重介子谱和混合现象

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Outline

- 介子的描述
- Bethe-Salpeter Eq. and Salpeter Eq.
- 波函数构造和解
- 重介子质量谱
- Summary

Meson and its description

- Usually using $^{2S+1}L_J$ or $J^{P(C)}$ 最常见两种方式
- nL or $^{2S+1}L_J$ from Atoms and non-relativistic

	$S=0$	$S=1$	$S=1$	$S=1$
S	$^1S_0\ (0^{-+})$	$^3S_1\ (1^{--})$		
P	$^1P_1\ (1^{+-})$	$^3P_0\ (0^{++})$	$^3P_1\ (1^{++})$	$^3P_2\ (2^{++})$
D	$^1D_2\ (2^{-+})$	$^3D_1\ (1^{--})$	$^3D_2\ (2^{--})$	$^3D_3\ (3^{--})$
F	$^1F_3\ (3^{+-})$	$^3F_2\ (2^{++})$	$^3F_3\ (3^{++})$	$^3F_4\ (4^{++})$
G	$^1G_4\ (4^{-+})$	$^3G_3\ (3^{--})$	$^3G_4\ (4^{--})$	$^3G_5\ (5^{--})$

- L=0, 1, 2, 3, 4 correspod to S, P, D, F, G waves

- S-D mixing： $|\psi(3770)\rangle = |1^3D_1\rangle\cos\theta + |2^3S_1\rangle\sin\theta,$

  没有混合需求： $|\psi(3686)\rangle = -|1^3D_1\rangle\sin\theta + |2^3S_1\rangle\cos\theta$

- ${}^1P_1 - {}^3P_1$ mixing

$$|D_1(2420)\rangle = |\frac{3}{2}\rangle = \cos\theta|^1P_1\rangle + \sin\theta|^3P_1\rangle,$$
$$|D_1'(2430)\rangle = |\frac{1}{2}\rangle = -\sin\theta|^1P_1\rangle + \cos\theta|^3P_1\rangle.$$

- 上面例子说明，${}^{2S+1}L_J$ 描述粒子波函数会有问题，有时不足以描述，因为是非相对论的近似法。

- 而用 $J^{P(C)}$ 描述粒子不会出问题，始终是对的，相对论情况也不会遇到问题。

Meson and its description

- $J^{P(C)}$ 确定波函数：不再是纯波，含有不同的分波

$J=0$	$0^{-(+)}\ (^1S_0)$	$0^{+(+)}\ (^3P_0)$		
$J=1$	$1^{-(-)}\ (^3S_1, {}^3S_1-{}^3D_1, {}^3D_1)$	$1^{++}\ (^3P_1)$	$1^{+-}\ (^1P_1)$	$1^{+}\ (^3P_1-{}^1P_1)$
$J=2$	$2^{+(+)}\ (^3P_2, {}^3P_2-{}^3F_2, {}^3F_2)$	$2^{--}\ (^3D_2)$	$2^{-+}\ (^1D_2)$	$2^{-}\ (^3D_2-{}^1D_2)$
$J=3$	$3^{-(-)}\ (^3D_3, {}^3D_3-{}^3G_3, {}^3G_3)$	$3^{++}\ (^3F_3)$	$3^{+-}\ (^1F_3)$	$3^{+}\ (^3F_3-{}^1F_3)$

- 3 categories，后两类含常见的混合。问：如何求解？

1. 0^- and 0^+ (横向)
2. Natural parity 1^-, 2^+ and 3^- ，第一类混合 (纵向)
3. Unnatural parity 1^+, 2^- and 3^+，第二类混合 (纵向)

Bethe-Salpeter方程

- 束缚态：求解薛定谔方程得到非相对论的波函数；若要得到相对论的波函数，则需求解Bethe-Salpeter方程，或者Salpeter方程（瞬时近似版本，适用于重介子）
- Relativistic Bethe-Salpeter equation （BS）

$$\chi_P(q) = iS(p_1)\int \frac{d^4k}{(2\pi)^4} V(P,k,q)\chi_P(k)S(-p_2)$$

- Instantaneous verson: Salpeter equation

$$\varphi(q_{P\perp}) \equiv i\int \frac{dq_P}{2\pi}\chi_P(q), \qquad \eta_P(q_{P\perp}) \equiv \int \frac{dk^3_{P\perp}}{(2\pi)^3} V(k_{P\perp}, q_{P\perp})\varphi(k_{P\perp})$$

$$\chi_P(q) = S(p_1)\eta_P(q_{P\perp})S(-p_2)$$

Salpeter euqation

- 正负能投影算符

$$\Lambda^{\pm}(p_{1_{P\perp}}) = \frac{1}{2\omega_1}\Big[\frac{\not{P}}{M}\omega_1 \pm (m_1 + \not{p}_{1_{P\perp}})\Big],$$

$$\Lambda^{\pm}(-p_{2_{P\perp}}) = \frac{1}{2\omega_2}\Big[\frac{\not{P}}{M}\omega_2 \pm (-m_2 + \not{p}_{2_{P\perp}})\Big]$$

