Effective range expansion with the long-range force

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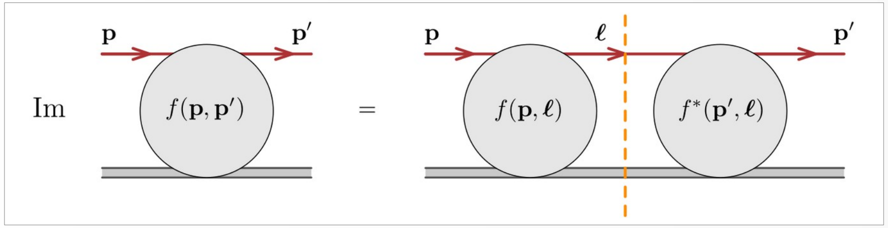
Based on: PRL 131, 131903 (2023)
& arXiv:2408.09375 [hep-ph]

Effective Range Expansion (ERE)

Unitarity: $SS^\dagger = 1, \quad \text{and} \quad S^\dagger S = 1.$ $S = 1 + iT$



$$-i(T - T^\dagger) = T^\dagger T$$

$$\mathrm{Im}\frac{1}{T_\ell} = -\rho(s)$$

$$2\mathrm{Im}\langle f|T|i\rangle = \sum_n \int \Big[\prod_{i=1}^{i_n} \frac{d^3 p_i}{2p_i^0 (2\pi)^3}\Big] \langle f|T^\dagger|n\rangle\langle n|T|i\rangle,$$

In general $$T_\ell = \frac{e^{i\delta_\ell}\sin\delta_\ell}{\rho} = \frac{1}{\rho\cot\delta_\ell - i\rho} = \frac{8\pi\sqrt{s}}{p\cot\delta_\ell - ip}$$

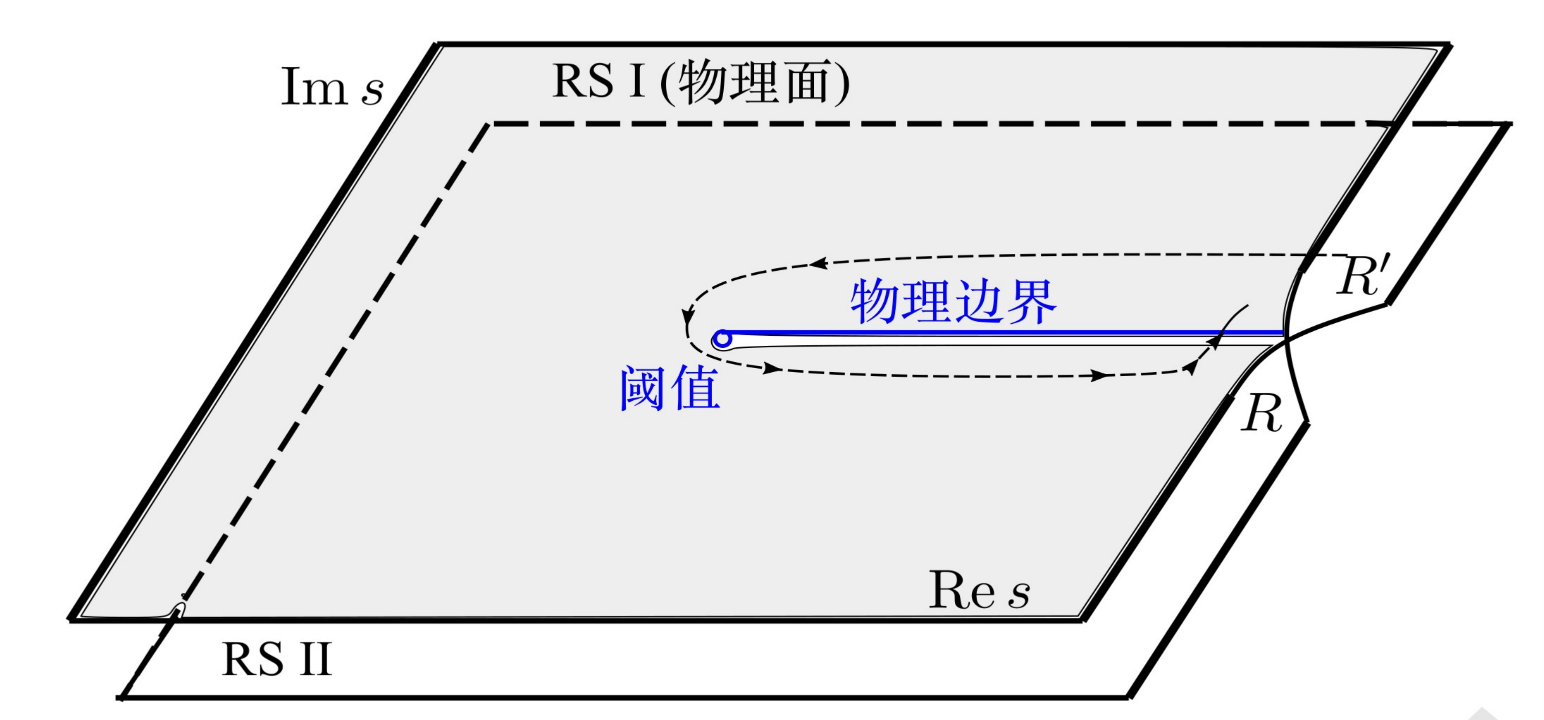
ERE

$$p\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2} r_0 p^2 + \mathcal{O}(p^4)$$

K-matrix

$$\frac{1}{T} = K^{-1} - i\rho \qquad K = \frac{\gamma}{s - s_0} + \alpha$$

Riemann sheets

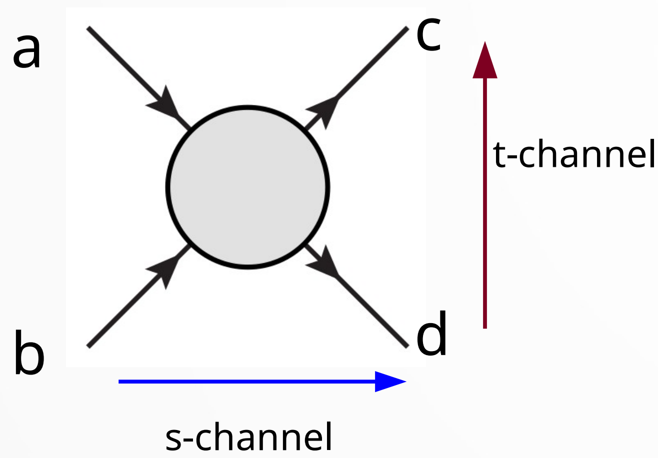


RS-I: $\mathrm{Im}\frac{1}{T_\ell} = -\rho(s)$

RS-II: $\frac{1}{T_\ell^{\mathrm{II}}} = \frac{1}{T_\ell} + 2i\rho$

$$\rho(s) = \frac{|\mathbf{p}|}{8\pi\sqrt{s}}\theta(\sqrt{s}-2m) = \frac{1}{16\pi}\sqrt{1-4m^2/s}$$

