第八届
强子谱和强子结构研讨会

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Dispersive Determination of Pion and Nucleon Gravitational Form Factors

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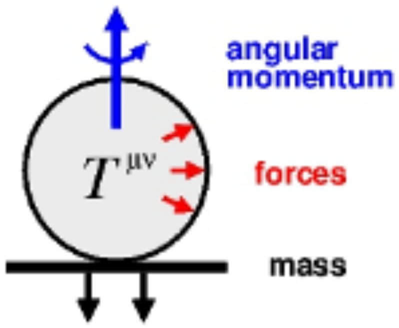
X.-H. Cao, F.-K. Guo, Q.-Z. Li & **DLY**, Dispersive determination of nucleon gravitational form factors, arXiv:2411.13398.

X.-H. Cao, F.-K. Guo, Q.-Z. Li, B.-W. Wu & **DLY**, Dispersive analysis of the gravitational form factors of the pions, kaons and nucleons, arXiv: 2507.05375.

Gravitational structure of hadrons

- Probes of hadron structure (nucleon as example)

em: $\partial_\mu J^\mu_{\mathbf{em}} = 0$	$\langle N'\vert J^\mu_{\mathbf{em}}\vert N\rangle$	$\longrightarrow$	$Q = 1.602176487(40) \times 10^{-19}\mathrm{C}$	
			$\mu = 2.792847356(23)\mu_N$	
weak: PCAC	$\langle N'\vert J^\mu_{\mathbf{weak}}\vert N\rangle$	$\longrightarrow$	$g_A = 1.2694(28)$	
			$g_p = 8.06(55)$	
gravity: $\partial_\mu T^{\mu\nu}_{\mathbf{grav}} = 0$	$\langle N'\vert T^{\mu\nu}_{\mathbf{grav}}\vert N\rangle$	$\longrightarrow$	$m = 938.272013(23)\,\mathrm{MeV}/c^2$	
			$J = \frac{1}{2}$	
Polyakov & Schweitzer, IJMPA (2018) 1830025			$\boldsymbol{D} = \boldsymbol{?}$	D-term: “Last unknown global property”

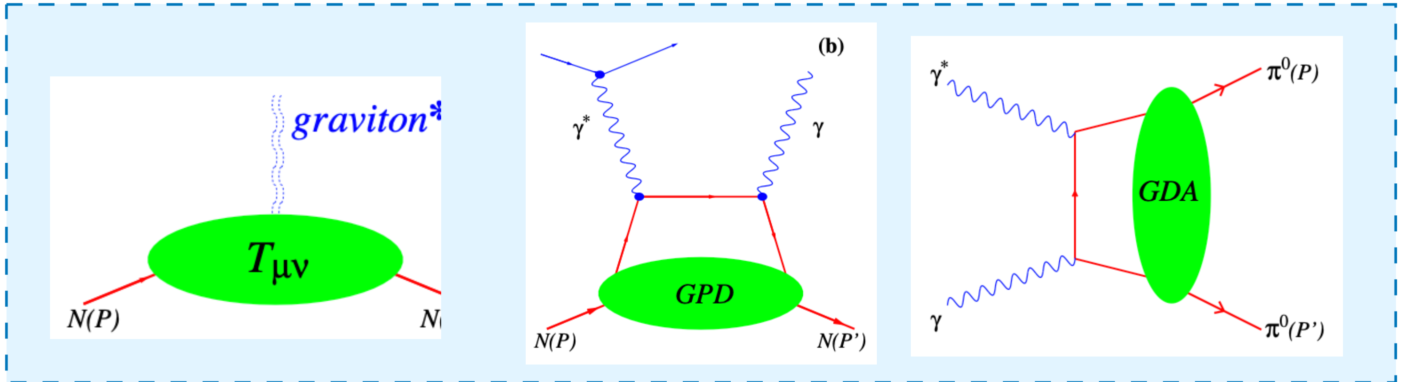


- Gravitational structure
  - Gravity couples to matter via QCD energy-momentum tensor (EMT)
  - Impractical probe by scattering off gravitons

✓ Nucleon as targets: accessible by hard-exclusive reactions like DVCS via GPDs

✓ Pions: GPDs →GDAs by crossing

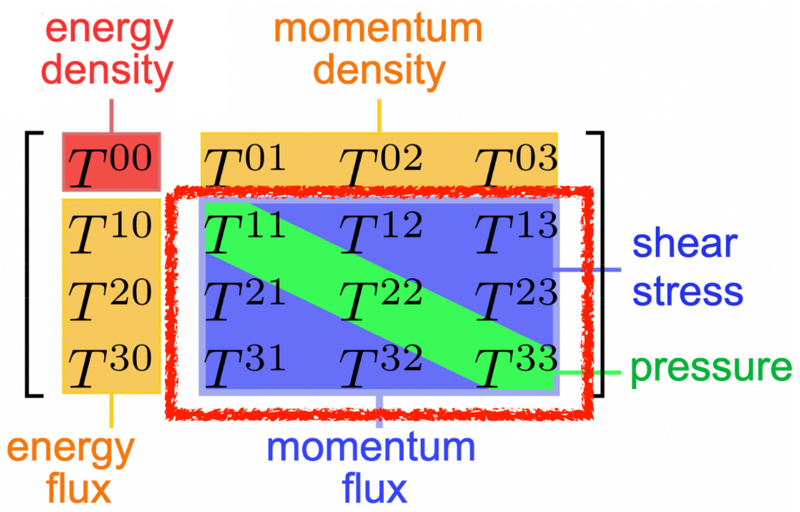
✓ Lattice QCD



The EMT QCD

- A weak classical background gravitational field $g_{\mu\nu}(x)$

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}, \quad S_{\text{grav}} = \int d^4x \sqrt{|g|}\mathscr{L}_{\text{QCD}}$$



- Symmetric Belinfante-improved total QCD EMT:
  - Total EMT = Quark + Gluon

$$\hat{T}^{\mu\nu} = \sum_q T_q^{\mu\nu} + T_g^{\mu\nu}, \quad T_q^{\mu\nu} = \bar{\psi}_q \left(\frac{i}{2}\gamma^{\{\mu}\mathscr{D}^{\nu\}} + \frac{1}{4}g^{\mu\nu}m_q \right)\psi_q, \quad T_g^{\mu\nu} = F^{A,\mu\eta}F^{A,}{}_{\eta}{}^{\nu} + \frac{1}{4}g^{\mu\nu}F^{A,\kappa\eta}F^{A,}{}_{\kappa\eta}$$

  - The total EMT is conserved: $\partial_\mu T^{\mu\nu} = 0$
  - Dilation current $j^\mu = x_\nu \hat{T}^{\mu\nu} \longrightarrow \partial_\mu j^\mu = \hat{T}^\mu_\mu = \sum m_q \bar{\psi}_q \psi_q$ (conserved in massless limit)
  - Quantum correction due to trace anomaly in non-Abelian gauge theory

$$\hat{T}_\mu{}^\mu \equiv \frac{\beta(g_s)}{2g_s} F^{A,\mu\nu}F^{A,}{}_{\mu\nu} + (1+\gamma_m)\sum_q m_q \bar{\psi}_q \psi_q$$

Nucleon sigma term $\sim$ 60 MeV from Higgs

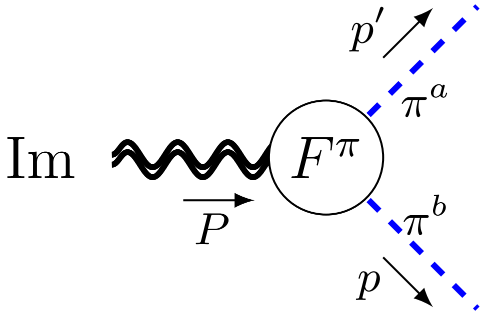
Pion and nucleon GFFs

- Definitions
  - Gravitational form factors (GFFs) for spin-0 particles, e.g., for pion:

$$\left\langle \pi^a(p') \middle| \hat{T}^{\mu\nu}(0) \middle| \pi^b(p) \right\rangle = \frac{\delta^{ab}}{2}\left[A^\pi(t)P^\mu P^\nu + D^\pi(t)\left(\Delta^\mu\Delta^\nu - t g^{\mu\nu}\right)\right]$$

$$P^\mu = p'^\mu + p^\mu, \qquad \Delta^\mu = p'^\mu - p^\mu$$

Crossing symmetry
for constructing dispersion relations



$$\left\langle \pi^a(p')\pi^b(p) \middle| \hat{T}^{\mu\nu}(0) \middle| 0 \right\rangle = \frac{\delta^{ab}}{2}\left[A^\pi(t)\Delta^\mu\Delta^\nu + D^\pi(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

  - Nucleon GFFs

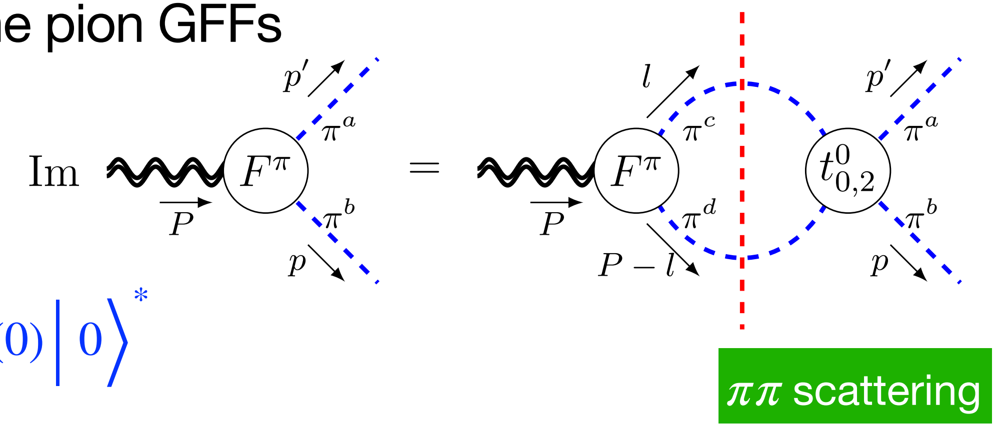
$$\left\langle N(p')\bar{N}(p) \middle| \hat{T}^{\mu\nu}(0) \middle| 0 \right\rangle = \frac{1}{4m_N}\bar{u}(p')\left[A(t)\Delta^\mu\Delta^\nu + J(t)\left(i\Delta^{\{\mu}\sigma^{\nu\}\rho}P_\rho\right) + D(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]u(p)$$

✓ Poincare symmetry and onshellness: $A(0) = 1$, $J(0) = 1/2$

✓ The last global unknown property: D-term (Druck term)

Unitarity relation for the pion GFFs

- Unitarity for full amplitude $\Rightarrow$ discontinuity (imaginary part) of the pion GFFs



DLY, Dai, Zheng, Zhou, Rept.Prog.Phys. 84 (2021)

$$
\begin{aligned}
&\mathrm{Disc}\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle \\
&= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \mathrm{d}\Omega_l \left\langle \pi^a(p')\pi^b(p)\,\middle|\, \pi^c(l)\pi^d(P-l)\right\rangle \left\langle \pi^c(l)\pi^d(P-l)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle^* \\
&= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \mathrm{d}\Omega_l \left[A(t,s,u)\delta^{ab}\delta^{cd} + A(s,t,u)\delta^{ac}\delta^{bd} + A(u,s,t)\delta^{ad}\delta^{bc}\right] \\
&\qquad \times \frac{\delta^{cd}}{2}\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right] \\
&= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \mathrm{d}\Omega_l \frac{\delta^{ab}}{2}A^{I=0}(s,t,u)\Big[\left(A^\pi(t)\right)^*\underbrace{(2l-P)^\mu(2l-P)^\nu}_{S\text{-},\,D\text{-waves}} + \left(D^\pi(t)\right)^*\underbrace{\left(P^\mu P^\nu - tg^{\mu\nu}\right)}_{S\text{-wave}}\Big]
\end{aligned}
$$

  - Mandelstam variables: $t = P^2$, $s = (p'-l)^2$, $u = (p-l)^2$; only two independent.

