma_0+m\right]\right\}}{\left[(l^0+\mu)^2-\boldsymbol{l}^2-m^2\right]\left[(l^0+q^0+\mu)^2-(\boldsymbol{l}+\boldsymbol{q})^2-m^2\right]} \\ &= \frac{-4ie^2}{(4\pi)^{D/2}}\left\{\left[g^\lambda{}_\rho g^\tau{}_\sigma-g^{\lambda\tau}g_{\rho\sigma}+g^\lambda{}_\sigma g^\tau{}_\rho\right]\left[\mathscr{B}^{\rho\sigma}(q;m,\mu;m,\mu;\beta)+\mathscr{B}^{\rho}(q;m,\mu;m,\mu;\beta)q^\sigma\right]\right. \\ &\quad +\mu\left[g^\lambda{}_\rho g^\tau{}_0-g^{\lambda\tau}g_{\rho 0}+g^\lambda{}_0 g^\tau{}_\rho\right]\left[2\mathscr{B}^{\rho}(q;m,\mu;m,\mu;\beta)+q^\rho\mathcal{B}_0(q;m,\mu;m,\mu;\beta)\right] \\ &\quad \left.+\left[2\mu^2 g^\lambda{}_0 g^\tau{}_0+(m^2-\mu^2)g^{\lambda\tau}\right]\mathcal{B}_0(q;m,\mu;m,\mu;\beta)\right\}. \end{aligned}$$

The analytical result after utilizing GPVR automatically satisfies the Ward identity

$$q_\rho\Pi^{\rho\sigma}(q;m,\mu;m,\mu;\beta)=0. \tag{3}$$

The second demonstration application of GPVR (2)

From explicit analytical expressions of $\Pi^{\rho\sigma}(q^0, |\boldsymbol{q}|, \mu)$ at zero temperature and finite density in Phys. Rev. B **97**, 075202 (2018),

- $\Pi^{00}(q^0, |\boldsymbol{q}|, \mu)$:

  Plasmon: Jianhui Zhou, Hao-Ran Chang, and Di Xiao, Phys. Rev. B **91**, 035114 (2015).

- $\Pi^{ij}(q^0, |\boldsymbol{q}|, \mu)$:

  Diagonal elements: $\delta_{ij}\Pi^{ij}(q^0, |\boldsymbol{q}|, \mu)$

  Plasmon and Optical conductivity: A. Thakur, K. Sadhukhan, and A. Agarwal, Phys. Rev. B **97**, 035403 (2018).

  Optical conductivity: Phillip E. C. Ashby and J. P. Carbotte, Phys. Rev. B **89**, 245121 (2014).

  Asymmetric part of off-diagonal elements: $\varepsilon_{ijk}\Pi^{ij}(q^0, |\boldsymbol{q}|, \mu)$

  Chiral magnetic conductivity: D.E. Kharzeev and H.J. Warringa, Phys. Rev. D **80**, 034028 (2009).

- $\Pi^{0j}(q^0, |\boldsymbol{q}|, \mu)$ and $\Pi^{j0}(q^0, |\boldsymbol{q}|, \mu)$:

  S. Ghosh and C. Timm, Phys. Rev. B **99**, 075104 (2019).

The second demonstration application of GPVR (3)

From explicit analytical expressions of $\Pi^{\rho\sigma}(q^0, |\boldsymbol{q}|, \mu)$ at zero temperature and finite density in Phys. Rev. B **97**, 075202 (2018),

- $\Pi^{\rho\sigma}(q^0, |\boldsymbol{q}|, \mu)$ under the hard dense loop approximation (where $q^0, |\boldsymbol{q}| \ll \mu$) :

  Dam Thanh Son (譚青山) and Naoki Yamamoto, Phys. Rev. D **87**, 085016 (2013).

- Chiral Plasma Instabilities:

  Yukinao Akamatsu and Naoki Yamamoto, PRL **111**, 052002 (2013).