Crossing Symmetry & Left-Hand Cut (LHC)

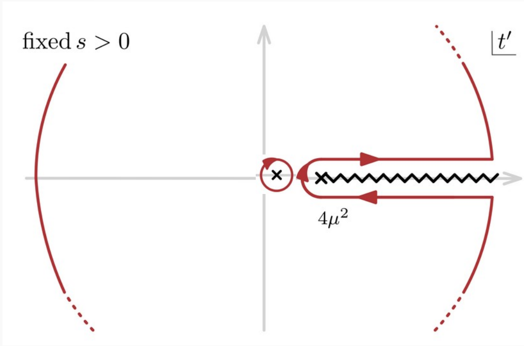


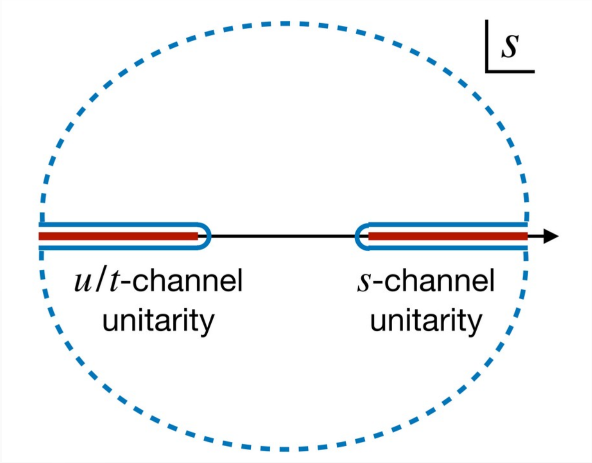
$T(s,t),\ T(s,\cos\theta)$

s-channel	$a(p_1)+b(p_2)\to c(p_3)+d(p_4),$
t-channel	$a(p_1)+\bar{c}(-p_3)\to \bar{b}(-p_2)+d(p_4),$
u-channel	$a(p_1)+\bar{d}(-p_4)\to c(p_3)+\bar{b}(-p_2).$

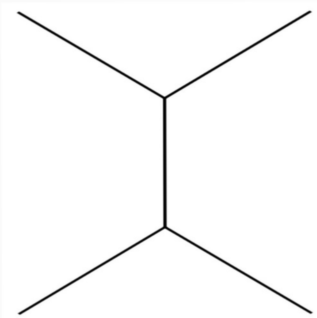
$$T_\ell(s)=\frac{1}{2}\int_{-1}^{1}P_\ell(\cos\theta)T(s,\cos\theta)d\cos\theta \propto \int dt P_\ell(\cos\theta(t))T(s,t)$$

t-channel unitarity





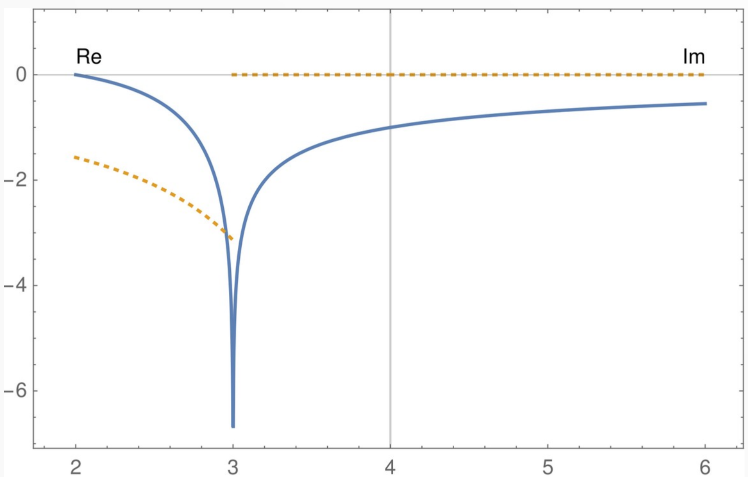
LHC from the one-particle exchange (OPE)

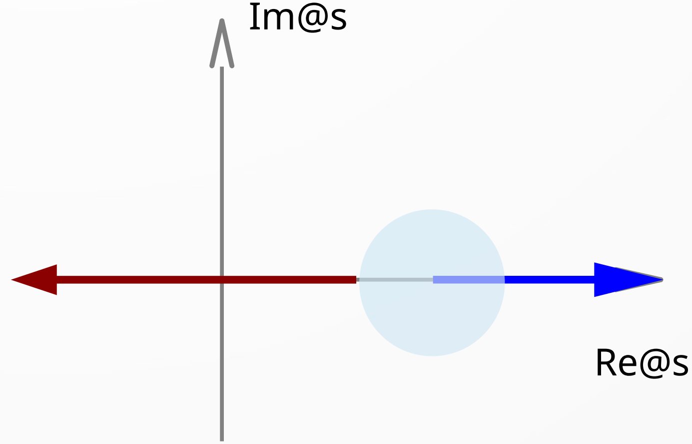


$$t = -2\mathbf{p}^2(1-\cos\theta)$$

S-wave 分波

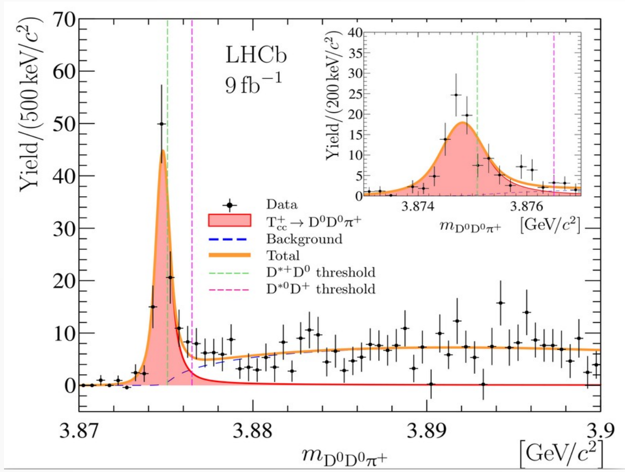
$$\frac{1}{2}\int_{-1}^{+1} d\cos\theta \frac{1}{-2\mathbf{p}^2(1-\cos\theta)-m_\rho^2+i\varepsilon} = -\frac{1}{4\mathbf{p}^2}\log\left(\frac{4\mathbf{p}^2+m_\rho^2}{m_\rho^2}+\frac{4\mathbf{p}^2}{m_\rho^4}i\varepsilon\right)$$

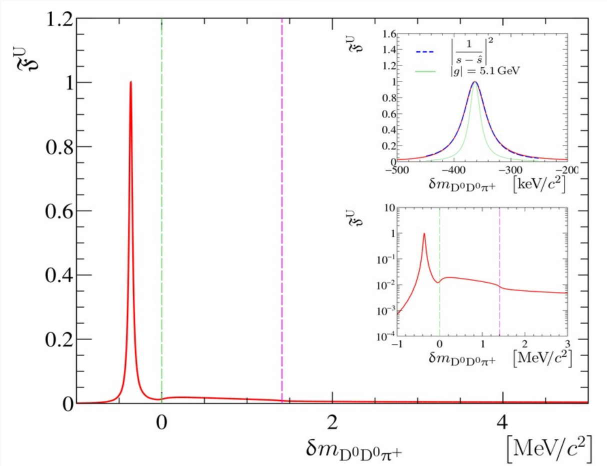




$$k\cot\delta = \frac{1}{a}+\frac{1}{2}rk^2+\mathcal{O}(k^4)$$

Doubly Charm Tetraquark (Tcc)





Breit-Wigner fit

LHCb, Nature Phys. 18, (2022) 751

Parameter	Value
N	117 ± 16
$\delta m_{\rm BW}$	-273 ± 61 keV
$\Gamma_{\rm BW}$	410 ± 165 keV

☞ $\mathfrak{R} \sim 400$ keV.