- Salpeter equation

$$\varphi(q_{P\perp}) = \frac{\Lambda^{+}(p_{1_{P\perp}})\eta_P(q_{P\perp})\Lambda^{+}(-p_{2_{P\perp}})}{(M-\omega_1-\omega_2)} - \frac{\Lambda^{-}(p_{1_{P\perp}})\eta_P(q_{P\perp})\Lambda^{-}(-p_{2_{P\perp}})}{(M+\omega_1+\omega_2)}$$

- Positive and negative wave functions

$$\varphi^{\pm\pm} = \Lambda^{\pm}(p_{1_{P\perp}})\frac{\not{P}}{M}\varphi\frac{\not{P}}{M}\Lambda^{\pm}(-p_{2_{P\perp}})$$

$$\varphi(q_{P\perp}) = \varphi^{++}(q_{P\perp}) + \varphi^{+-}(q_{P\perp}) + \varphi^{-+}(q_{P\perp}) + \varphi^{--}(q_{P\perp})$$

Salpeter euqation

- Salpeter euqation 第二种表述

$$\boxed{\varphi^{++}(q_{P\perp}) = \frac{\Lambda^{+}(p_{1_{P\perp}})\eta(q_{P\perp})\Lambda^{+}(-p_{2_{P\perp}})}{(M-\omega_1-\omega_2)}},$$

$$\varphi^{--}(q_{P\perp}) = -\frac{\Lambda^{-}(p_{1_{P\perp}})\eta(q_{P\perp})\Lambda^{-}(-p_{2_{P\perp}})}{(M+\omega_1+\omega_2)},$$

$$\varphi^{+-}(q_{P\perp}) = \varphi^{-+}(q_{P\perp}) = 0\,,$$

- 一般 $M+\omega_1+\omega_2 \gg M-\omega_1-\omega_2$ (近似不能用)

$$\varphi^{++}(q_{P\perp}) \gg \varphi^{--}(q_{P\perp}), \quad \varphi(q_{P\perp}) \approx \varphi^{++}(q_{P\perp})$$

- 四个方程可求解最多8个未知量的波函数

依据J^P构建波函数

- 对介子进行P变换

$$P' = (P_0, -\vec{P}) \text{ and } q' = (q_0, -\vec{q})$$

$$\varphi_P(q) = \eta_P \gamma_0 \varphi_{P'}(q') \gamma_0,$$

- C变换

$$\varphi_P(q) = \eta_C C \varphi_P^T(-q) C^{-1},$$

$$C\gamma_5^T C^{-1} = \gamma_5 \text{ and } C\gamma_\mu^T C^{-1} = -\gamma_\mu$$

质心系：P_0=M

瞬时近似：q_0=0 (Salpeter Eq.)

依据J^P构建波函数表示

- The wave function of a meson can be composed meson mass M, momentum P, internal relative momentum q between quark and antiquark, and Dirac matrix, possible polarization vector or tensor, etc.
- Universal wave function for a 0^- meson in Instantaneous approximation ($P\cdot q_{\perp}=0$), $q_{\parallel}^{\mu}=(P\cdot q/M^2)P^{\mu}$, $q_{\perp}^{\mu}=q^{\mu}-q_{\parallel}^{\mu}$

$$\varphi_{P}^{0^-}(q_{\perp})=\left(a_1 M+a_2\, \not{P}+a_3\, \not{q}_{\perp}+a_4\frac{\not{q}_{\perp}\, \not{P}}{M}\right)\gamma^5$$

Four unknown radial wave functions $a_i \equiv a_i(-q_{\perp}^2)$

赝标0^-介子波函数

- Non-relativistic wave function： pure S wave

$$\varphi_P^{{}^1S_0}(q_\perp) = (a_1 M + a_2 \not{P})\gamma^5 \qquad \text{(a1=a2)}$$

- Relativistic wave function： S wave + P wave

$$\varphi_P^{0^-}(q_\perp) = \left(a_1 M + a_2 \not{P} + a_3 \not{q}_\perp + a_4 \frac{\not{q}_\perp \not{P}}{M}\right)\gamma^5$$

- 球坐标系，用球谐函数表示

$$\varphi_P^{0^-}(q_\perp) = \sqrt{4\pi}\left[M Y_{00}\left(a_1 + a_2\gamma^0\right) - \frac{|\vec{q}|}{\sqrt{3}}(Y_{1-1}\gamma^+ + Y_{11}\gamma^- - Y_{10}\gamma^3)(a_3 + a_4\gamma^0)\right]\gamma^5$$

a1 and a2 terms are S waves, non-relativistic

a3 and a4 terms are P waves, relativistic correction

波函数结果

利用Salpeter方程求解得径向波函数a1，a2，和本征值

$$\varphi_P^{0^-}(q_\perp) = M\left(a_1 + a_2\frac{\not{P}}{M} - a_1 x_- \not{q}_\perp + a_2 x_+ \frac{\not{q}_\perp \not{P}}{M}\right)\gamma^5$$

$$x_+ = \frac{\omega_1+\omega_2}{m_1\omega_2+m_2\omega_1},\quad x_- = \frac{\omega_1-\omega_2}{m_1\omega_2+m_2\omega_1}\qquad \omega_1 = \sqrt{m_1^2-q_\perp^2},\ \omega_2 = \sqrt{m_2^2-q_\perp^2}$$