- Partial-wave amplitudes of $\pi\pi$ scattering: $A^I(t,s) = 32\pi\sum_J (2J+1)P_J(\cos\theta)t_J^I(t)$

Unitarity relation for the pion GFFs

- Discontinuity of A^π and D^π: associated only with t_0^0, t_2^0 partial waves.

$$\mathrm{Disc}\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \frac{\delta^{ab}}{2}\left[\mathrm{Disc}\,A^\pi(t)\Delta^\mu\Delta^\nu + \mathrm{Disc}\,D^\pi(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

$$= 2i\frac{2p_\pi}{\sqrt{t}}\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*\left(\frac{4}{3t}p_\pi^2(t_0^0(t) - t_2^0(t))\left(P^\mu P^\nu - t g^{\mu\nu}\right) + t_2^0(t)\Delta^\mu\Delta^\nu\right) + \left(D^\pi(t)\right)^* t_0^0(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

- Partial-wave unitarity

$$\mathrm{Im}\,A^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_2^0(t)\right)^* A^\pi(t)$$

$$\mathrm{Im}\,D^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left[\frac{4}{3}\frac{p_\pi^2}{t}\left(t_0^0(t) - t_2^0(t)\right)^* A^\pi(t) + \left(t_0^0(t)\right)^* D^\pi(t)\right]$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements (conserved separately):

$$\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \delta^{ab}\left\{\frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^\pi(t) + \left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]A^\pi(t)\right\}$$

Trace part: $\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^\mu{}_\mu(0)\right|0\right\rangle = \delta^{ab}\Theta^\pi(t)$, $\Theta^\pi(t) = -\frac{1}{2}\left(4p_\pi^2 A^\pi(t) + 3tD^\pi(t)\right)$

Unitarity relation for the pion GFFs

- Discontinuity of A^π and D^π: associated only with t_0^0, t_2^0 partial waves.

$$\mathrm{Disc}\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \frac{\delta^{ab}}{2}\left[\mathrm{Disc}\,A^\pi(t)\Delta^\mu\Delta^\nu + \mathrm{Disc}\,D^\pi(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

$$= 2i\frac{2p_\pi}{\sqrt{t}}\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*\left(\frac{4}{3t}p_\pi^2(t_0^0(t)-t_2^0(t))\left(P^\mu P^\nu - t g^{\mu\nu}\right) + t_2^0(t)\Delta^\mu\Delta^\nu\right) + \left(D^\pi(t)\right)^* t_0^0(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

- Partial-wave unitarity

$$\mathrm{Im}\,A^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_2^0(t)\right)^* A^\pi(t)$$

$$\mathrm{Im}\,D^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left[\frac{4}{3}\frac{p_\pi^2}{t}\left(t_0^0(t)-t_2^0(t)\right)^* A^\pi(t) + \left(t_0^0(t)\right)^* D^\pi(t)\right] \quad \Rightarrow \quad \mathrm{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_0^0(t)\right)^*\Theta^\pi(t)$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements (conserved separately):

$$\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \delta^{ab}\left\{\frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^\pi(t) + \left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]A^\pi(t)\right\}$$

Trace part: $\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^\mu{}_\mu(0)\right|0\right\rangle = \delta^{ab}\Theta^\pi(t)$, $\Theta^\pi(t) = -\frac{1}{2}\left(4p_\pi^2 A^\pi(t) + 3tD^\pi(t)\right)$

$\pi\pi$-$K\bar{K}$ coupled channels

- $\pi\pi$ phase shifts known precisely from Roy(-like) equation analyses   Bern group; Madrid-Krakow group
- Generalisation to coupled channels: isoscalar, scalar $\pi\pi$-$K\bar{K}$; $f_0(500), f_0(980)$ mesons
  - Unitarity relation for $\Theta^\pi(t)$ $\Rightarrow$ Matrix relation for coupled channels (both pion and kaon trace GFFs):

$$\mathrm{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_0^0(t)\right)^*\Theta^\pi(t)$$

Omnes solution

$$\mathrm{Im}\boldsymbol{\Theta}(t) = [\mathbf{T}_0^0(t)]^*\,\boldsymbol{\Sigma}_0^0(t)\,\boldsymbol{\Theta}(t)$$

$$\boldsymbol{\Theta}(t) = \begin{pmatrix} \Theta^\pi(t) \\ \frac{2}{\sqrt{3}}\Theta^K(t) \end{pmatrix}$$

$\pi\pi$-$K\bar{K}$ T-matrix

Phase-space factor

$$\boldsymbol{\Sigma}_0^0(t) \equiv \mathrm{diag}(\sigma_\pi\theta(t-t_\pi), \sigma_K\theta(t-t_K))$$

With $\sigma_i(t) = \sqrt{1-4m_i^2/t}\ (i=\pi, K)$

$$\mathbf{T}_0^0(t) = \begin{pmatrix} \frac{\eta_0^0(t)e^{2i\delta_0^0(t)}-1}{2i\sigma_\pi} & |g_0^0(t)|e^{i\Psi_0^0(t)} \\ |g_0^0(t)|e^{i\Psi_0^0(t)} & \frac{\eta_0^0(t)e^{2i(\Psi_0^0(t)-\delta_0^0(t))}-1}{2i\sigma_K} \end{pmatrix}.$$

$$\eta_0^0(t) = \sqrt{1-4\sigma_\pi\sigma_K|g_0^0(t)|^2\theta\left(t-t_K\right)}$$

Muskhelishvili-Omnes problem

J. Donoghue, J. Gasser, H. Leutwyler, NPB 343 (1991) 341

Muskhelishvili-Omnes representation

- **Single-channel**: Watson's theorem ⇒ phase of FF = scattering phase shift

K.M. Watson (1952)

$$\mathrm{disc}\, A^\pi(t) = 2iA^\pi(t)\theta(t-t_\pi)\sin\delta_2^0(t)e^{-i\delta_2^0(t)}$$

$$|t_2^0|e^{i\phi_2^0} = \frac{\eta_2^0 e^{2i\delta_2^0}-1}{2i\sigma_\pi}$$

❖ Omnes solution

$$A^\pi(t) = P_2^\pi(t)\Omega_2^0(t)\ , \qquad \Omega_2^0(t) \equiv \exp\left\{\frac{t}{\pi}\int_{t_\pi}^\infty \frac{\mathrm{d}t'}{t'}\frac{\phi_2^0(t')}{t'-t}\right\}$$

δ_2^0 replaced by the phase of $\pi\pi$ partial wave ϕ_2^0 to account for inelasticity

M. Hoferichter et al., Phys. Rept. 625 (2016) 1

- δ_2^0, η_2^0 up to $E_0 \simeq 2$ GeV from latest dispersive analysis

P. Bydžovský et al., PRD 94 (2016) 116013

- Beyond matching point E_0, B. Moussallam, EPJC 14 (2000) 111

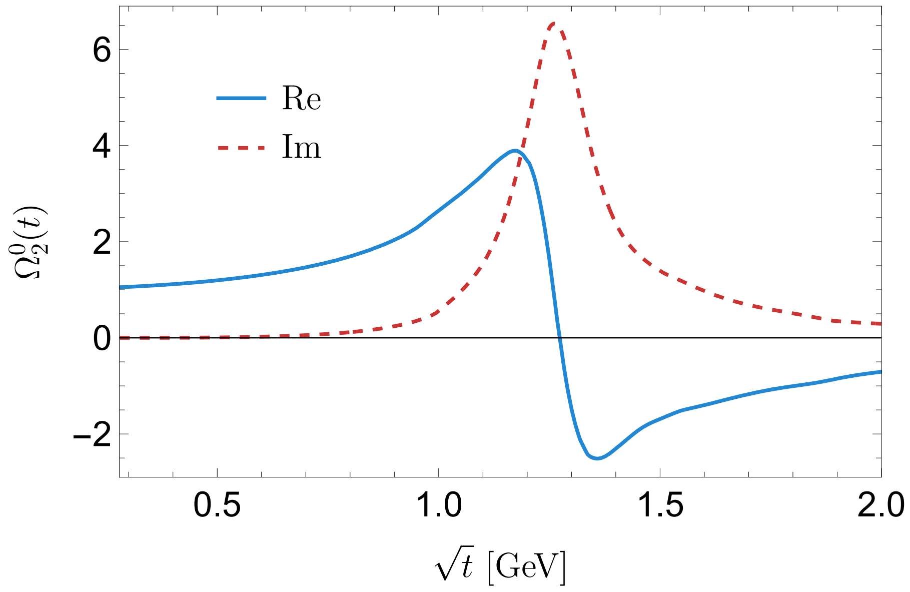
$$\delta_2^0(t) = \pi + \left(\delta_2^0(E_2^2)-\pi\right)\frac{2}{1+(\sqrt{t}/E_0)^3}\ , \quad (\text{also for } \eta_2^0)$$

- Polynomial: $P_2^\pi(t) = 1 + \alpha t$ matched to NLO ChPT

$$A^\pi(t) = 1 - \frac{2L_{12}^r}{F_\pi^2}t$$

Tensor-meson dominance estimate: $L_{12}^r = -\dfrac{F_\pi^2}{2m_{f_2}^2}$

$$P_2^\pi(t) = 1 + \left[\frac{1}{m_{f_2}^2} - \dot{\Omega}_2^0(0)\right]t \simeq 1-(0.01\ \mathrm{GeV}^{-2})t$$