Unitarized and analytical

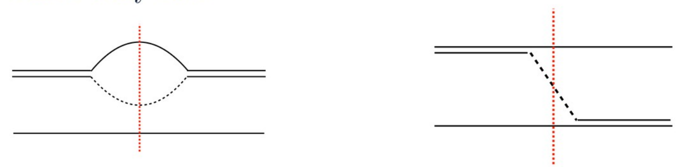
LHCb, Nature Commun. 13 (2022), 3351

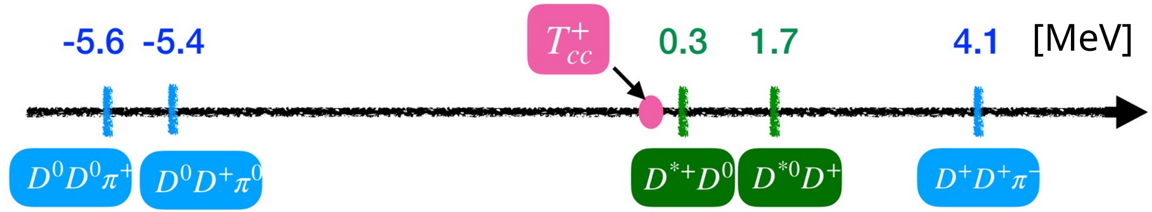
$\delta m = m_{T_{cc}^+} - m_{D^{*+}} - m_{D^0}$

$$\delta m_{\rm pole} = -360 \pm 40^{+4}_{-0} \text{ keV}$$

$$\Gamma_{\rm pole} = 48 \pm 2^{+0}_{-14} \text{ keV}$$

☞ Three-body cuts

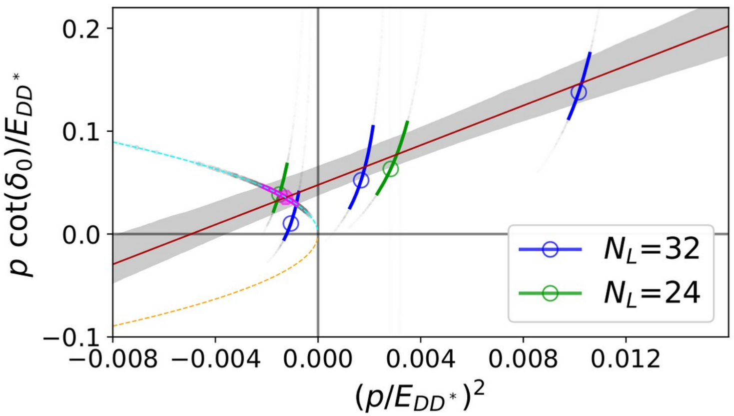




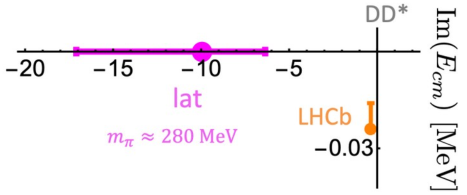
Doubly Charm Tetraquark on the Lattice

Padmanath *et al*, PRL129,032002(2022)

	m_D (MeV)	m_{D^*} (MeV)	M_{av} (MeV)	$a_{l=0}^{(J=1)}$ (fm)	$r_{l=0}^{(J=1)}$ (fm)	$\delta m_{T_{cc}}$ (MeV)	T_{cc}
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(h)}$)	1927(1)	2049(2)	3103(3)	1.04(29)	$0.96(^{+0.18}_{-0.20})$	$-9.9^{+3.6}_{-7.2}$	Virtual bound st.
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(l)}$)	1762(1)	1898(2)	2820(3)	0.86(0.22)	$0.92(^{+0.17}_{-0.19})$	$-15.0(^{+4.6}_{-9.3})$	Virtual bound st.
Experiment [2,41]	1864.85(5)	2010.26(5)	3068.6(1)	−7.15(51)	[−11.9(16.9), 0]	−0.36(4)	Bound st.

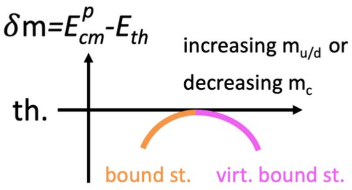


$$\delta m_{T_{cc}} = \mathrm{Re}(E_{cm}) - m_{D^0} - m_{D^{*+}} \ [\mathrm{MeV}]$$