Summary and Remark

- The most significant in the reduction for one-loop tensor Feynman integrals is to construct a complete set of tensor structures based on continuous spacetime symmetry.
- In relativistic QFTs at zero temperature and zero density, the Lorentz symmetry is respected. Consequently, the tensor structures are Lorentz covariant and the master integrals are Lorentz invariant.
- In relativistic QFTs at finite temperature and finite density, the Lorentz symmetry is broken. Consequently, the Lorentz-covariant tensor structures are incomplete and the master integrals are not Lorentz-invariant any longer.
- A complete set of (symmetric) tensor structures can be constructed by combining the constant vector $u^\rho = (1, 0, \cdots, 0)$ with Lorentz-covariant tensors, where u^ρ in D-dimensional spacetime denotes the rest reference frame due to finite temperature and finite density.
- The reason for introducing this extra constant vector $u^\rho = (1, 0, \cdots, 0)$ to the Lorentz-covariant tensors here is similar to that for imposing a gauge condition to the eletromagnetic field A_ρ.
- At finite temperature and finite density, the purely-temporal tensor master integrals $\mathscr{B}^0, \mathscr{B}^{00}, \mathscr{C}^0, \mathscr{C}^{00}, \mathscr{C}^{000}, \cdots$ must be introduced, and all the master integrals are non-Lorentz invariant.
- GPVR goes back to PVR after artificially removing terms containing u^ρ and simultaneously setting $\mu_a = 0$ and $T = 0$. The constant vector u^ρ is not defined at zero temperature and zero density. According to the third law of thermodynamics zero temperature is qualitatively different from finite temperature.
- Both GPVR and PVR are valid for tensor Feynman integrals in relativistic QFTs at finite temperature and finite density when $D \geq (1 + 1)$.

Outlook

- Reduction up to N-point one-loop tensor Feynman integrals.
- Generalization to multi-loop tensor Feynman integrals
  "Auxiliary mass flow method" (Yan-Qing Ma).
- Generalization to more efficient reduction than GPVR:
  Incorporating Effects of finite temperature and finite density with
  "Improved PV-reduction method with auxiliary vector" (Bo Feng).
- Generalization to other energy-momentum relations:
  Pseudo-relativistic QFTs: $E(\boldsymbol{p}) = \pm\sqrt{\boldsymbol{p}^2 v_a^2 + \Delta_a^2}$ V.S. $E(\boldsymbol{p}) = \pm\sqrt{\boldsymbol{p}^2 c^2 + m^2 c^4}$
  Non-relativistic QFTs: $E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m}$ (V. Shtabovenko).
- Generalization to nonequilibrium processes.
- Generalization to AdS/dS spacetime (Bo Feng).
- Computer program packages for automatic algebraic calculation.
  Calculation of scalar master integrals for purely boson internal lines:
  $\mathcal{A}_0^{(f/b)}, \mathcal{B}_0^{(f/b)}, \mathcal{C}_0^{(f/b)}, \mathcal{D}_0^{(f/b)}, \cdots$
  Calculation of tensor master integrals for purely fermion/boson internal lines:
  $\mathscr{A}_{(f/b)}^{0}, \mathscr{B}_{(f/b)}^{0;00}, \mathscr{C}_{(f/b)}^{0;00;000}, \mathscr{D}_{(f/b)}^{0;00;000;0000}, \cdots$

Thanks for your attention !

Contribution of finite temperature and finite density

Let us evaluate the photon self-energy at the one-loop level. From (5.40),

$$\Pi^{\mu\nu} = e^2 T \sum_l \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}\left(\gamma^\nu \frac{1}{\not p - m} \gamma^\mu \frac{1}{\not p + \not k - m}\right) \tag{5.49}$$

Here $p^0 = (2l+1)\pi Ti + \mu$ and $k^0 = 2n\pi Ti$. We can always write $\Pi^{\mu\nu} = \Pi^{\mu\nu}_{\text{vac}} + \Pi^{\mu\nu}_{\text{mat}}$, where

$$\Pi^{\mu\nu}_{\text{vac}} = \lim_{\substack{T\to 0\\ \mu\to 0}} \Pi^{\mu\nu} \tag{5.50}$$

is the vacuum self-energy and $\Pi^{\mu\nu}_{\text{mat}}$ is the remainder due to the presence of matter. The vacuum part is discussed in many textbooks on field theory, such as Peskin and Schroeder [2]. The matter part is readily evaluated:

$$\Pi^{00}_{\text{mat}} = -\frac{e^2}{\pi^2} \operatorname{Re} \int_0^\infty \frac{dp\, p^2}{E_p} N_{\mathrm{F}}(p) \left[1 + \frac{4E_p k^0 - 4E_{\mathrm{P}}^2 - k^2}{4p\omega} \ln\left(\frac{R_+}{R_-}\right)\right]$$

$$\Pi^{\mu}_{\text{mat}\,\mu} = -2\frac{e^2}{\pi^2} \operatorname{Re} \int_0^\infty \frac{dp\, p^2}{E_p} N_{\mathrm{F}}(p) \left[1 - \frac{2m^2 + k^2}{4p\omega} \ln\left(\frac{R_+}{R_-}\right)\right] \tag{5.51}$$