Bc介子波函数

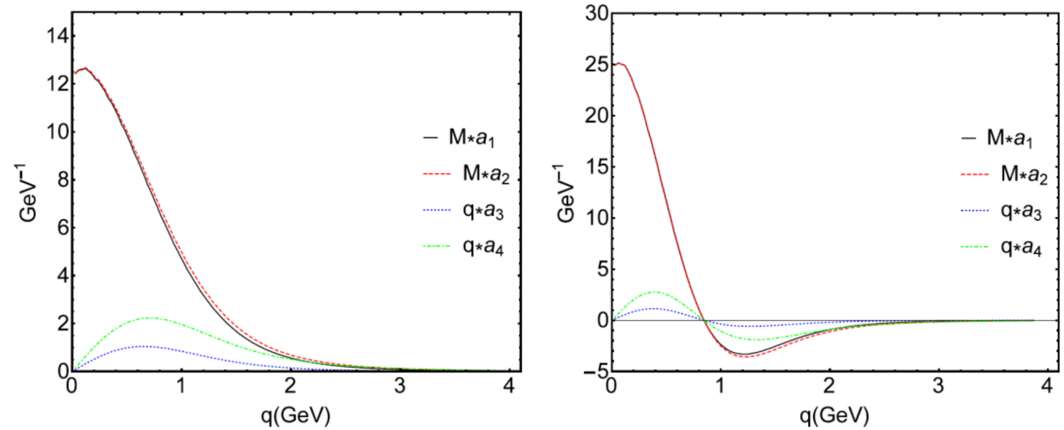


Figure 1. The 0^- wave functions of the ground state $B_c(1S)$ (left) and the first excited state $B_c(2S)$ (right). a_1 and a_2 terms are S waves; a_3 and a_4 terms are P waves.

Bc介子S、P波占比： S:P=1:0.082; 1:0.091; 1:0.097 for 1S, 2S, 3S

矢量1^-介子的波函数S-P-D混合

- 1^- $$\varphi_{1^-}(q_\perp) = q_\perp.\epsilon\left(D_1 + \frac{\not P}{M}D_2 + \frac{\not q_\perp}{M}D_3 + \frac{\not P\not q_\perp}{M^2}D_4\right)$$
$$+M\not\epsilon\left(D_5 + \frac{\not P}{M}D_6 + \frac{\not q_\perp}{M}D_8 + \frac{\not P\not q_\perp}{M^2}D_7\right)$$

D_5, D_6 terms are S waves, D_1, D_2, D_7 and D_8 terms are P waves.

D_3 and D_4 terms are $S-D$ mixture.

The D waves are

$$\left(\epsilon\cdot q_{\perp}\not q_{\perp} - \frac{1}{3}q_{\perp}^2\not\epsilon\right)\left(\frac{1}{M}D_3 - \frac{\not P}{M^2}D_4\right),$$

and the complete S waves are

$$M\not\epsilon\left(D_5 + \frac{\not P}{M}D_6\right) + \frac{1}{3}q_{\perp}^2\not\epsilon\left(\frac{1}{M}D_3 - \frac{\not P}{M^2}D_4\right).$$