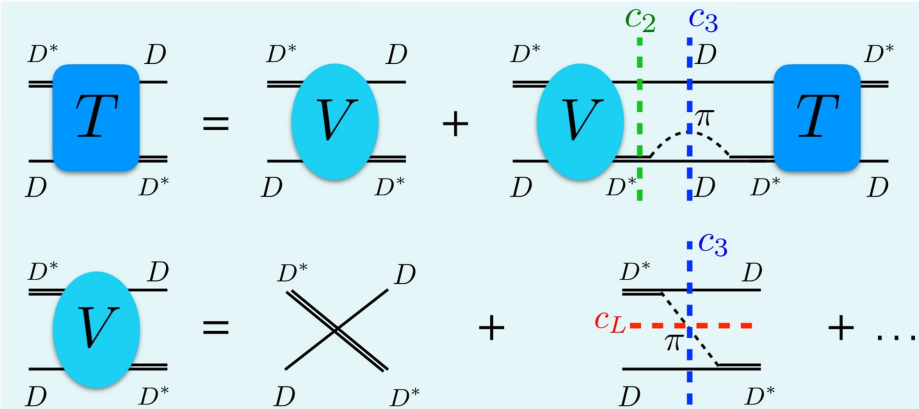


$$t = \frac{E_{\mathrm{cm}}}{2}\frac{1}{p\cot\delta - ip},$$

$$p\cot\delta = \frac{1}{a_0} + \frac{1}{2}r_0 p^2,$$

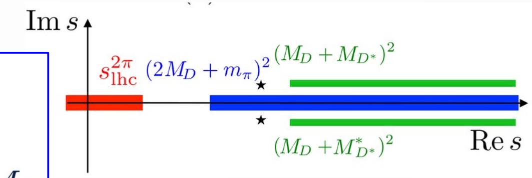


Three-body cut vs Left-hand cut



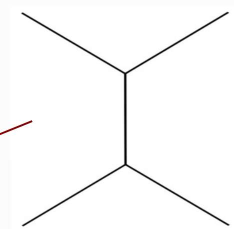
☞ three-body cut

$E > M_D + M_D + M_\pi$

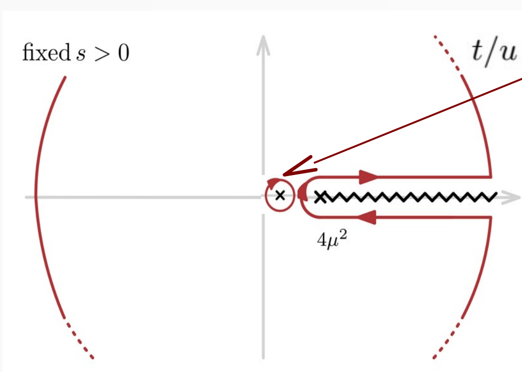


☞ left-hand cut

$$\int_{-1}^{1} d\cos\theta\, G_\pi(E,p,p)$$

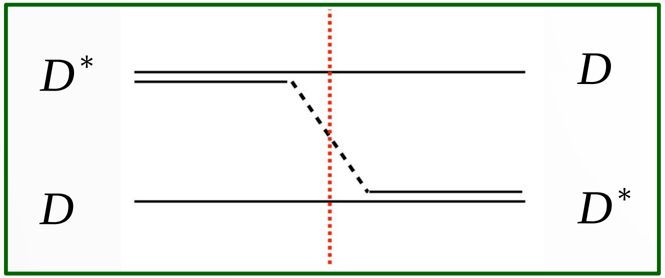


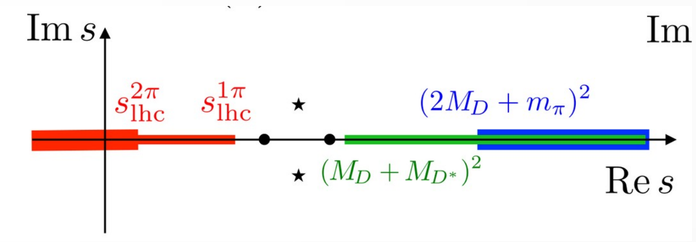
t/*u*-channel unitarity

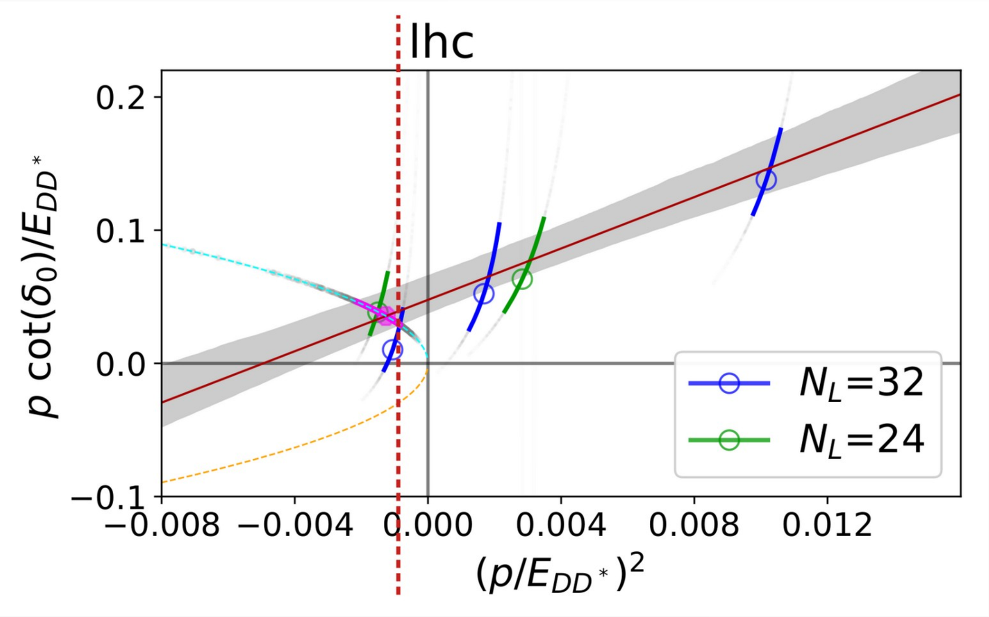


$$G_\pi^{-1}(E,\mathrm{k},\mathrm{k}') \xrightarrow[\text{on shell: } k=k'=p]{\cos\theta=\pm1} E_{D*}(p^2) - E_D(p^2) - \omega_\pi(4p^2/0) = 0$$

The left-hand cut







$m_\pi = 280$ MeV

☞ two-body branch point:

$E = M_D + M_{D^*}$

$\Longrightarrow p^2_{\rm rhc_2} = 0$

☞ three-body branch point:

$E = M_D + M_D + m_\pi$

$\Longrightarrow \left(\frac{p_{\rm rhc_3}}{E_{DD^*}}\right)^2 = +0.019$

☞ left-hand cut branch point:

$\Longrightarrow \left(\frac{p^{1\pi}_{\rm lhc}}{E_{DD^*}}\right)^2 = -0.001$

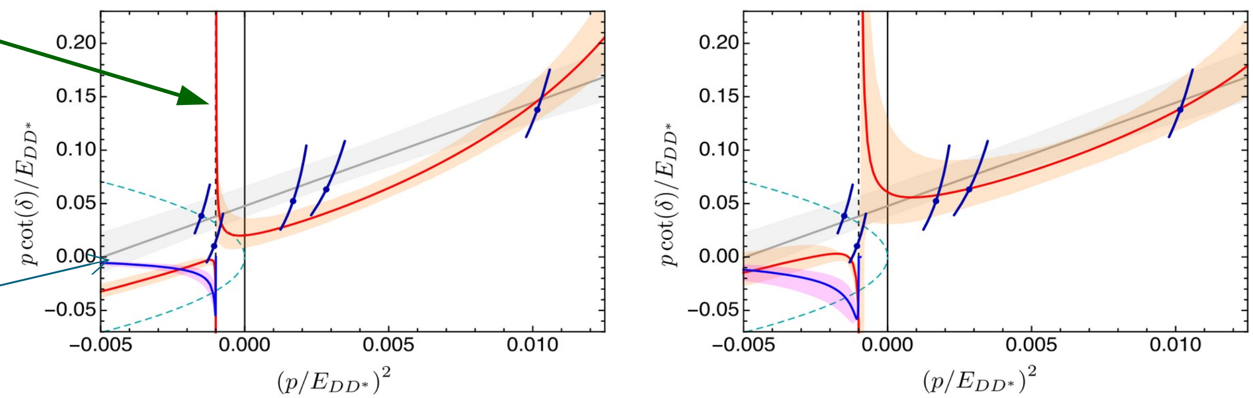
$\left(\frac{\tilde{p}^{1\pi}_{\rm lhc}}{E_{DD^*}}\right)^2 = -0.190$