Here

$$\omega = |\mathbf{k}| \qquad k^2 = k_0^2 - \omega^2 \qquad E_p = \sqrt{\mathbf{p}^2 + m^2}$$

$$N_{\mathrm{F}}(p) = \frac{1}{\mathrm{e}^{\beta(E_p-\mu)} + 1} + \frac{1}{\mathrm{e}^{\beta(E_p+\mu)} + 1}$$

$$R_\pm = k^2 - 2k_0 E_p \pm 2p\omega$$

Josehp I. KAPUSTA and Charles GALE, Finite-temperature field theory: Principles and Applications.

Reducing form factors to master integrals at zero temperature and zero density

After integration over l, Lorentz tensor index ρ of l^ρ must be inherited by a complete set of rank-one Lorentz tensor. In the present case, the Lorentz tensor has no choice but to be p_1^ρ. As a result,

$$\int \frac{d^D l}{i\pi^{D/2}} \frac{l^\rho}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]} = \tilde{\mathscr{B}}^\rho(p_1; m_1; m_2) = p_1^\rho \tilde{\mathcal{B}}_1(p_1; m_1; m_2). \tag{4}$$

Contracting $p_{1\rho}$ with the left-handed side of $\tilde{\mathscr{B}}^\rho(p_1; m_1; m_2)$ gives rise to

$$\begin{aligned} p_{1\rho}\tilde{\mathscr{B}}^\rho(p_1; m_1; m_2) &= \int \frac{d^D l}{i\pi^{D/2}} \frac{p_1 \cdot l}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]} \\ &= \int \frac{d^D l}{i\pi^{D/2}} \frac{\frac{1}{2}\left\{\left[(l+p_1)^2 - m_2^2\right] - \left[l^2 - m_1^2\right] + (m_2^2 - m_1^2 - p_1^2)\right\}}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]} \\ &= \frac{\tilde{\mathcal{A}}_0(0; m_1)}{2} - \frac{\tilde{\mathcal{A}}_0(0; m_2)}{2} + \frac{(m_2^2 - m_1^2 - p_1^2)}{2}\tilde{\mathcal{B}}_0(p_1; m_1; m_2). \end{aligned} \tag{5}$$

Contracting $p_{1\rho}$ with the right-handed side of $\tilde{\mathscr{B}}^\rho(p_1; m_1; m_2)$ gives rise to

$$p_{1\rho}\tilde{\mathscr{B}}^\rho(p_1; m_1; m_2) = p_{1\rho}p_1^\rho \tilde{\mathcal{B}}_1(p_1; m_1; m_2) = p_1^2 \tilde{\mathcal{B}}_1(p_1; m_1; m_2). \tag{6}$$

From the above two equations, the form factor $\tilde{\mathcal{B}}_1$ can be expressed in terms of $\tilde{\mathcal{A}}_0$ and $\tilde{\mathcal{B}}_0$ as

$$\tilde{\mathcal{B}}_1(p_1; m_1; m_2) = \left[\frac{\tilde{\mathcal{A}}_0(0; m_1) - \tilde{\mathcal{A}}_0(0; m_2)}{2p_1^2} + \frac{(m_2^2 - m_1^2 - p_1^2)}{2p_1^2}\tilde{\mathcal{B}}_0(p_1; m_1; m_2)\right] \sim \{\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0\}. \tag{7}$$

Reducing form factors to master integrals at finite temperature and finite density

After integration over l, Lorentz tensor index ρ of l^ρ must be inherited by a complete set of rank-one Lorentz tensor. In the present case, the tensor structures have no choice but to be p_1^ρ and u^ρ. As a result,

$$\begin{aligned}
&\mathscr{B}^\rho(p_1; m_1, \mu_1; m_2, \mu_2; \beta) = p_1^\rho \mathcal{B}_1(p_1; m_1, \mu_1; m_2, \mu_2; \beta) + u^\rho \mathcal{B}_2(p_1; m_1, \mu_1; m_2, \mu_2; \beta) \\
&= \int \frac{d^D l}{i\pi^{D/2}} \frac{l^\rho}{\left\{(l^0+\mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2 - \left[(\boldsymbol{l}+\boldsymbol{p}_1)^2 + m_2^2\right]\right\}}.
\end{aligned} \tag{8}$$