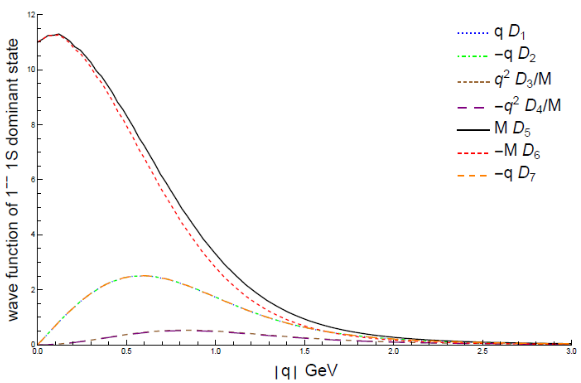
第一类：1^- 矢量波函数

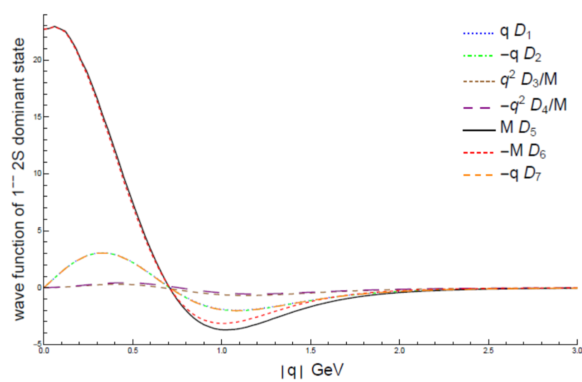
- 完整解：1S，2S，1D，3S，2D，4S ...（混合）

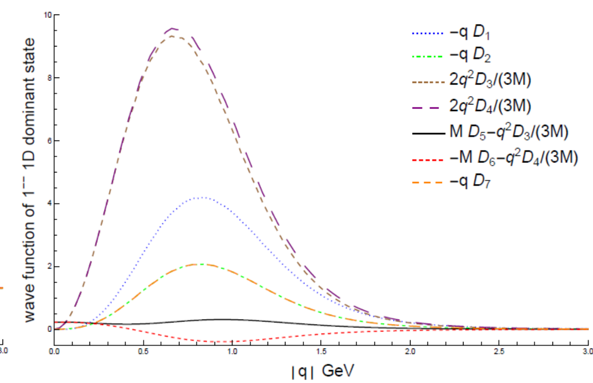
  粲偶素质量谱：3097, 3688, 3779, 4057, 4111,4329 MeV