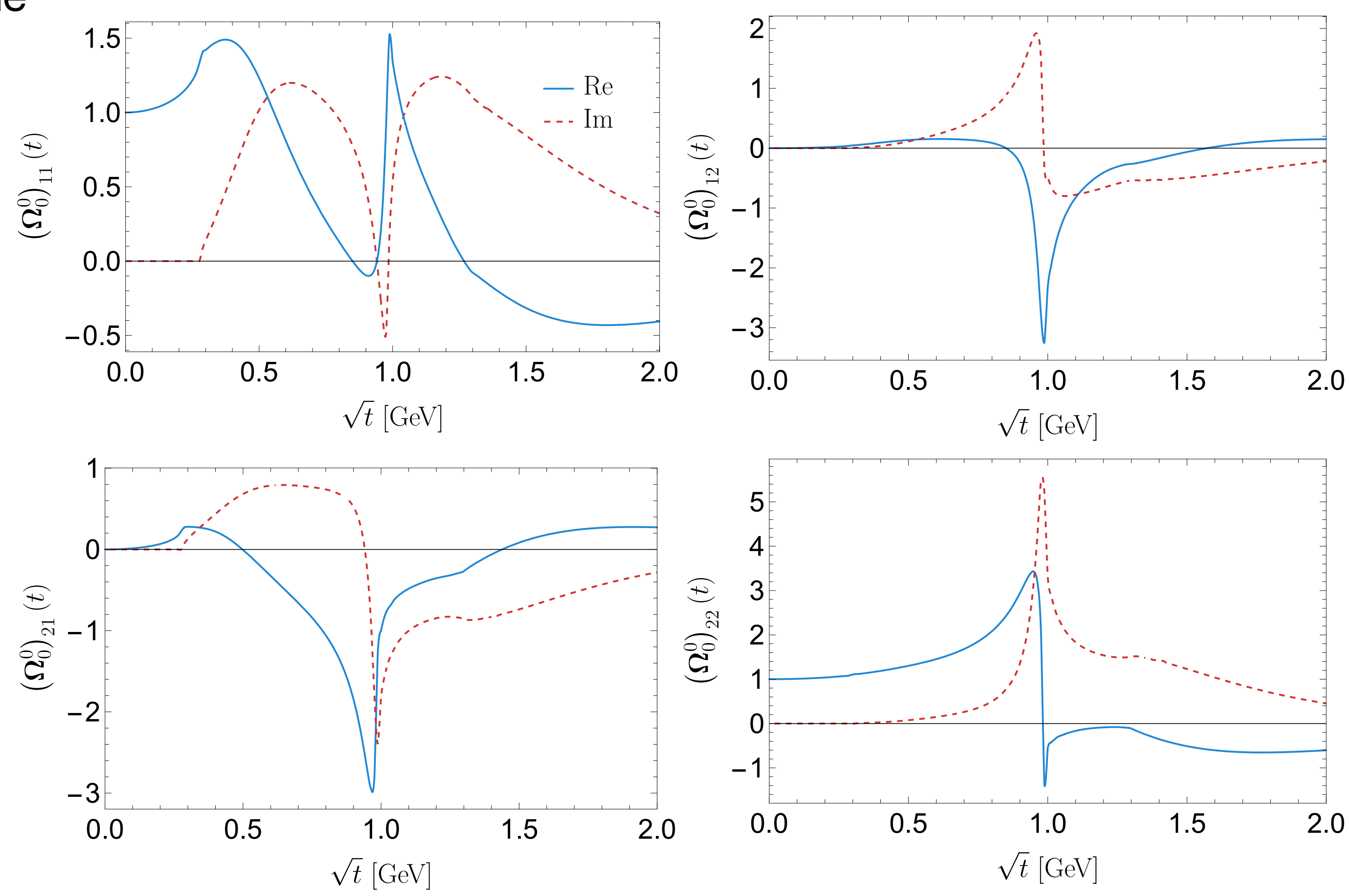
PDG average: $m_{f_2} = (1275.4 \pm 0.8)$ MeV

We take $m_{f_2} = (1275.4 \pm 20)$ MeV for a conservative error estimate.

Muskhelishvili-Omnes representation

- **Coupled-channel**: solution known as the Muskhelishvili-Omnes (MO) representation
  - The above can be generalised to $\pi\pi$-$K\bar{K}$ coupled channels (matching point: ~ 1.3 GeV)
  - Take isoscalar scalar $\pi\pi$-$K\bar{K}$ for example

$$\mathbf{\Omega}_0^0(t) = \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{dt'}{t'-t}[\mathbf{T}_0^0(t)]^* \, \mathbf{\Sigma}_0^0(t) \, \mathbf{\Omega}_0^0(t')$$



$\pi\pi$ phase shifts: Roy equation

I. Caprini et al. (2012);

$\pi\pi \to K\bar{K}$: Roy-Steiner equation

P. Büttiker et al. (2004);

M. Hoferichter et al., JHEP 06 (2012) 063

Muskhelishvili-Omnes representation

- Pion and Kaon trace GFFs

$$[\boldsymbol{\Theta}(t)]^T = [\boldsymbol{P}_0(t)]^T \boldsymbol{\Omega}_0^0(t) \ , \quad \boldsymbol{P}_0(t) = \begin{pmatrix} 2m_\pi^2 + \beta_\pi t \\ \frac{2}{\sqrt{3}}\left(2m_K^2 + \beta_K t\right) \end{pmatrix}$$

$$\beta_\pi = \dot{\Theta}^\pi(0) - 2m_\pi^2 \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{11}(0) - \frac{4m_K^2}{\sqrt{3}} \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{12}(0) \ ,$$

$$\beta_K = \dot{\Theta}^K(0) - \sqrt{3} m_\pi^2 \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{21}(0) - 2m_K^2 \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{22}(0) \ ,$$

- Matching to NLO ChPT

$$\dot{\Theta}^\pi(0) = 1 - 4L_{12}^r \frac{m_\pi^2}{F_\pi^2} - 24(L_{11}^r - L_{13}^r)\frac{m_\pi^2}{F_\pi^2} - \frac{3}{2}\frac{m_\pi^2}{F_\pi^2} I_\pi + \frac{m_\pi^2}{2F_\pi^2} I_\eta = 0.98(2) \ ,$$

$$\dot{\Theta}^K(0) = 1 - 4L_{12}^r \frac{m_K^2}{F_\pi^2} - 24(L_{11}^r - L_{13}^r)\frac{m_K^2}{F_\pi^2} - \frac{m_K^2}{F_\pi^2} I_\eta = 0.94(14) \ .$$

$\Rightarrow$ chiral logs: $I_i = \frac{1}{48\pi^2}\left[\ln\frac{\mu^2}{m_i^2} - 1\right]$

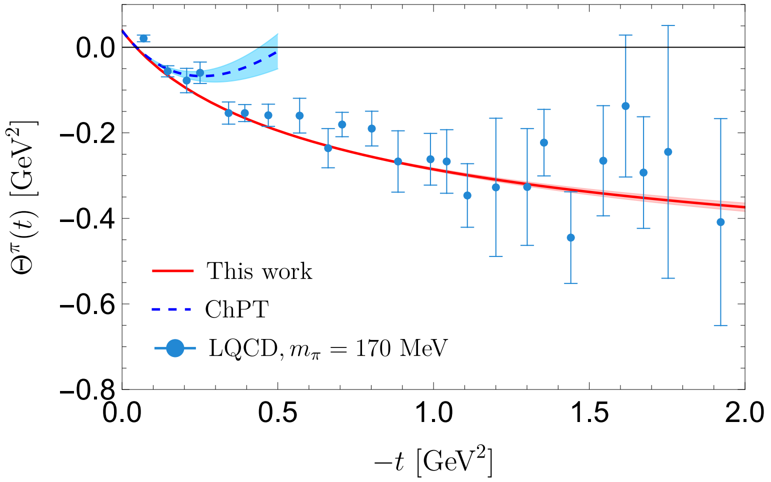
- Similar coupled-channel analysis for D-wave $\pi\pi$-$K\bar{K}$

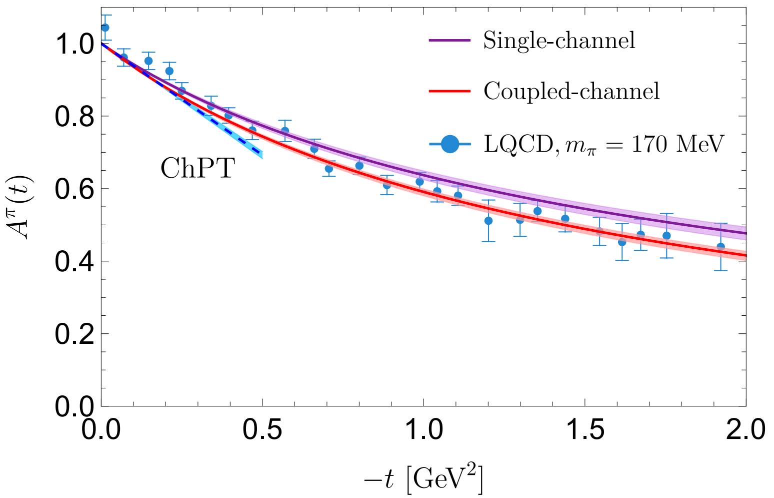
  $\Rightarrow$ coupled channel results for A^π, A^K and D^π, D^K