Phase shift with the LHC: Solving Lippmann-Schwinger Equation

Limit ERE

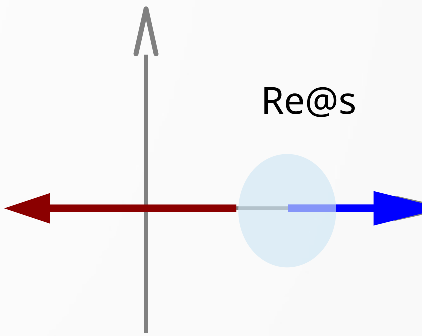
$$p\cot\delta = -\frac{2\pi}{\mu}\frac{1}{T} + ik$$

$M_D = 1927$ MeV, $M_{D^*} = 2049$ MeV, $m_\pi = 280$ MeV



Traditional ERE

$$k\cot\delta = \frac{1}{a} + \frac{1}{2}rk^2 + \mathcal{O}(k^4)$$



Related recent works on FV w/ LHC…

Plane-wave basis to treat long-range interactions — Meng and Epelbaum, JHEP (2021); Meng et al., PRD (2024)

Project to irep. of the cubic to avoid the lhc associated to the partial wave projection — Mai and Döring, EPJA (2017), PRL (2019)

Generalization of the Lüscher + *K*-matrix — Hansen and Raposo, JHEP (2024)

Three-body framework (automatically includes lhc) — Dawid et al., PRD (2023); Hansen et al., PRD (2024)

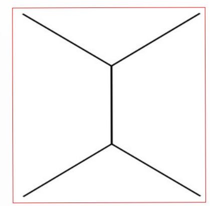
Modify the Lüscher formula via “modified effective range expansion” — Bubna et al., JHEP (2024)

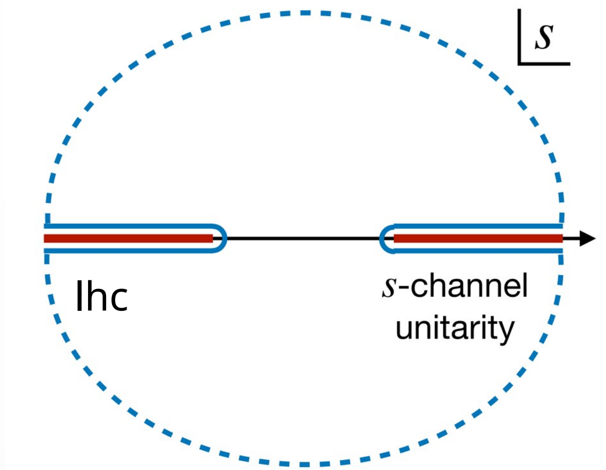
The N/D method

$$T(s)=\frac{N(s)}{D(s)}$$

$$\mathrm{Im}D=\mathrm{Im}\frac{N}{T}=N\mathrm{Im}\frac{1}{T}=\begin{cases}-N\rho, & s>s_{\rm thr}\\ 0, & s<s_{\rm thr}\end{cases}$$

$$\mathrm{Im}N=\begin{cases}\mathrm{Im}TD, & s<s_{\rm lhc}\\ 0, & s>s_{\rm lhc}\end{cases}$$





$$D(s)=\sum_i\frac{\gamma_i}{s-s_i}+\sum_{m=0}^{n-1}a_m s^m-\frac{(s-s_0)^n}{\pi}\int_{s_{\rm thr}}^{\infty}ds'\frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s)=\sum_{m=0}^{n-\ell-1}b_m s^m+\frac{(s-s_0)^{n-\ell}}{\pi}\int_{-\infty}^{s_{\rm left}}ds'\frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

N=1 ⇨

$$D(s)=f(s)+G(s)$$

$$T(s)=\frac{1}{f(s)+G(s)}$$

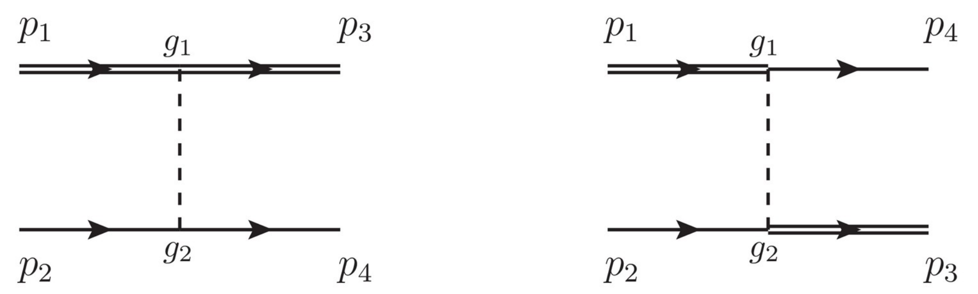
$f(s)$ =1/V

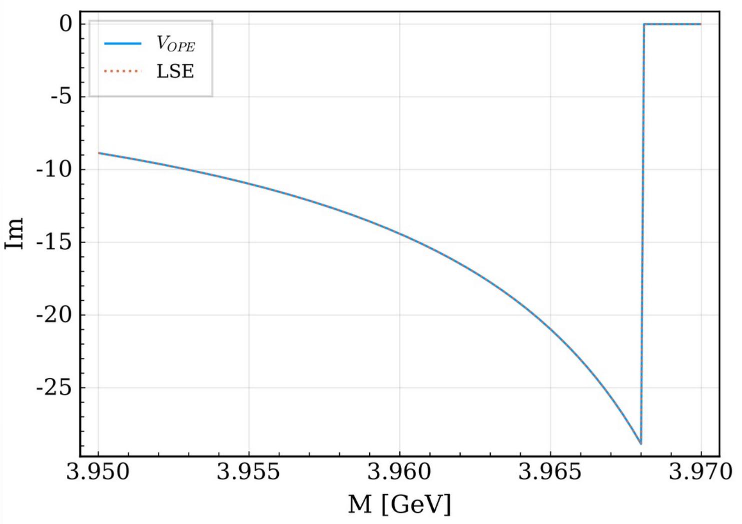
On-Shell factorization

$$\frac{1}{T_\ell^{\rm II}}=\frac{1}{T_\ell}+2i\rho \;\Rightarrow\; T^{\rm II}=\frac{1}{\frac{D}{N}+2i\rho}=\frac{N}{D+2i\rho N}$$

Along the lhc, ρ and D is real, N has imaginary part. ⇨ $D+2i\rho N\neq 0$

The left-hand cut arising from OPE





$$\mathrm{Im}\, f(k^2) = c\, \mathrm{Im}\, L(k^2) = -\frac{c}{4k^2}\pi, \quad \text{for } k^2 < k^2_{\mathrm{lhc}}$$

$$T(E,k,k') = V(E,k,k') - \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} V(E,k,q)G(E,q)T(E,q,k')$$

Solving LSE could be time-consuming.