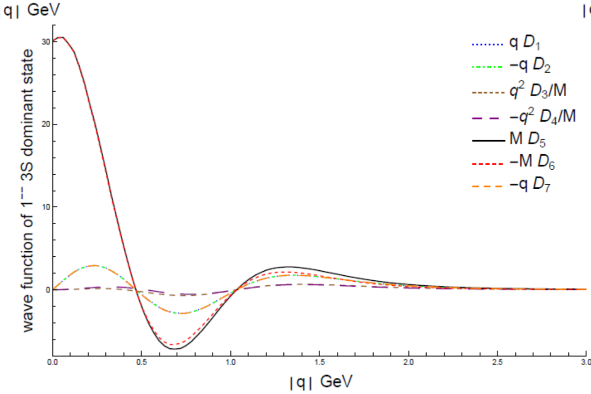
- 两类：（1）1S、2S、3S：S波为主，少量P波，D波最小

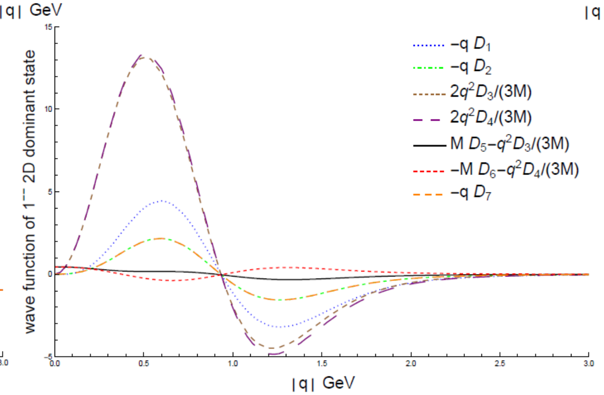
  （2）1D、2D、3D：D波为主，少量P波，S波最小



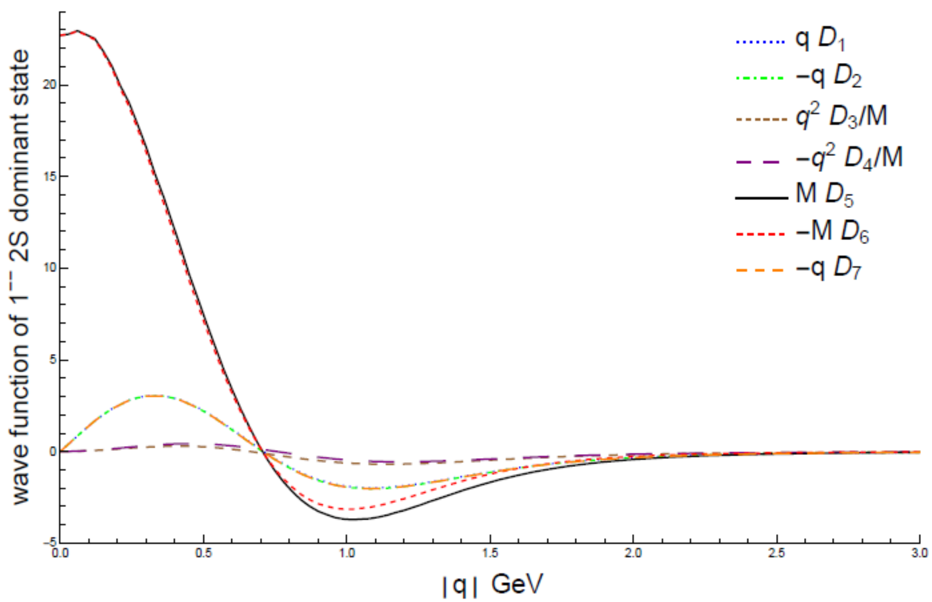


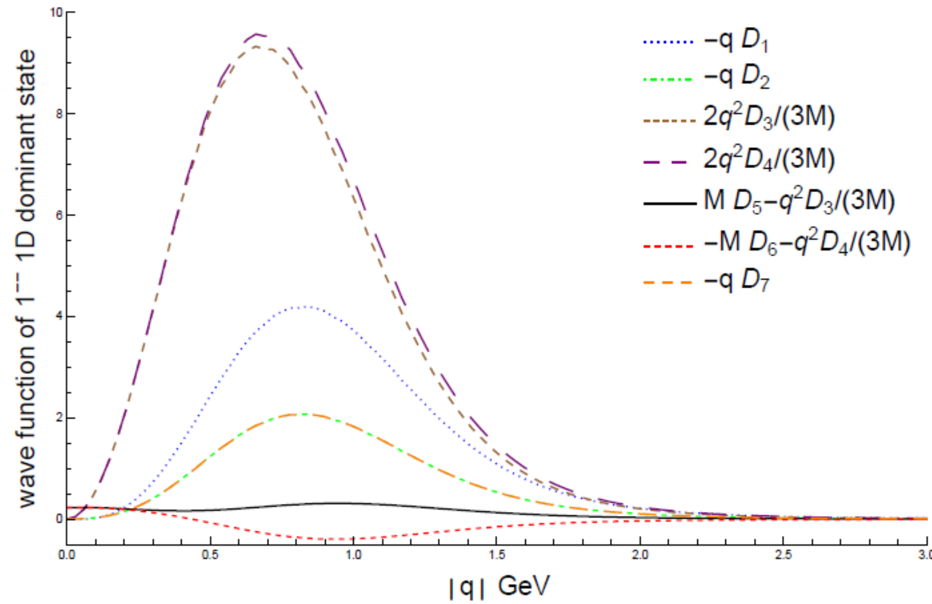






2S和1D径向波函数（放大）





Wave function of a 1^+ state and its partial waves

$$\varphi_P^{1^+}(q_\perp) = \epsilon\cdot q_\perp\left(f_1 + f_2\frac{\not P}{M} + f_3\frac{\not q_\perp}{M} + f_4\frac{\not q_\perp \not P}{M^2}\right)\gamma^5$$
$$+\frac{i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu q_\perp^\rho \epsilon^\sigma}{M}\left(g_1 + g_2\frac{\not P}{M} + g_3\frac{\not q_\perp}{M} + g_4\frac{\not q_\perp \not P}{M^2}\right),$$

f_i terms are 1^{+-} $({}^1P_1)$, g_i are 1^{++} $({}^3P_1)$, so it is ${}^1P_1 -{}^3 P_1$ mixing state.

$$\varphi_P^{1^+}(q_\perp) = \epsilon\cdot q_\perp\left(f_1 + f_2\frac{\not P}{M} - f_1 x_- \not q_\perp + f_2 x_+\frac{\not q_\perp \not P}{M}\right)\gamma^5$$
$$+\frac{i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu q_\perp^\rho \epsilon^\sigma}{M}\left(g_1 + g_2\frac{\not P}{M} - g_1 x_- \not q_\perp + g_2 x_+\frac{\not q_\perp \not P}{M}\right).$$

$$\textit{Wave function of a } 1^+ \textit{ state and its partial waves}$$

- Normalization

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1\omega_2\vec{q}^2}{3M(m_1\omega_2+m_2\omega_1)}(f_1f_2-2g_1g_2) \equiv \cos^2\theta+\sin^2\theta=1.$$

where the mixing angle is defined by wave function

- There is also P and D partial waves, f1,f2,g1,g2 terms are non-relativistic P waves, others are relativistic D waves.
- When m1=m2, the 1^+ decouples into two states: 1^{++} and 1^{+-}. This can be seen from the normalization formula. Later, we will use the 2^- as an.

$Wave\ function\ of\ a\ 1^+\ state\ and\ its\ partial\ waves$

- Solutions appear in pairs, first and second are 1P, third and fourth are 2P, etc
- First two $1^1P_1\ :\ 1^3P_1 = 0.284 : 0.716$ and $0.716 : 0.284$
  with mixing angle $\theta_{1P} = -57.8^\circ$ or 32.2°
- 3, 4 $2^1P_1 : 2^3P_1 = 0.263 : 0.737$ and $0.737 : 0.263$,

$$\theta_{2P} = -59.1^\circ \text{ or } 30.9^\circ.$$

- Pure P wave: $\varphi_{nP} = \theta_{nP}$
- $P\ :\ D = 1\ :\ 0.0971$ for two 1P
  $P : D = 1 : 0.0936$ for two 2P