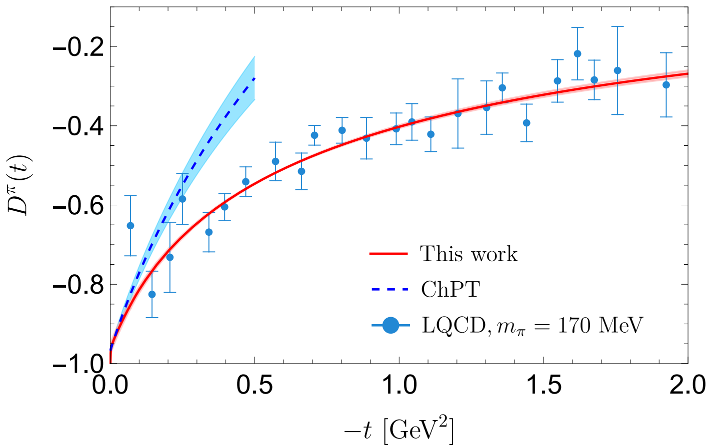
Pion and Kaon GFFs

Predictions: $\Theta(t)$ $A(t)$ $D(t)$

Pion

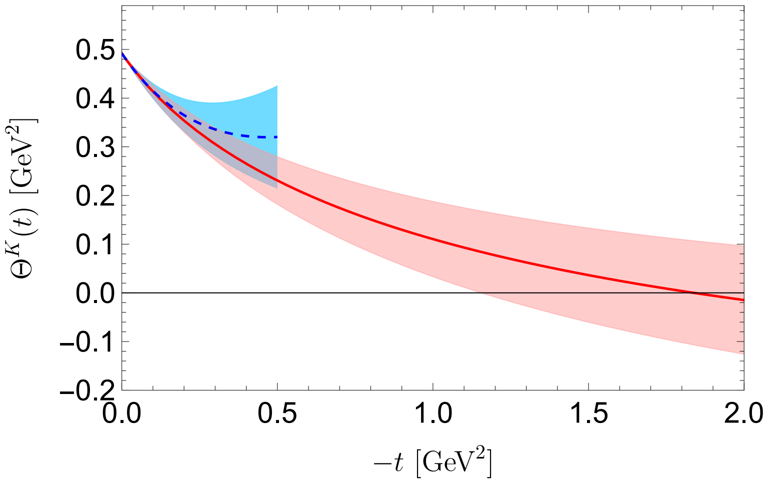


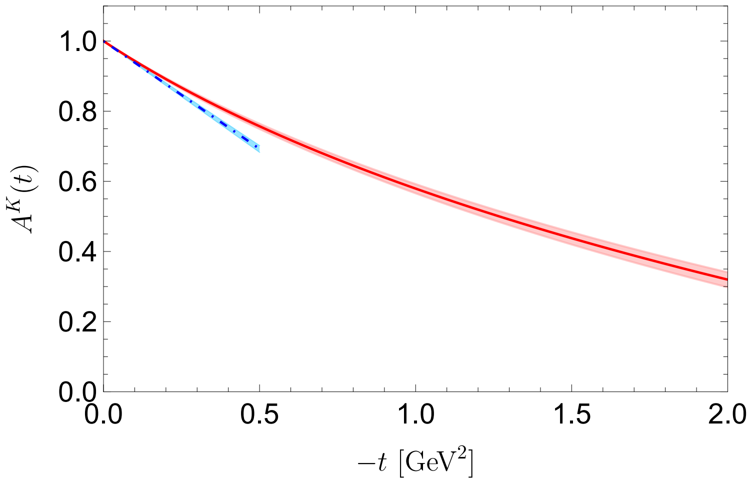


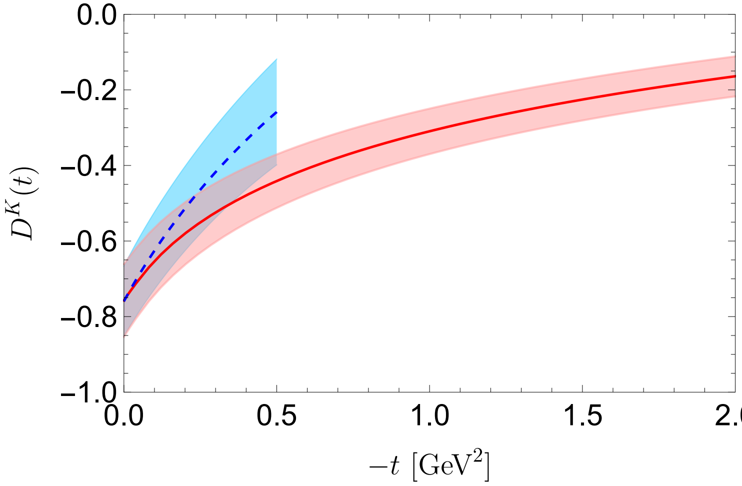


LQCD ($m_\pi = 170$ MeV): D.C. Hackett et al., PRL 132 (2024) 251904

Kaon



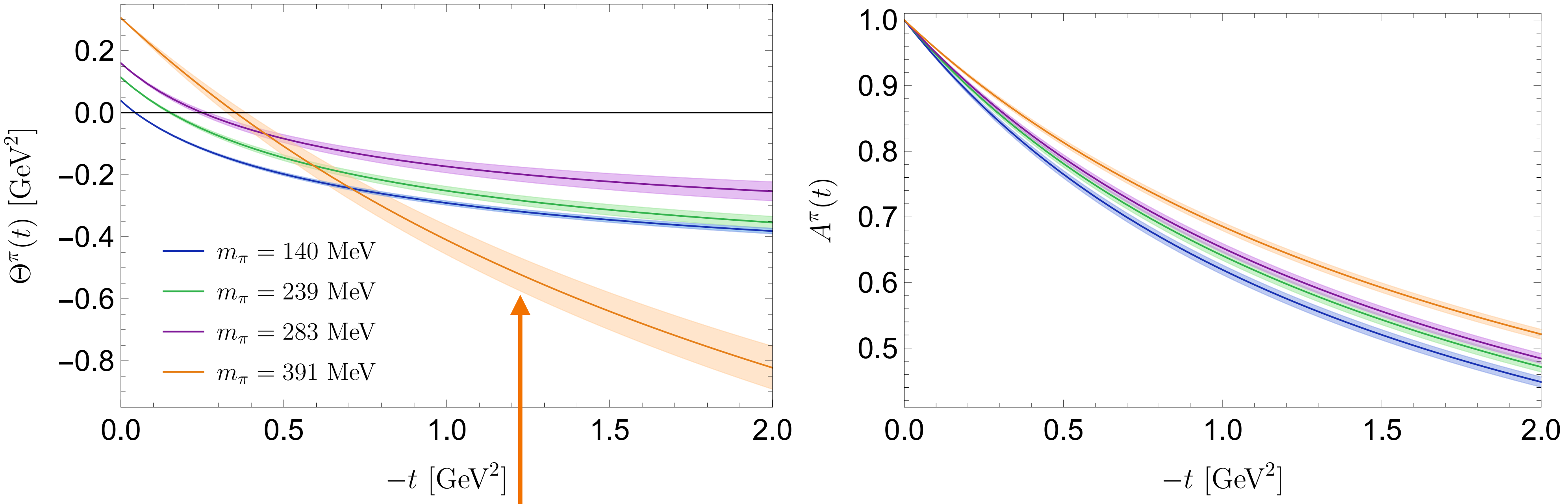




Pion GFFs: pion mass dependence

- Using the $\pi\pi$ scattering phase shifts at unphysical pion masses (239 MeV, 283 MeV, 391 MeV) obtained from Roy equation analyses

X.-H. Cao et al., PRD 108 (2023) 034009; A. Rodas et al., PRD 109 (2024) 034513



Fast change before $m_\pi = 391$ MeV:
σ meson becomes a $\pi\pi$ bound state!

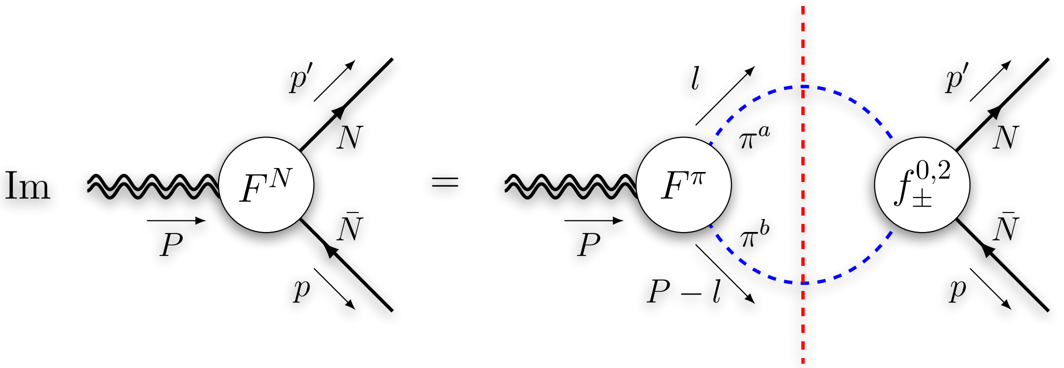
R.A. Briceno et al. [HadSpec], PRL 118 (2017) 022002

Unitarity relation for nucleon GFFs

- Discontinuity

$$\text{Disc}\langle N(p')\bar{N}(p)\,|\,\hat{T}^{\mu\nu}(0)\,|\,0\rangle$$

$$\propto \sum_n \langle N(p')\bar{N}(p)\,|\,n\rangle\langle n\,|\,\hat{T}^{\mu\nu}(0)\,|\,0\rangle^* \delta^4(p+p'-p_n)$$



- In the region $t \in (t_\pi, 16t_\pi)$, only $\pi\pi$ intermediate state

$\pi\pi/K\bar{K} \to N\bar{N}$ scattering

$$\text{Disc}\left\langle N(p')\bar{N}(p)\middle|\hat{T}^{\mu\nu}(0)\middle|\,0\right\rangle$$

$$= \frac{1}{4m_N}\bar{u}(p')\left[\text{Disc}\,\hat{A}(t)\Delta^\mu\Delta^\nu + \text{Disc}\,\hat{J}(t)\left(i\Delta^{\{\mu}\sigma^{\nu\}\rho}P_\rho\right) + \text{Disc}\,\hat{D}(t)\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]v(p)$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \text{d}\Omega_l \left\langle N(p')\bar{N}(p)\middle|\,\pi^a(l)\pi^b(P-l)\right\rangle\left\langle \pi^a(l)\pi^b(P-l)\middle|\hat{T}^{\mu\nu}(0)\middle|\,0\right\rangle^*$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \text{d}\Omega_l\ \bar{u}(p')\left[\delta^{ab}\mathbf{1}\left(A^+ + \frac{(\not{P}-2\not{l})}{2}B^+\right) + i\epsilon_{bac}\tau^c\left(A^- + \frac{(\not{P}-2\not{l})}{2}B^-\right)\right]v(p)$$

$$\times\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \text{d}\Omega_l\ \bar{u}(p')\frac{3}{2}\left(A^+ + \frac{(\not{P}-2\not{l})}{2}B^+\right)v(p)\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right],$$

$A^\pm, B^\pm$ Lorentz invariant πN scattering amplitudes

G. Höhler (1983)

Unitarity relation for nucleon GFFs

- Partial-wave amplitudes for $\pi\pi \to N\bar{N}$

W. Frazer, J. Fulco (1960); G. Höhler (1983)

$$A^I(t,s) = -\frac{8\pi}{p_N^2}\sum_{J=0}^{\infty}(J+\frac{1}{2})(p_\pi p_N)^J\left[P_J(\cos\theta)f_+^J(t) - \frac{m_N\cos\theta}{\sqrt{J(J+1)}}P'_J(\cos\theta)f_-^J(t)\right]$$

$$B^I(t,s) = 8\pi\sum_{J=1}^{\infty}\frac{J+\frac{1}{2}}{\sqrt{J(J+1)}}(p_\pi p_N)^{J-1}P'_J(\cos\theta)f_-^J(t)$$

$I = +/-$ for even/odd J;
$f_\pm^J$: $\pi\pi \to N\bar{N}$ partial-wave amp. with $+/-$ for parallel/anti-parallel $N\bar{N}$ helicities.