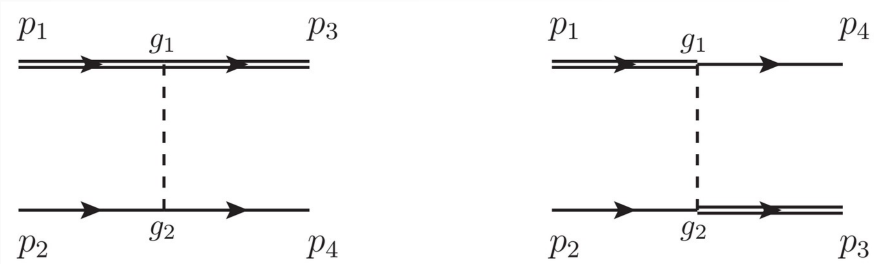
For a t-channel exchange at low-energies, an S-wave amplitude reads

$$L_t(s) = \frac{1}{2}\int \frac{1}{t-m_5^2} d\cos\theta = -\frac{s}{\lambda(s,m_1^2,m_2^2)} \log\left(\frac{s-2(m_1^2+m_2^2)+m_5^2+\frac{(m_1^2-m_2^2)^2}{s}}{m_5^2}\right),$$

with m_5 the mass of changed particle. Likewise, the u-channel exchanged S-wave amplitude reads

$$L_u(s) = \frac{1}{2}\int \frac{1}{u-m_5^2} d\cos\theta = -\frac{s}{\lambda(s,m_1^2,m_2^2)}\left(\log(s+m_5^2-2(m_1+m_2)^2) - \log(m_5^2 - \frac{(m_1^2-m_2^2)^2}{s})\right).$$

The left-hand cut: nonrelativistic



$$\eta = |m_1 - m_2|/(m_1 + m_2)$$

$$\mu_{\rm ex}^2 = m_{\rm ex}^2 - (m_1 - m_2)^2$$

$$\mu_+^2 = 4\mu\mu_{\rm ex}^2/(m_1 + m_2)$$

Exchanged-particle: relativistic

$$L_t(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4 + k^2}{m_{\rm ex}^2/4},$$

$$L_u(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2},$$

$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_3)^2}{t - m_{\rm ex}^2}\mathrm{d}\cos\theta = -\frac{m_{\rm ex}^2}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} - 1$$

$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_4)^2}{u - m_{\rm ex}^2}\mathrm{d}\cos\theta \approx -\frac{\mu_{\rm ex}^2}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} - 1$$

$\mathcal{F}_\ell/2$

$$f(k^2) = \frac{n(k^2)}{d(k^2)}$$

$$\mathrm{Im}\, d(k^2) = -k\, n(k^2), \qquad \text{for } k^2 > 0,$$

$$\mathrm{Im}\, n(k^2) = d(k^2)\ \mathrm{Im}\, f(k^2), \qquad \text{for } k^2 < k_{\rm lhc}^2.$$

The N/D method: nonrelativistic

$\propto \mathrm{Im}L$

$$n(k^2) = n_m(k^2) + \frac{(k^2)^m}{\pi}\int_{-\infty}^{k^2_{\mathrm{lhc}}} \frac{d(k'^2)\,\mathrm{Im}\,f(k'^2)}{(k'^2-k^2)(k'^2)^m}\mathrm{d}k'^2$$

No singularity along lhc

$$n(k^2) = n'_m(k^2) + \frac{P(k^2)}{\pi}\int_{-\infty}^{k^2_{\mathrm{lhc}}} \frac{\mathrm{Im}f(k'^2)}{k'^2-k^2}\mathrm{d}k'^2 = n'_m(k^2) + P(k^2)\tilde{g}L(k^2)$$

$$\begin{aligned} n(k^2) &= n_0 + n_1 k^2 + \frac{k^2}{\pi}\int_{-\infty}^{k^2_{\mathrm{lhc}}} \frac{(d_0+d_1k'^2)\mathrm{Im}f(k'^2)}{(k'^2-k^2)k'^2}\mathrm{d}k'^2 \\ &= n_0 + n_1k^2 - cL_0 + (d_0+d_1k^2)\frac{c}{\pi}\int_{-\infty}^{k^2_{\mathrm{lhc}}}\frac{\mathrm{Im}f(k'^2)}{k'^2-k^2}\mathrm{d}k'^2 \\ &= n'_0 + n_1k^2 + (d_0+d_1k^2)cL(k^2) \end{aligned}$$

$$n(k^2) = \tilde{n}(k^2) + \tilde{g}(L(k^2) - L_0)$$

$$L_0 = L(k^2=0) = -1/\mu^2_{\mathrm{ex}}$$

The N/D method: nonrelativistic

$$d(k^2) = d_n(k^2) - \frac{(k^2 - k_0^2)^n}{\pi} \int_0^\infty \frac{k' n(k'^2)\, \mathrm{d}k'^2}{(k'^2 - k^2)(k'^2 - k_0^2)^n}$$

$$d(k^2) = \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi} \int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2} \mathrm{d}k'^2$$

$$= \tilde{d}(k^2) - ik\, n(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

It is worth stressing that $d(k^2)$ is free of lhc, as the lhc associated with $n(k^2)$ below the threshold is counterbalanced by $d^R(k^2)$, which is crucial to ensure that $f(k^2)$ exhibits the correct lhc behavior. Along the rhc, both $n(k^2)$ and $d^{\mathrm{R}}(k^2)$ are real such that $\mathrm{Im}d(k^2) = -k\, n(k^2)$.

Effective range expansion with the left-hand cut

$$\frac{1}{f(k^2)} = \frac{\tilde{d}(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)}{\tilde{n}(k^2) + \tilde{g}\left(L(k^2) - L_0\right)} - ik$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log\frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k}\right)$$

$$L(k^2) = -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2}$$

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}(L(k^2) - L_0)} - ik$$

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik\right]^{-1}$$

$\tilde{g} \to 0$

$$\frac{1}{f(k^2)} = \frac{1}{a} + \frac{1}{2} r k^2 - ik$$

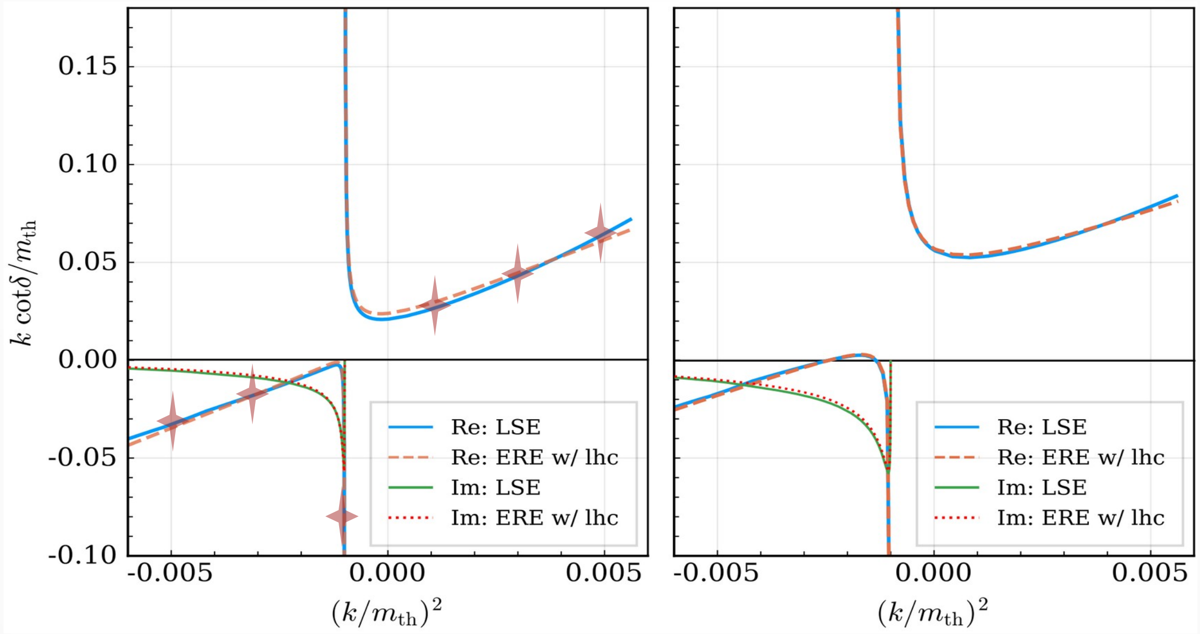
Scattering length $$a = f(k^2 = 0) = \left[\tilde{d}_0 + \frac{\tilde{g}}{\mu_+}(1 - \eta)\right]^{-1}$$