1^+ 态Bc波函数和质量谱

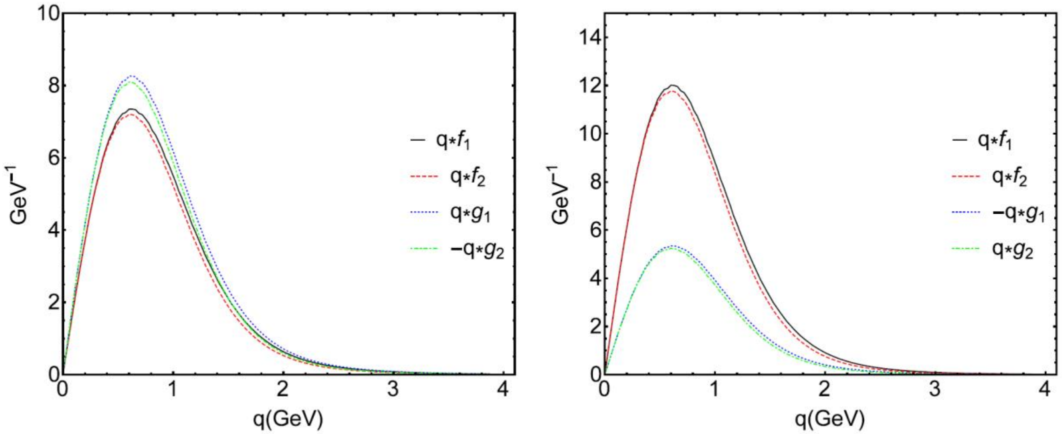


Figure 6. The 1^+ wave functions of the $1^1P_1 - 1^3P_1$ mixing states $B_{c1}(1P)$ (left) and $B'_{c1}(1P$ (right). f_1 and f_2 terms are $^1P_1$ waves; g_1 and g_2 terms are $^3P_1$ waves.

$1\ P_1$	1^+	6739 (input)
$1\ P'_1$	1^+	6748
θ_{1P}		$32.2°(-57.8°)$
$2\ P_1$	1^+	7144
$2\ P'_1$	1^+	7149
θ_{2P}		$30.9°(-59.1°)$

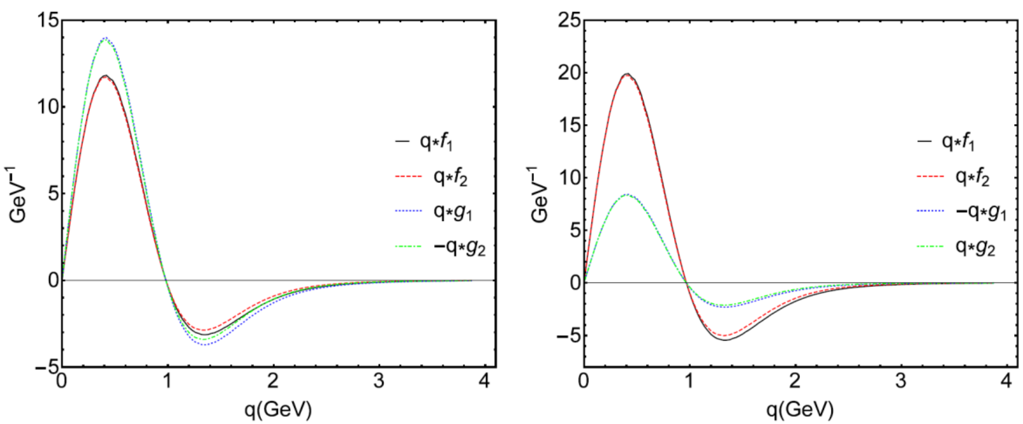


Figure 7. The 1^+ wave functions of the $2^1P_1 - 2^3P_1$ mixing states $B_{c1}(2P)$ (left) and $B'_{c1}(2P)$ (right). f_1 and f_2 terms are $^1P_1$ waves; g_1 and g_2 terms are $^3P_1$ waves.

2^-等质量退耦为2^{--}和2^{-+}两个态

- 2^-波函数

$$\varphi_{2^-}(q_\perp) = \epsilon_{\mu\nu} q_\perp^\mu q_\perp^\nu \left(a_1 + \frac{\not{P}}{M} a_2 + \frac{\not{q}_\perp}{M} a_3 + \frac{\not{P}\not{q}_\perp}{M^2} a_4 \right) \gamma^5 \qquad {}^1D_2$$

$$+ \frac{i\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu P^\nu q_\perp^\alpha \epsilon^{\beta\delta} q_{\perp\delta}}{M} \left(h_1 + \frac{\not{P}}{M} h_2 + \frac{\not{q}_\perp}{M} h_3 + \frac{\not{P}\not{q}_\perp}{M^2} h_4 \right) \qquad {}^3D_2$$

- 等质量时a3和h3的C宇称不对

$$a_3 = -a_1 \frac{M(\omega_1 - \omega_2)}{m_1\omega_2 + m_2\omega_1}, \; a_4 = -a_2 \frac{M(\omega_1 + \omega_2)}{m_1\omega_2 + m_2\omega_1},$$

但约束关系去掉了这两项

$$h_3 = h_1 \frac{M(\omega_1 - \omega_2)}{m_1\omega_2 + m_2\omega_1}, \; h_4 = h_2 \frac{M(\omega_1 + \omega_2)}{m_1\omega_2 + m_2\omega_1}.$$

- 可以求解完整2-态(取等质量)=单独求解两个态

${}^1D_2 \;\; 2^{-+}$

$$\varphi_{2^{-+}}(q_\perp) = \epsilon_{\mu\nu} q_\perp^\mu q_\perp^\nu \left(a_1 + \frac{\not{P}}{M} a_2 + \frac{\not{P}\not{q}_\perp}{M^2} a_4 \right) \gamma^5$$

${}^3D_2 \;\; 2^{--}$

$$\varphi_{2^{--}}(q_\perp) = i\epsilon_{\mu\nu\alpha\beta} \frac{P^\nu}{M} q_\perp^\alpha \epsilon^{\beta\delta} q_{\perp\delta} \gamma^\mu \left(f_1 + \frac{\not{P}}{M} f_2 + \frac{\not{P}\not{q}_\perp}{M m_c} f_2 \right)$$

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等质量时求解完整的2^-态解得2^{--}和2^{-+}态

- 质量

TABLE I. Our predictions of the mass spectra of 2^{--} and 2^{-+} charmonium in unit of MeV.