- Discontinuity of the nucleon GFFs

$$\text{Im}\, A^s(t) = \frac{3p_\pi^5}{\sqrt{6t}}\left[f_-^2(t) + \sqrt{\frac{3}{2}}\frac{m_N}{p_N^2}\Gamma^2(t)\right]^* A^\pi(t)$$

$$\text{Im}\, J^s(t) = \frac{3p_\pi^5}{2\sqrt{6t}}\left[f_-^2(t)\right]^* A^\pi(t)$$

$$\text{Im}\, D^s(t) = -\frac{3m_N p_\pi}{2p_N^2\sqrt{t}}\left[\frac{4p_\pi^2}{3t}\left(\left(f_+^0(t)\right)^* - (p_\pi p_N)^2\left(f_+^2(t)\right)^*\right)A^\pi(t) + \left(f_+^0(t)\right)^* D^\pi(t)\right]$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements

$$\left\langle N(p')\bar{N}(p)\middle|\hat{T}^{\mu\nu}(0)\middle|0\right\rangle = \bar{u}(p')\left(T_S^{\mu\nu} + T_T^{\mu\nu}\right)v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^s(t)$$

$$T_T^{\mu\nu} = \frac{1}{4m_N}\left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]A^s(t) + \left[i\Delta^{\{\mu}\sigma^{\nu\}\rho}P_\rho + \frac{2i\sigma^{\rho\kappa}\Delta_\rho P_\kappa}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]J^s(t)$$

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$$\Theta^s(t) = \frac{1}{4m_N}\left[-4p_N^2A^s(t) + 2tJ^s(t) - 3tD^s(t)\right]$$

$$\Rightarrow \quad \mathrm{Im}\,\Theta^s(t) = -\frac{3p_\pi}{4p_N^2\sqrt{t}}\left[f_+^0(t)\right]^*\Theta^\pi(t)$$

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$$\Theta^s(t) = \frac{1}{4m_N}\left[-4p_N^2 A^s(t) + 2tJ^s(t) - 3tD^s(t)\right]$$

$$\mathrm{Im}\,\Theta^s(t) = -\frac{3p_\pi}{4p_N^2\sqrt{t}}\left[f_+^0(t)\right]^*\Theta^\pi(t)$$

Coupled-channel

$$\mathrm{Im}\,\Theta^s(t) = -\frac{3}{4p_N^2\sqrt{t}}\left\{p_\pi\left[f_+^0(t)\right]^*\Theta^\pi(t)\theta(t-t_\pi) + \frac{4}{3}p_K\left[h_+^0(t)\right]^*\Theta^K(t)\theta(t-t_K)\right\}$$

$K\bar{K} \to N\bar{N}$ amplitude

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements

$$\left\langle N(p')\bar{N}(p)\middle|\hat{T}^{\mu\nu}(0)\middle|0\right\rangle = \bar{u}(p')\left(T_S^{\mu\nu} + T_T^{\mu\nu}\right)v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^s(t)$$

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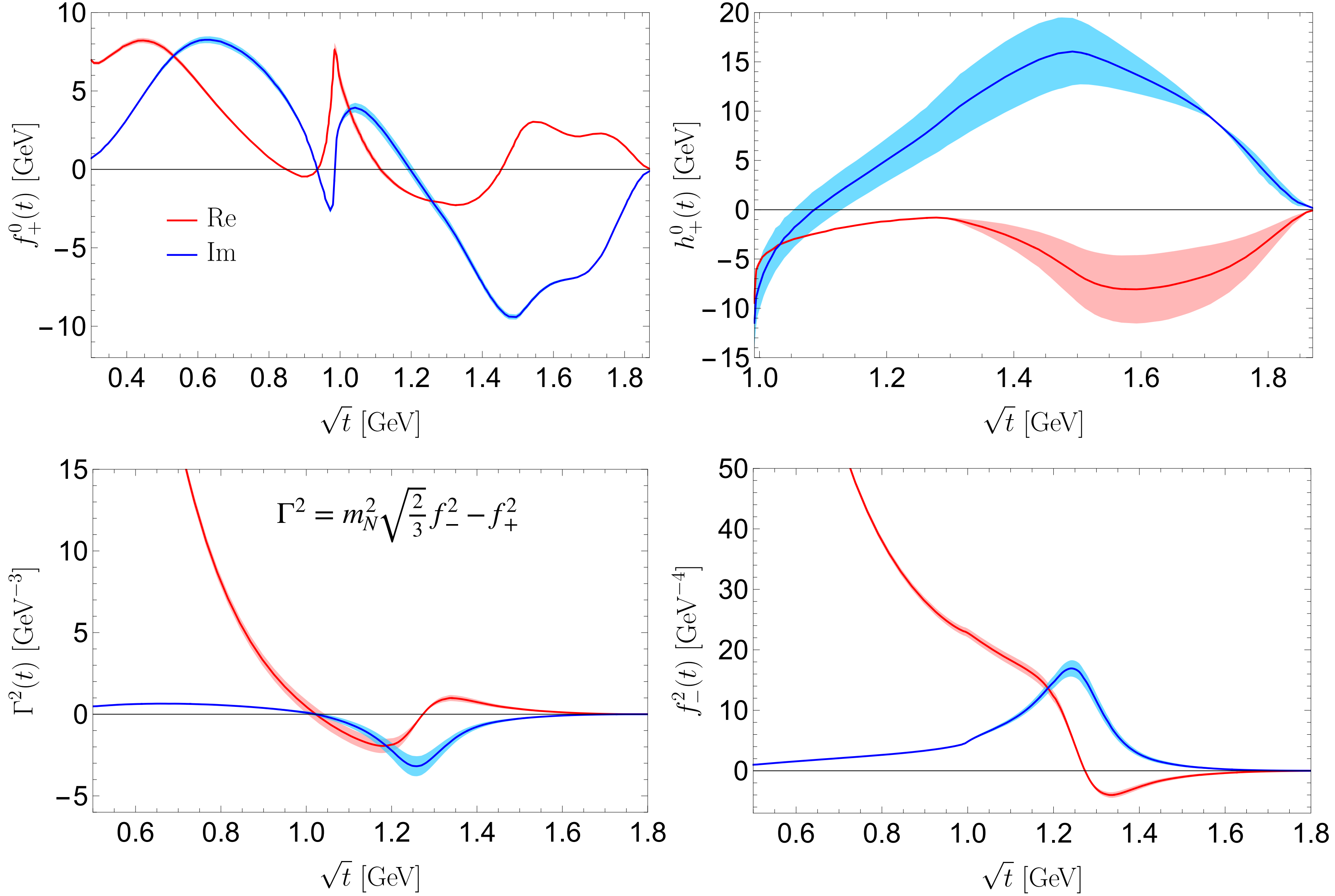
$\pi\pi/K\bar{K} \to N\bar{N}$ S-wave amplitudes

- Inputs: $\pi\pi/K\bar{K} \to N\bar{N}$ partial-wave amplitudes $f_{\pm}^{0,2}$, h_{+}^{0} from Roy-Steiner analyses G.E. Hite ,F. Steiner (1973)

M. Hoferichter et al., Phys. Rept. 625 (2016) 1; PLB 853 (2024) 138698; X.-H. Cao et al, JHEP 12 (2022) 073



Nucleon GFFs

- Dispersive relations (DR) for the nucleon GFFs

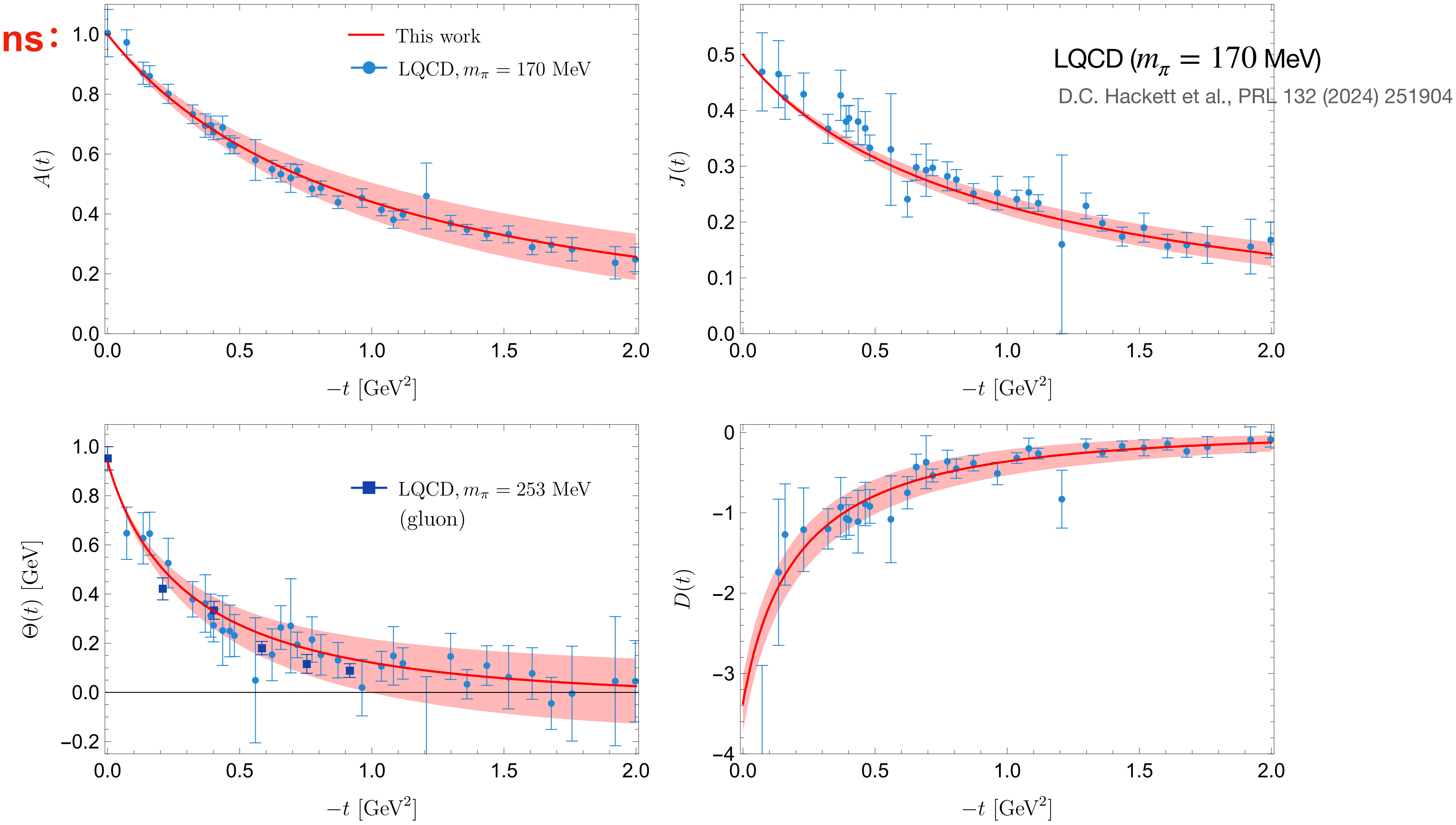
$$(A, J, \Theta)(t) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}(A, J, \Theta)(t')}{t' - t}$$

  - Un-subtracted DR: pQCD ⇒ $\lim_{t\to\pm\infty}(A, J, \Theta)(t) \sim 1/|t|$ X.-B. Tong, J.-P. Ma and F. Yuan, PLB823(2021) & JHEP10(2022)
  - Normalisation ⇒ sum rules M.A. Belushkin et al., PRC 75 (2007) 035202; M. Hoferichter et al., EPJA 52 (2016) 331

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}(A, J, \Theta)(t')}{t'} = \left(1, \frac{1}{2}, m_N\right)$$