Contracting $p_{1\rho}$ and u_ρ with both sides of $\mathscr{B}^\rho(p_1; m_1, \mu_1; m_2, \mu_2; \beta)$ gives rise to

$$\begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot u \\ u \cdot p_1 & u \cdot u \end{pmatrix} \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} = \begin{pmatrix} p_{1\rho}\mathscr{B}^\rho \\ u_\rho \mathscr{B}^\rho \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
\mathcal{F}_1 \equiv p_{1\rho}\mathscr{B}^\rho &= \frac{\mathcal{A}_0(0; m_1, \mu_1; \beta) - \mathcal{A}_0(0; m_2, \mu_2; \beta)}{2} - (\mu_2 - \mu_1)\mathscr{B}^0(p_1; m_1, \mu_1; m_2, \mu_2; \beta) \\
&+ \frac{m_2^2 - (m_1^2 - {\mu_1}^2) - \left[(\mu_2 + p_1^0)^2 - \boldsymbol{p}_1^2\right]}{2} \mathcal{B}_0(p_1; m_1, \mu_1; m_2, \mu_2; \beta), \\
\mathcal{F}_2 \equiv u_\rho \mathscr{B}^\rho &= \mathscr{B}^0(p_1; m_1, \mu_1; m_2, \mu_2; \beta).
\end{aligned} \tag{10}$$

The form factors can be expressed as a combination of $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathscr{B}^0$, namely,

$$\begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} \sim \left\{\mathcal{A}_0, \mathcal{B}_0, \mathscr{B}^0\right\}. \tag{11}$$

At finite temperature and finite density,

$$\mathscr{B}^{\rho}(p_1;m_1,\mu_1;m_2,\mu_2;\beta) = p_1^{\rho}\mathcal{B}_1(p_1;m_1,\mu_1;m_2,\mu_2;\beta) + u^{\rho}\mathcal{B}_2(p_1;m_1,\mu_1;m_2,\mu_2;\beta)$$

$$= \int \frac{d^D l}{i\pi^{D/2}} \frac{l^{\rho}}{\left\{(l^0+\mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0+p_1^0+\mu_2)^2 - \left[(\boldsymbol{l}+\boldsymbol{p}_1)^2 + m_2^2\right]\right\}}.$$

$$\begin{pmatrix} p_1\cdot p_1 & p_1\cdot u \\ u\cdot p_1 & u\cdot u \end{pmatrix}\begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} = \begin{pmatrix} p_{1\rho}\mathscr{B}^{\rho} \\ u_{\rho}\mathscr{B}^{\rho} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix},$$

where

$$\mathcal{F}_1 \equiv p_{1\rho}\mathscr{B}^{\rho} = \frac{\mathcal{A}_0(0;m_1,\mu_1;\beta) - \mathcal{A}_0(0;m_2,\mu_2;\beta)}{2} - (\mu_2-\mu_1)\mathscr{B}^0(p_1;m_1,\mu_1;m_2,\mu_2;\beta)$$
$$+ \frac{m_2^2 - (m_1^2 - {\mu_1}^2) - \left[(\mu_2+p_1^0)^2 - \boldsymbol{p}_1^2\right]}{2}\mathcal{B}_0(p_1;m_1,\mu_1;m_2,\mu_2;\beta),$$
$$\mathcal{F}_2 \equiv u_{\rho}\mathscr{B}^{\rho} = \mathscr{B}^0(p_1;m_1,\mu_1;m_2,\mu_2;\beta).$$

Directly setting $\mu_a = 0$, $\beta = \infty$ and artificially removing terms containing u_{ρ} can give rise to the expression at zero temperature and zero density,

$$\tilde{\mathscr{B}}^{\rho}(p_1;m_1;m_2) = p_1^{\rho}\tilde{\mathcal{B}}_1(p_1;m_1;m_2) = \int \frac{d^D l}{i\pi^{D/2}} \frac{l^{\rho}}{\left[l^2 - m_1^2\right]\left[(l+p_1)^2 - m_2^2\right]},$$

where

$$\tilde{\mathcal{B}}_1(p_1;m_1;m_2) = \frac{\tilde{\mathcal{A}}_0(0;m_1) - \tilde{\mathcal{A}}_0(0;m_2)}{2p_1^2} + \frac{m_2^2 - m_1^2 - p_1^2}{2p_1^2}\tilde{\mathcal{B}}_0(p_1;m_1;m_2).$$