	2^{--}	2^{-+}
1D	3823.0 (input)	3828.2
2D	4153.7	4157.7
3D	4407.5	4410.7

- 波函数 3823MeV 3828.2

0.0000000000000000	0.0000000000000000	-7.636350659883E-002	7.5867462354225E-002
0.0000000000000000	0.0000000000000000	-0.30274732682145383	0.30069257145332684
0.0000000000000000	0.0000000000000000	-0.68037529849201783	0.67546879995471865
0.0000000000000000	0.0000000000000000	-1.1966593473083946	1.1872824633099552
0.0000000000000000	0.0000000000000000	-1.8322941595454281	1.8163917517492894
0.0000000000000000	0.0000000000000000	-2.5813336740654158	2.5563165589770049
0.0000000000000000	0.0000000000000000	-3.4228895665345571	3.3855444187070400
0.0000000000000000	0.0000000000000000	-4.3838628026473270	4.3301185882252371
0.0000000000000000	0.0000000000000000	-5.3617211631158774	5.2875150680711744
0.0000000000000000	0.0000000000000000	-6.3884576718862967	6.2888089004942511
0.0000000000000000	0.0000000000000000	-7.4017351053125937	7.2715708315447793
0.0000000000000000	0.0000000000000000	-8.5142885295266542	8.3464102283880326
0.0000000000000000	0.0000000000000000	-9.5208954457716821	9.3108560285864836
0.0000000000000000	0.0000000000000000	-10.502177505435727	10.243856628686087
0.0000000000000000	0.0000000000000000	-11.357393144543629	11.046548161414339
0.0000000000000000	0.0000000000000000	-12.299321220328759	11.926334605682644
0.0000000000000000	0.0000000000000000	-13.039975857692486	12.603529206472503
0.0000000000000000	0.0000000000000000	-13.707368577420588	13.202641828019113
0.0000000000000000	0.0000000000000000	-14.171894309646952	13.599292474808779
0.0000000000000000	0.0000000000000000	-14.723073338477983	14.071956251517925
0.0000000000000000	0.0000000000000000	-15.025817209481056	14.301539214542826
0.0000000000000000	0.0000000000000000	-15.235104169100920	14.436733212863446
0.0000000000000000	0.0000000000000000	-15.216085999073901	14.351544858299579
0.0000000000000000	0.0000000000000000	-15.296492125541139	14.355190881198757
0.0000000000000000	0.0000000000000000	-15.144244832687697	14.138970427942551
0.0000000000000000	0.0000000000000000	-14.921242533765668	13.854937636514066

-9.302185425020E-002	-9.23114227022E-002	5.1566068364009863E-013	-1.3967860403864375E-013
-0.36861703707916194	-0.3655679807351633	9.4872177839947850E-014	-1.3939553018328617E-013
-0.82826744007506348	-0.8209649654840550	1.6554649942520213E-013	-1.2902477878020497E-013
-1.4567175015350755	-1.4429153064296665	8.6349896139325040E-014	-1.1833691272914959E-013
-2.2303621414256818	-2.2072264587327730	8.5189901924930513E-014	-1.2486787429650342E-013
-3.1420808254689576	-3.1061969662506685	1.4202965864824957E-013	-1.7694583280762917E-013
-4.1665477170883127	-4.1138166237637970	1.0894567705099579E-013	-1.8349423617282226E-013
-5.3365187938043466	-5.2617306007446931	2.1274750717858029E-013	-2.1796758952632505E-013
-6.5271853823356247	-6.4252682974511037	2.3081781564552717E-013	-1.9710654323835709E-013
-7.7777185736425878	-7.6425522871697362	3.6892630221110988E-013	-3.7656519741673755E-013
-9.0122535467182807	-8.8376720774330693	4.9888776573142411E-013	-5.0660876154567237E-013
-10.368203449728179	-10.145360030210655	5.6103556225507090E-013	-5.6159075934483362E-013
-11.595558885157446	-11.319171583181495	6.7410945869329991E-013	-6.1869834858444208E-013
-12.792533043772305	-12.455154190948297	7.3937326647968909E-013	-6.7587951630411969E-013
-13.836587471515111	-13.433285594469723	7.1157338667502679E-013	-6.6563909461133732E-013
-14.987268259899491	-14.506477833007052	7.8061380309764793E-013	-7.4102971255103848E-013
-15.892942818130956	-15.333153198995973	7.7730377902327512E-013	-7.5356528071375926E-013
-16.710233321671787	-16.065910043500288	8.2018452104026988E-013	-8.3452860639058514E-013
-17.280724874741924	-16.552597171936068	8.3399376264604091E-013	-7.9299227232316744E-013
-17.957758861130472	-17.132939317336664	8.4794843331535058E-013	-8.8033169621656284E-013
-18.332067087370625	-17.417326607591768	8.8803102430813197E-013	-8.7313789122863651E-013
-18.593000094467818	-17.587451439352161	9.7512516979355856E-013	-9.1274625287849784E-013
-18.575789586495706	-17.489430678988665	9.9273837182891178E-013	-9.7384505765640345E-013
-18.680400384347685	-17.500181314074350	9.4113787199254048E-013	-8.5993766267994100E-013
-18.501256340710572	-17.243184283774134	1.0169622802364849E-012	-8.9708113528656717E-013
-18.235807452898918	-16.903496154500989	9.4969481350920449E-013	-8.5848349219867697E-013
-17.726913391627090	-16.338814178087969	9.0326831834056376E-013	-8.1989973696249287E-013
-17.359795271699543	-15.903245464482591	7.9963879071963708E-013	-7.1234935421463791E-013