  - Introduce S-wave (0^{++}) and D-wave (2^{++}) poles to the spectral functions: $\pi c_{S,D} m_{S,D}^2 \delta(t - m_{S,D}^2)$ to satisfy the sum rules
    - ✓ 0^{++}: $m_S \in (1.5, 1.8)$ GeV to cover $f_0(1500)$ and $f_0(1710)$
    - ✓ 2^{++}: $m_D \in (1.5, 2.2)$ GeV to cover $f_2(1500)$, $f_2(1950)$ and $f_2(2010)$

Predictions：



LQCD ($m_\pi = 253$ MeV), gluon part only: B. Wang et al. [χQCD], PRD 109 (2024) 094504

Spatial density profiles

- Consider various densities

M. Polyakov, PLB 555 (2003) 57; M. Polyakov, P. Schweitzer, IJMPA 33 (2018) 183005;
C. Lorcé et al., EPJC 79 (2019) 89; C. Lorcé et al., PLB 776 (2018) 38;
X. Ji, Front. Phys. (Beijing) 16 (2021) 64601; D.E. Kharzeev, PRD 104 (2021) 054015; …

$$\rho_{\rm Ener}(r) = m_N \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} \left[A(t) - \frac{t}{4m_N^2}\left[A(t) - 2J(t) + D(t)\right]\right] = \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} \left[\Theta(t) + \frac{t}{2m_N} D(t)\right] ,$$

$$\rho_{\Theta}(r) = m_N \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} \left[A(t) - \frac{t}{4m_N^2}\left[A(t) - 2J(t) + 3D(t)\right]\right] = \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} \Theta(t) ,$$

$$\rho_J(r) = \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} \left[J(t) + \frac{2}{3} t \frac{\mathrm{d}}{\mathrm{d}t} J(t)\right] ,$$

$$p_r(r) = m_N \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} \left[-\frac{1}{r^2} \frac{1}{\sqrt{t m_N^2}} \frac{\mathrm{d}}{\mathrm{d}t} t^{\frac{3}{2}} D(t)\right] \equiv p(r) + \frac{2}{3} s(r) ,$$

$$p(r) = \frac{1}{6m_N} \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}}{\mathrm{d}r} \tilde{D}(r) , \quad s(r) = -\frac{1}{4m_N} r \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \tilde{D}(r)$$

Zero-average-momentum frame (ZAMF)

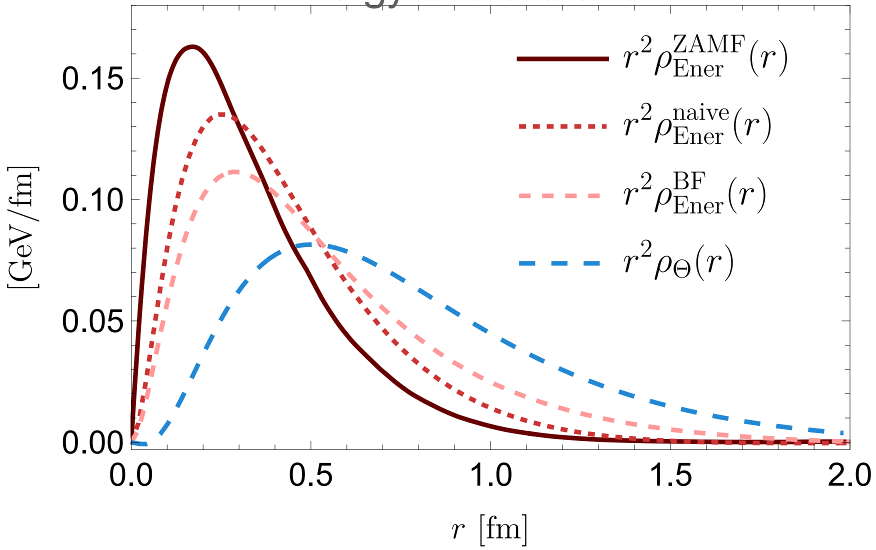
E. Epelbaum et al., PRL 129 (2022) 012001; J.Y. Panteleeva et al., EPJC 83 (2023) 617; …

$$\rho^{\rm ZAMF}_{\rm Ener}(r) = \frac{m_N}{4\pi r} \int_0^\infty \mathrm{d}\Delta \; \Delta \sin(\Delta r) \int_{-1}^{1} \mathrm{d}\alpha \; A\left[(\alpha^2 - 1)\Delta^2\right]$$

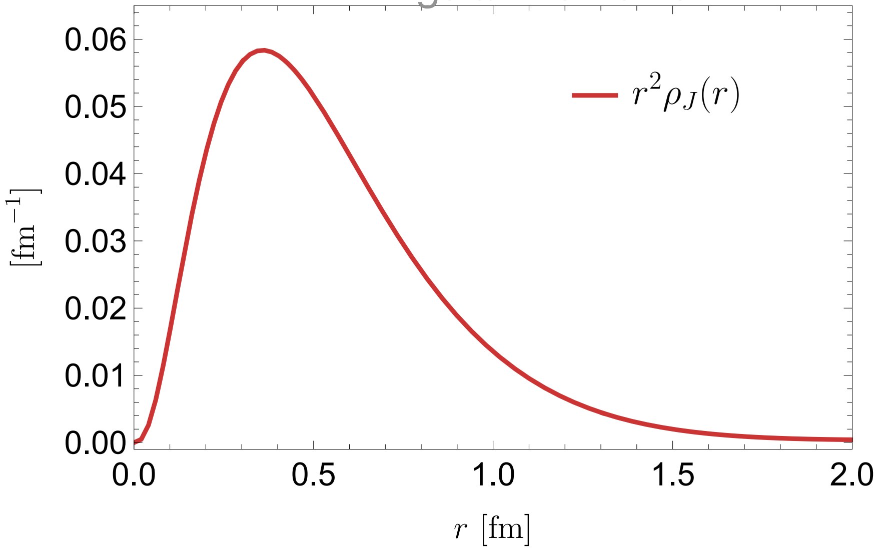
$$\rho^{\rm naive}_{\rm Ener}(r) = m_N \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{r}\cdot\boldsymbol{\Delta}} A(t)$$

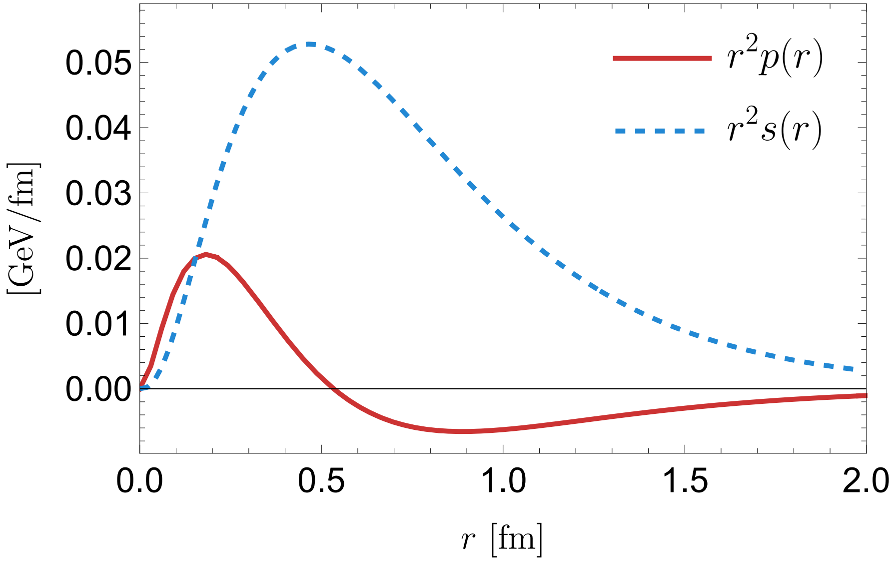
Spatial density profiles

Energy and dilatation

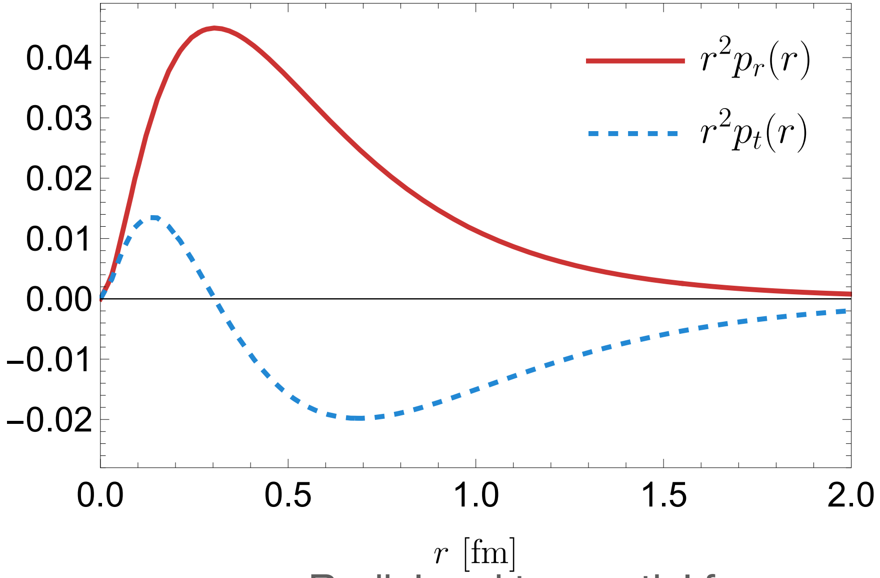


Angular momentum





Pressure and shear force



Radial and tangential force

Nucleon GFFs: D-term & radii

- *D*-term: $D \equiv D(0)$
- **Various radii in the Breit frame**
  - From the trace FF:

$$\langle r_\Theta^2 \rangle = 6A'(0) - \frac{9D}{2m_N^2}$$

  - Radius of energy density:

$$\langle r_{\text{Ener}}^2 \rangle = 6A'(0) - \frac{3D}{2m_N^2}$$

  - Mechanical radius

$$\langle r_{\text{Mech}}^2 \rangle = \frac{6D}{\int_{-\infty}^0 \mathrm{d}t\ D(t)}$$