Effective range $$r = \left.\frac{\mathrm{d}^2(1/f + ik)}{\mathrm{d}k^2}\right|_{k=0} = 2\tilde{d}_1 - \frac{8\tilde{g}}{3\mu_+^3}(1 - \eta^3) - \frac{4\tilde{g}}{\mu_+^4 a_u}(1 - \eta^4)$$

Example: Tcc on the Lattice [3 parameters]

$f_{[0,1]}$

Du *et al.*, 2408.09375 [hep-ph]



$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1}$$

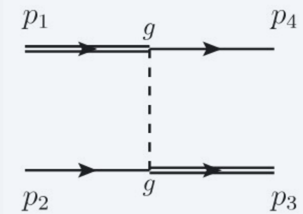
$$-\frac{2\pi}{\mu}\mathrm{Im}f = \mathrm{Im}T = \mathrm{Im}V_{\mathrm{OPE}}(k^2) = g_P\frac{-\pi}{4k^2}\mathcal{F}_\ell, \quad \text{for } k^2 < k_{\mathrm{lhc}}^2$$

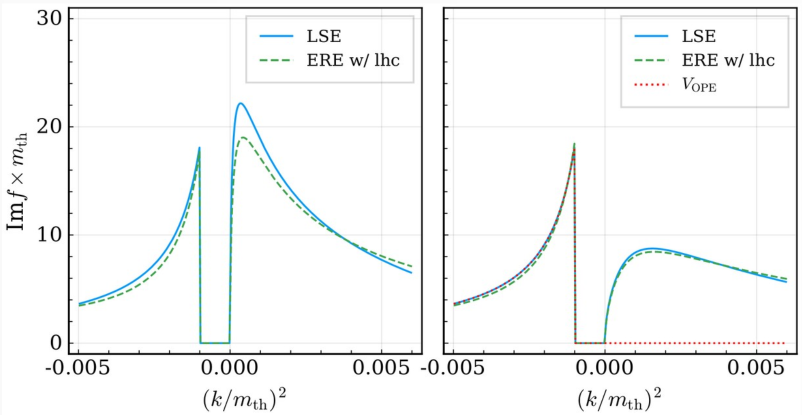
$$\mathrm{Im}\, n(k^2) = -\tilde{g}\frac{\pi}{4k^2}, \quad \text{for } k^2 < k_{\mathrm{lhc}}^2$$

$$d_u^{0,\mathrm{lhc}} = \tilde{d}_0 - \frac{\tilde{d}_1\mu_+^2}{4} + \frac{\mu_+}{2}\left(1+\frac{\tilde{g}}{\mu_{\mathrm{ex}}^2}\right) + \frac{\tilde{g}\log[2/(1+\eta)]}{\mu_+}$$

$$g_P = -\frac{2\pi\tilde{g}}{\mu d^{0,\mathrm{lhc}}\mathcal{F}_\ell}$$

$$g_{D^*D\pi}^2/(4F^2) = 9.2\ \mathrm{GeV}^{-2}$$





$9.0\ \mathrm{GeV}^{-2}$ $9.4\ \mathrm{GeV}^{-2}$

The amplitude zero

At leading order, i.e., $\tilde{n}(k^2) = 1$,

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g}d^{\mathrm{R}}(k^2)}{1+\tilde{g}(L(k^2)-L_0)} - ik\right]^{-1}$$

For a general u-channel exchange,

$$1 + \tilde{g}\left[L_u\left(k_{u,\mathrm{zero}}^2\right) + \frac{1}{\mu_{\mathrm{ex}}^2}\right] = 0,$$

$$k_{t,\mathrm{zero}}^2 = -\frac{m_{\mathrm{ex}}^2}{4}\left[1+\frac{1}{y}W(-e^{-y}y)\right]$$

$$y \equiv 1 + m_{\mathrm{ex}}^2/\tilde{g}$$

Application to the NN scattering: 1S0

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}(L(k^2) - L_0)} - ik$$

ERE w/lhc	$g^2_{\pi NN}/4\pi$	$E_{\text{pole}} - m_{\text{thr}}$
d_0, d_1, d_2, n_1, g_0	$12.4^{+2.2}_{-2.3}$	-66.4 keV
d_0, d_1, n_1, n_2, g_0	$11.0^{+2.6}_{-2.8}$	-66.6 keV
$d_0, d_1, d_2, n_1, n_2, g_0$	$13.0^{+3.0}_{-4.1}$	-66.3 keV

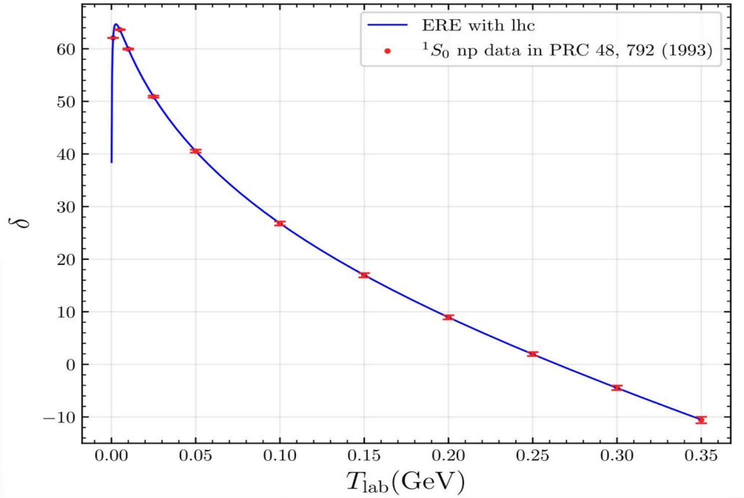
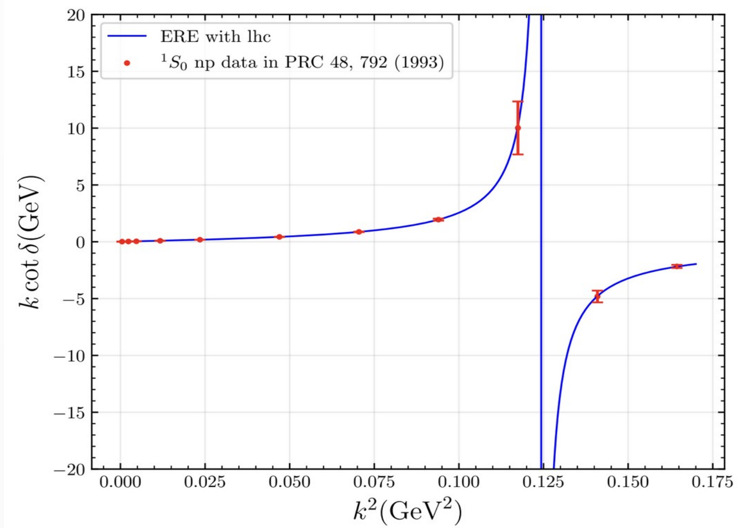