Alternative GPVR (1)

In relativistic QFTs at finite temperature and finite density,

- The continuous spacetime symmetry is spatial SO$(D-1)$ symmetry rather than spacetime SO$(1, D-1)$ symmmtry.
- Only the reduction for the spatial component(s) of one-loop tensor Feynman integrals are needed.
- The Lorentz-covariant tensors are over-complete to expand the one-loop tensor Feynman integrals.
- A complete set of tensor structures can be constructed based on SO$(D-1)$ symmetry.
- Treat the spatial components in the FIRST GPVR just as the temporal-spatial components in the PVR.

Jianhui Zhou and Hao-Ran Chang, Phys. Rev. B **97**, 075202 (2018)

Alternative GPVR (2)

Based on the SO$(D-1)$ symmetry, a complete set of (symmetric) SO$(D-1)$-covariant tensors are constructed to expand the one-loop spatial tensor Feynman integrals.

$$\begin{aligned}
&\mathscr{A}^i(p; m_1, \mu_1; \beta) \\
&= \int \frac{d^D l}{i\pi^{D/2}} \frac{l^i}{(l^0 + p^0 + \mu_1)^2 - \left[(\boldsymbol{l} + \boldsymbol{p})^2 + m_1^2\right]} \\
&= -p^i \mathcal{A}_0(0; m_1, \mu_1; \beta), \\
&\mathscr{B}^i(p_1; m_1, \mu_1; m_2, \mu_2; \beta) \\
&= \int \frac{d^D l}{i\pi^{D/2}} \frac{l^i}{\left\{(l^0 + \mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0 + p_1^0 + \mu_2)^2 - \left[(\boldsymbol{l} + \boldsymbol{p}_1)^2 + m_2^2\right]\right\}} \\
&= p_1^i \mathcal{B}_1(p_1; m_1, \mu_1; m_2, \mu_2; \beta), \\
&\mathscr{B}^{0i}(p_1; m_1, \mu_1; m_2, \mu_2; \beta) = \mathscr{B}^{i0}(p_1; m_1, \mu_1; m_2, \mu_2; \beta) \\
&= \int \frac{d^D l}{i\pi^{D/2}} \frac{l^0 l^i}{\left\{(l^0 + \mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0 + p_1^0 + \mu_2)^2 - \left[(\boldsymbol{l} + \boldsymbol{p}_1)^2 + m_2^2\right]\right\}} \\
&= p_1^i \mathcal{B}_{01}(p_1; m_1, \mu_1; m_2, \mu_2; \beta), \\
&\mathscr{B}^{ij}(p_1; m_1, \mu_1; m_2, \mu_2; \beta) \\
&= \int \frac{d^D l}{i\pi^{D/2}} \frac{l^i l^j}{\left\{(l^0 + \mu_1)^2 - \left[\boldsymbol{l}^2 + m_1^2\right]\right\}\left\{(l^0 + p_1^0 + \mu_2)^2 - \left[(\boldsymbol{l} + \boldsymbol{p}_1)^2 + m_2^2\right]\right\}} \\
&= -\delta^{ij} \mathcal{B}_{00}(p_1; m_1, \mu_1; m_2, \mu_2; \beta) + p_1^i p_1^j \mathcal{B}_{11}(p_1; m_1, \mu_1; m_2, \mu_2; \beta),
\end{aligned}$$

Alternative GPVR (3)

$$\mathscr{C}^{i}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta),$$

$$\mathscr{C}^{0i}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta) = \mathscr{C}^{i0}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta),$$

$$\mathscr{C}^{ij}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta),$$

$$\begin{aligned} \mathscr{C}^{00i}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta &= \mathscr{C}^{0i0}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta) \\ &= \mathscr{C}^{i00}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta), \end{aligned}$$

$$\begin{aligned} \mathscr{C}^{0ij}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta) &= \mathscr{C}^{i0j}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta) \\ &= \mathscr{C}^{ij0}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta), \end{aligned}$$

$$\mathscr{C}^{ijk}(p_1, p_2; m_1, \mu_1; m_2, \mu_2; m_3, \mu_3; \beta),$$

$$\cdots\cdots$$

- Too many generic one-loop tensor Feynman integrals.
- Not in line with the habits of high-energy theorists.