  - Radius of the density $J(t) + \frac{2}{3}t\frac{\mathrm{d}}{\mathrm{d}t}J(t)$

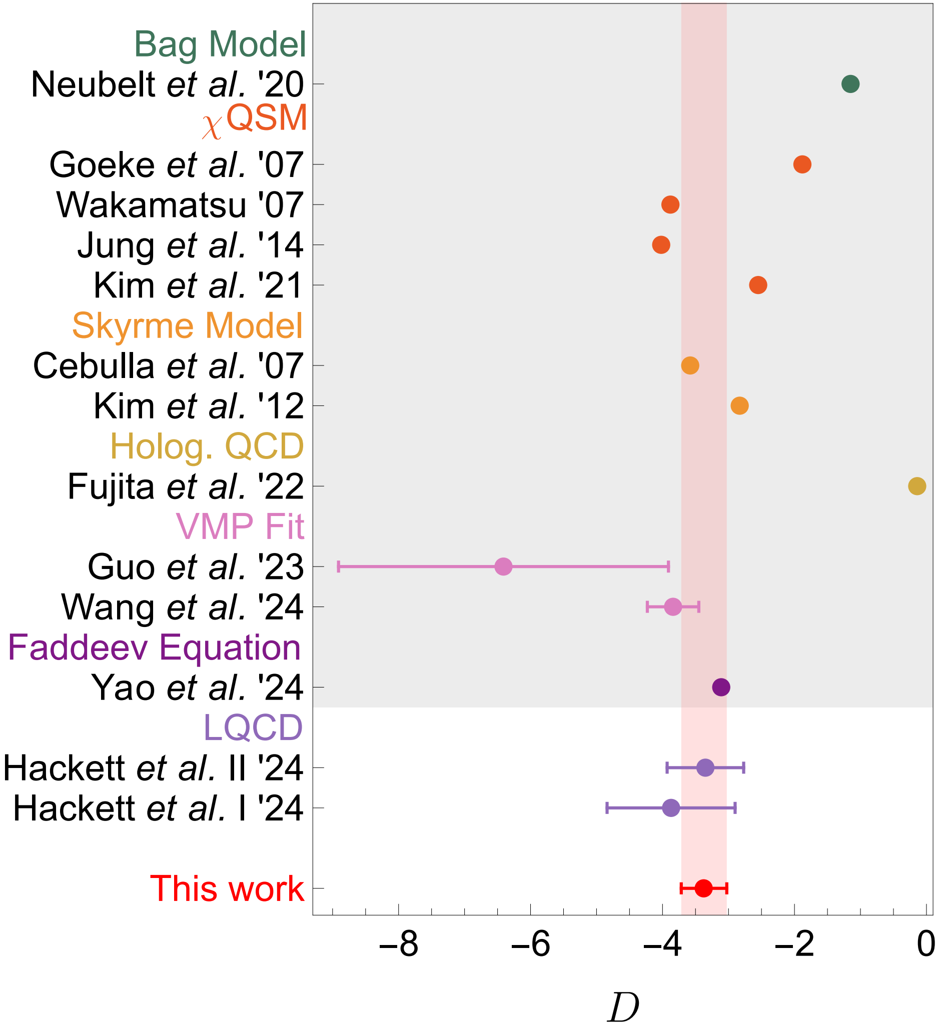
$$\langle r_J^2 \rangle = 20J'(0)$$

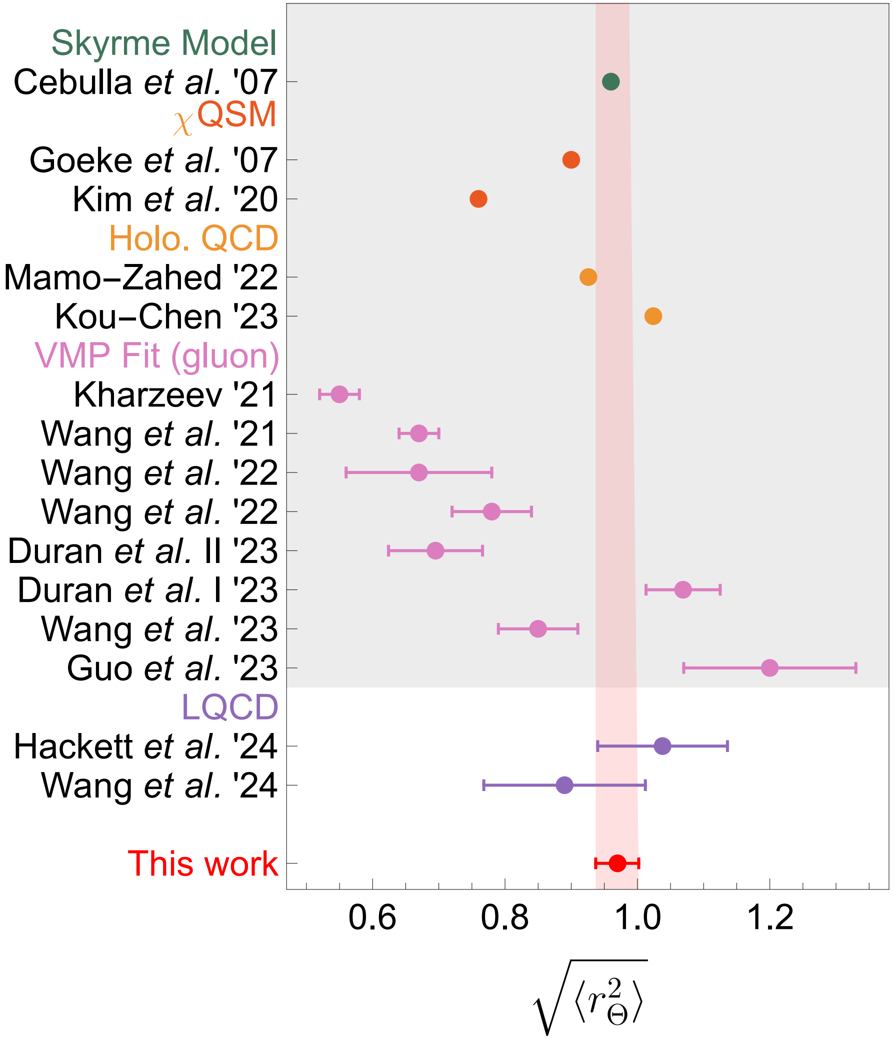
Quantity	Result	Error budget
D-term	$-3.38^{+0.34}_{-0.35}$	+(0.18)$_{\text{ChPT}}$(0.12)$_{\text{PWA}}$(0.26)$_{\text{eff}}$
		-(0.16)$_{\text{ChPT}}$(0.12)$_{\text{PWA}}$(0.29)$_{\text{eff}}$
$\langle r_\Theta^2 \rangle$ [fm]	$0.97^{+0.03}_{-0.03}$	+(0.01)$_{\text{ChPT}}$(0.01)$_{\text{PWA}}$(0.03)$_{\text{eff}}$
		-(0.02)$_{\text{ChPT}}$(0.01)$_{\text{PWA}}$(0.26)$_{\text{eff}}$
$\langle r_{\text{Ener}}^2 \rangle$ [fm]	$0.70^{+0.03}_{-0.04}$	+(0.02)$_{\text{ChPT}}$(0.01)$_{\text{PWA}}$(0.02)$_{\text{eff}}$
		+(0.02)$_{\text{ChPT}}$(0.01)$_{\text{PWA}}$(0.03)$_{\text{eff}}$
$\langle r_{\text{Mech}}^2 \rangle$ [fm]	$0.72^{+0.09}_{-0.08}$	+(0.02)$_{\text{ChPT}}$(0.00)$_{\text{PWA}}$(0.09)$_{\text{eff}}$
		-(0.03)$_{\text{ChPT}}$(0.01)$_{\text{PWA}}$(0.07)$_{\text{eff}}$
$\langle r_J^2 \rangle$ [fm]	$0.70^{+0.02}_{-0.02}$	+(0.01)$_{\text{ChPT}}$(0.01)$_{\text{PWA}}$(0.01)$_{\text{eff}}$
		-(0.01)$_{\text{ChPT}}$(0.00)$_{\text{PWA}}$(0.02)$_{\text{eff}}$

- ChPT: NLO ChPT inputs
- paw: $\pi\pi/K\bar{K} \to N\bar{N}$
- eff: effective poles m_S, m_D