TABLE I. Some deduced values for the πNN coupling constant. The quoted uncertainty are those quoted by the authors and usually do not include systematic uncertainties.

Source	Year	System	$g^2_{\pi NN}/4\pi$
Karlsruhe-Helsinki [3]	1980	πp	14.28(18)[a]
Kroll [4]	1981	pp	14.52(40)[a]
Nijmegen [6]	1993	pp,np	13.58 (5)[a]
VPI [8]	1994	pp,np	13.70
Nijmegen [7]	1997	pp,np	13.54 (5)[a]
Timmermans [12]	1997	π^+p	13.45(14)[a]
VPI [9]	1994	GMO, πp	13.75(15)[a]
Uppsala [2]	1998	$np\rightarrow pn$	14.52(26)
Pavan *et al.* [11]	1999	πp	13.73 (9)
Schröder *et al.*, corrected [14,10]	1999	GMO, $\pi^{\pm}p$	13.77(18)
Present work	2001	GMO, $\pi^{\pm}p$	14.11(20)

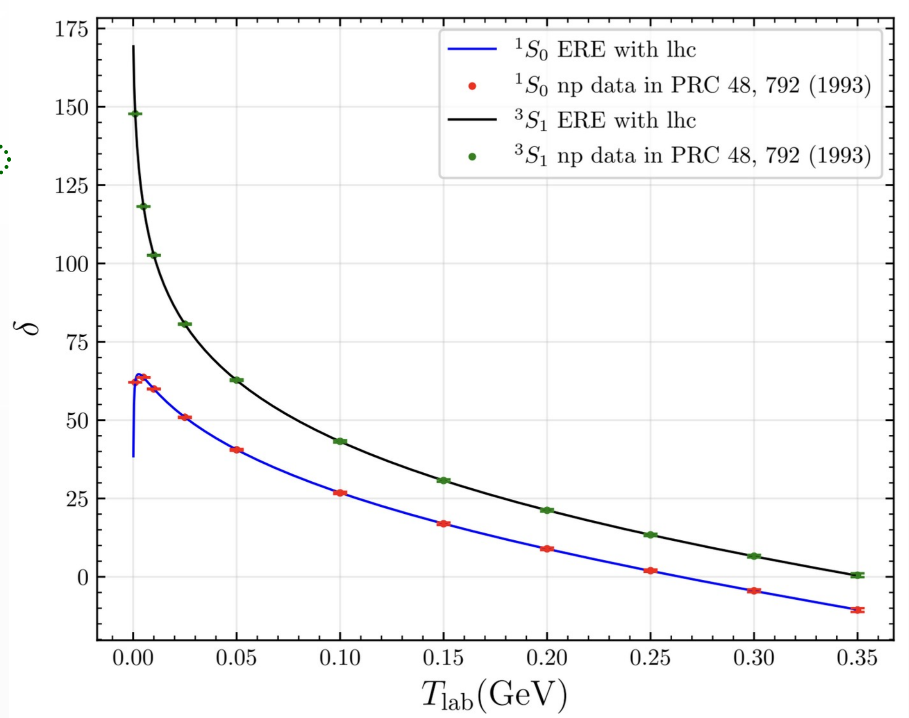
[a]Statistical uncertainty only.

PRC66(2002)014005



Application to the NN scattering: 1S0 + 3S1

ERE w/lhc	$g^2_{\pi NN}/4\pi$	$E_{\text{pole}} - m_{\text{thr}}$	a_t	r_t	$\frac{\chi^2}{\text{d.o.f}}$
$\vert {}^1S_0, I=1\rangle$ $d_0^1, d_1^1, d_2^1, n_1^1, g_0^1$	$13.2^{+2.0}_{-2.1}$	-66.18 keV(RSII)			$\frac{1.73}{22-9} \approx 0.13$[a]
$\vert {}^3S_1, I=0\rangle$ $d_0^3, d_1^3, d_2^3, n_1^3, g_0^3$		-2.21 MeV(RSI)			
$\vert {}^1S_0, I=1\rangle$ $d_0^1, d_1^1, n_1^1, n_2^1, g_0^1$	$11.7^{+2.4}_{-2.6}$	-66.42 keV(RSII)			$\frac{1.40}{22-9} \approx 0.11$[b]
$\vert {}^3S_1, I=0\rangle$ $d_0^3, d_1^3, n_1^3, n_2^3, g_0^3$		-2.2 MeV(RSI)			
$\vert {}^1S_0, I=1\rangle$ $d_0^1, d_1^1, d_2^1, n_1^1, g_0^1$	$12.8^{+2.1}_{-2.2}$	-66.3keV(RSII)			$\frac{0.74}{22-9} \approx 0.06$[c]
$\vert {}^3S_1, I=0\rangle$ $d_0^3, d_1^3, n_1^3, n_2^3, g_0^3$		-2.2MeV(RSI)			
$\vert {}^1S_0, I=1\rangle$ $d_0^1, d_1^1, n_1^1, n_2^1, g_0^1$	$12.3^{+2.4}_{-2.5}$	-66.3 keV(RSII)			$\frac{2.58}{22-9} \approx 0.20$[d]
$\vert {}^3S_1, I=0\rangle$ $d_0^3, d_1^3, d_2^3, n_1^3, g_0^3$		-2.2 MeV(RSI)			



Summary & Outlook

★ Unphysical pion masses on the Lattice

$M_D = 1927$ MeV, $M_{D^*} = 2049$ MeV, $m_\pi = 280$ MeV

↪ the three-body cut above the two-body cut ($\sqrt{s_{\mathrm{lhc}}} = 3968$ MeV)

↪ The traditional ERE valid only in a very limited range

↪ An accurate extraction of the pole requires the OPE implemented

★ The ERE with the left-hand cut

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik \right]^{-1}$$

↪ correct behavior of the left-hand cut

↪ can be used to extract the couplings of the exchanged particle to the

↪ amplitude zeros caused by the interplay between the short- and

Put the new parameterization on LATTICE!

Thank you very much for your attention!