Nucleon GFFs: results

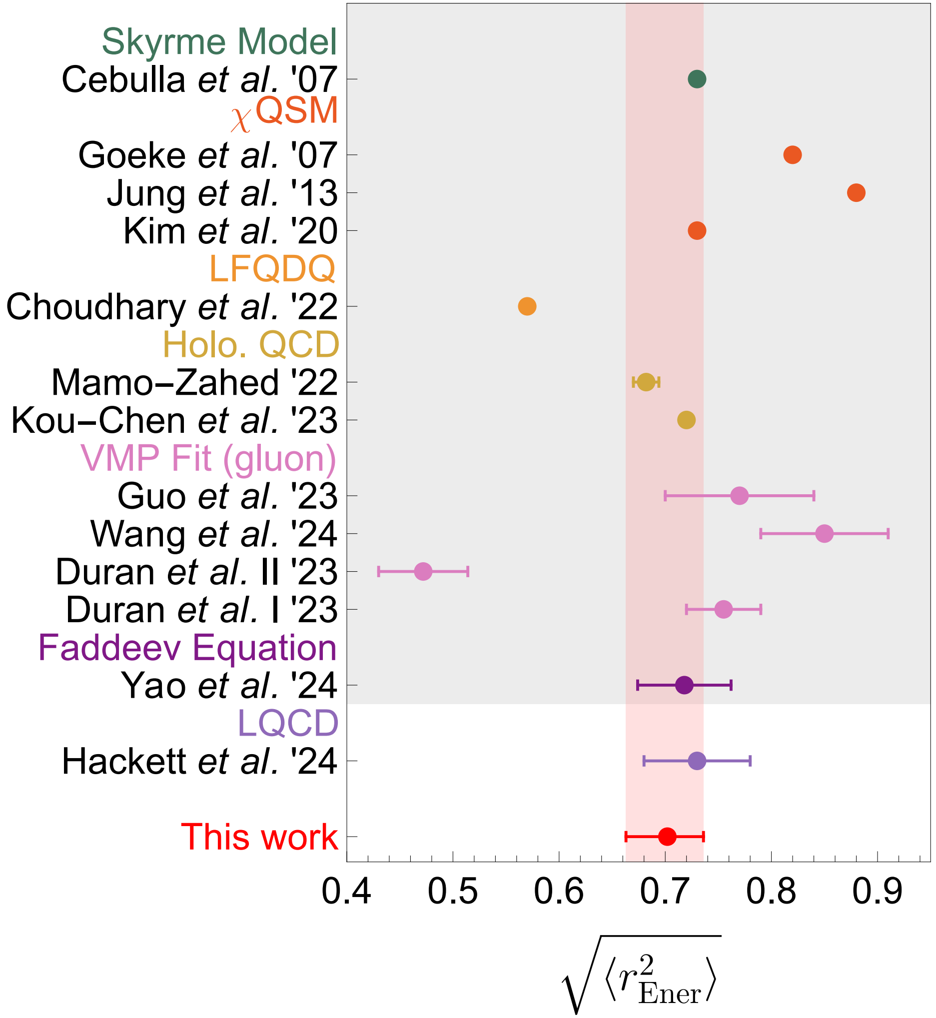
- Comparison with other results

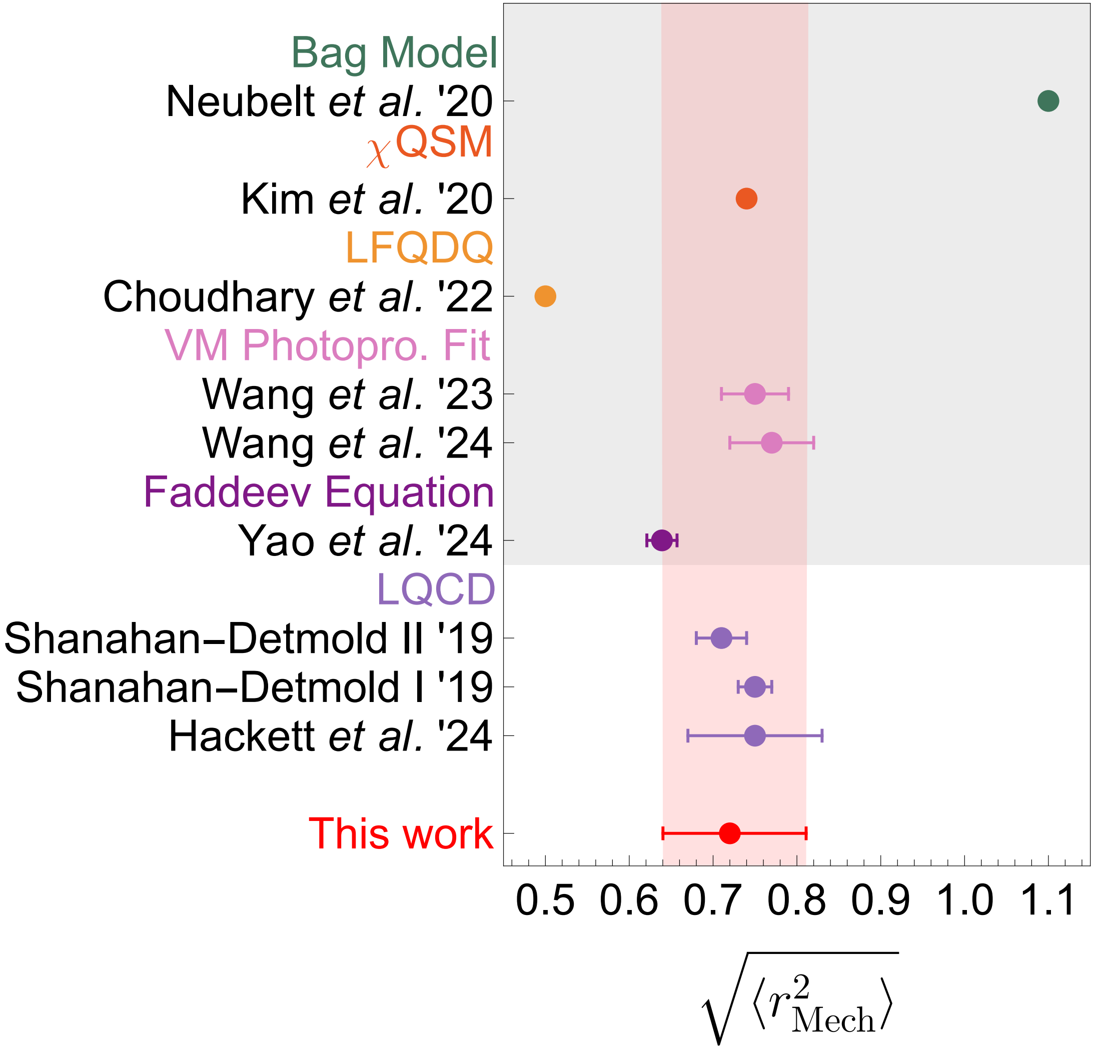




Nucleon GFFs: results

- Comparison with other results





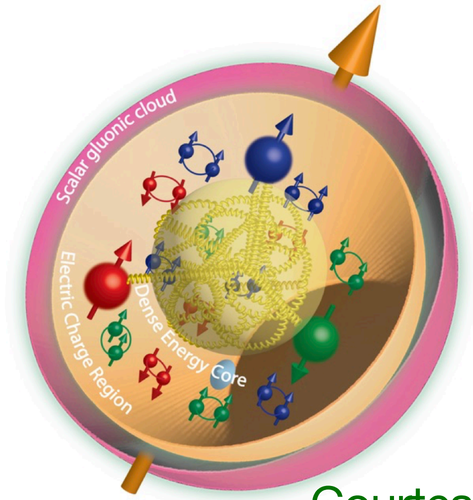
Summary and outlook

- The pion, kaon and nucleon GFFs are precisely determined using dispersive method with inputs:
  - Coupled-channel $\pi\pi$-$K\bar{K}$, $\pi\pi/K\bar{K} \to N\bar{N}$ amplitudes
  - Low energy: NLO ChPT with LECs estimated using resonance saturation, improvable with lattice calculations
  - High energy: highly excited meson resonances
- Nucleon static properties:

$$D^N = -3.38^{+0.34}_{-0.35}\,, \qquad \sqrt{\langle r_\Theta^2\rangle} = 0.97^{+0.03}_{-0.03}\ \text{fm} \quad > \sqrt{\langle r_{E,p}^2\rangle} \simeq 0.84\ \text{fm} > \quad \sqrt{\langle r_{\text{Ener}}^2\rangle} = 0.70^{+0.03}_{-0.04}\ \text{fm}$$

proton electric charge

- More results to come…
- **Outlook**:
  - Pion mass dependence
  - Extension to hyperons



Courtesy of Meziani

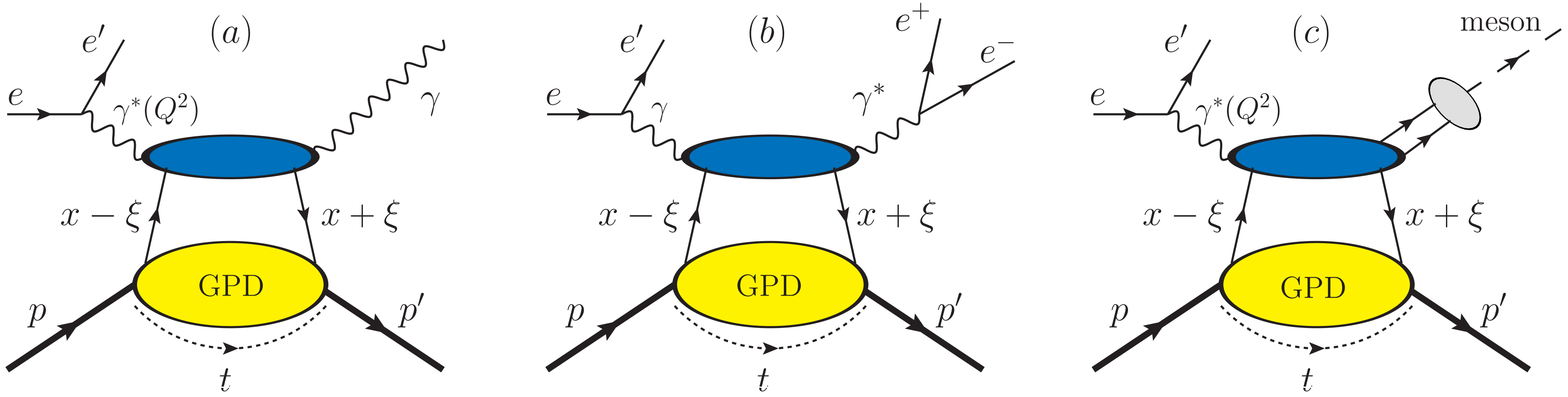
Thank you for your attention!

- Deeply virtual Compton scattering (DVCS) $eN \to e'N'\gamma$: H^a, E^a (GPD)
- Definition of the n-th order Mellin moments:

$$\int_{-1}^{1} dx\, x^{n-1} H^a(x,\xi,t), \quad \int_{-1}^{1} dx\, x^{n-1} E^a(x,\xi,t),$$

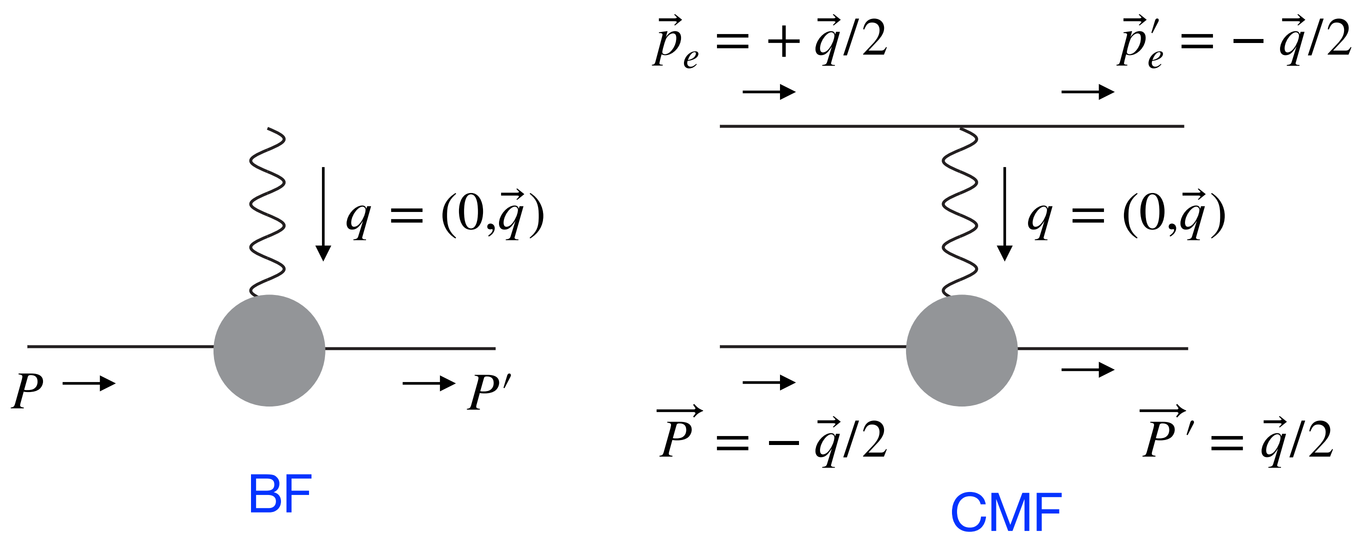
  - The first Mellin moments → Pauli and Dirac form factors
  - The second Mellin moments → Gravitational form factors

$$\int_{-1}^{1} dx\, x\, H^a(x,\xi,t) = A^a(t) + \xi^2 D^a(t), \qquad \int_{-1}^{1} dx\, x\, E^a(x,\xi,t) = B^a(t) - \xi^2 D^a(t)\,.$$



Breit frame (BF)

- The BF, for elastic electron scattering, coincides with electron-nucleon center-of-mass frame (CMF)



- BF matrix elements → Three-dimensional Fourier transformation → Charge distribution & Magnetic Density

$$\langle N_{s'}(\vec{q}/2)\,|\,J^0_{\rm EM}(0)\,|\,N_s(-\vec{q}/2)\rangle = 2m_N\, G_E(\vec{q}^2)\,\delta_{s',s},$$

$$\langle N_{s'}(\vec{q}/2)\,|\,\vec{J}_{\rm EM}(0)\,|\,N_s(-\vec{q}/2)\rangle = G_M(\vec{q}^2)\,\chi^\dagger_{s'}\left(i\vec{\sigma}\times\vec{q}\right)\chi_s\ .$$