重强子谱：粲偶素和底偶素

Table 2 Mass spectra of $(c\bar{c})$ and $(b\bar{b})$ systems with quantum numbers $J^{PC} = 0^{-+}, 1^{--}$. Here ${}^{(2S+1)}L_J$ denotes the dominant component in the state. 'Ex' means the experimental results from PDG [17] and the data for η_b come from ref. [18]

$n\ J^{PC}({}^{(2S+1)}L_J)$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $0^{-+}({}^1S_0)$	2980.3(input)	2980.3	9390.2(input)	9388.9
2 $0^{-+}({}^1S_0)$	3576.4	3637	9950.0	
3 $0^{-+}({}^1S_0)$	3948.8		10311.4	
1 $1^{--}({}^3S_1)$	3096.9(input)	3096.916	9460.5(input)	9460.30
2 $1^{--}({}^3S_1)$	3688.1	3686.09	10023.1	10023.26
3 $1^{--}({}^3D_1)$	3778.9	3772.92	10129.5	
4 $1^{--}({}^3S_1)$	4056.8	4039	10368.9	10355.2
5 $1^{--}({}^3D_1)$	4110.7	4153	10434.7	
6 $1^{--}({}^3S_1)$	4329.4	4421	10635.8	10579.4
7 $1^{--}({}^3S_1)$	4545.9		10852.1	10865

Table 3 Mass spectra of $(c\bar{c})$ and $(b\bar{b})$ systems with quantum numbers $J^{PC} = 0^{++}, 1^{++}, 2^{++}, 1^{+-}$ in unit of MeV. Here ${}^{(2S+1)}L_J$ denotes the dominant component in the state. 'Ex' means the experimental results from PDG [17]

$n\ J^{PC}({}^{2S+1})L_J$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $0^{++}({}^3P_0)$	3414.7(input)	3414.75	9859.0	9859.44
2 $0^{++}({}^3P_0)$	3836.8		10240.6	10232.5
3 $0^{++}({}^3P_0)$	4140.1		10524.7	
1 $1^{++}({}^3P_1)$	3510.3(input)	3510.66	9892.2	9892.78
2 $1^{++}({}^3P_1)$	3928.7		10272.7	10255.46
3 $1^{++}({}^3P_1)$	4228.8		10556.2	
1 $2^{++}({}^3P_2)$	3556.1(input)	3556.20	9914.4	9912.21
2 $2^{++}({}^3P_2)$	3972.4		10293.6	10268.65
3 $2^{++}({}^3F_2)$	4037.9		10374.4	
4 $2^{++}({}^3P_2)$	4271.0		10561.5	
1 $1^{+-}({}^1P_1)$	3526.0(input)	3525.93	9900.2	
2 $1^{+-}({}^1P_1)$	3943.0		10280.4	
3 $1^{+-}({}^1P_1)$	4242.4		10562.0	

重强子谱：Bc介子

$n\ ^{2S+1}L_J$	J^P	ours	[38]	[39, 40]	[41]	[42]	[55]	Exp
$1\ ^1S_0$	0^-	6277 (input)	6271	6272	6275	6271 (input)	6276	6274.9±0.8 [56]
$1\ ^3S_1$	1^-	6332 (input)	6338	6333	6329	6326 (input)	6331	6333 [27]
$2\ ^1S_0$	0^-	6867	6855	6842	6867	6871 (input)		6871.6±1.1 [56]
$2\ ^3S_1$	1^-	6911	6887	6882	6898	6890		6900.1 [27]
$3\ ^1S_0$	0^-	7228	7250	7226	7254	7239		
$3\ ^3S_1$	1^-	7272	7272	7258	7280	7252		
$1\ ^3P_0$	0^+	6705 (input)	6706	6699	6693	6714	6712	
$1\ P_1$	1^+	6739 (input)	6741	6743	6731	6757	6736	
$1\ P_1'$	1^+	6748	6750	6750	6739	6776		
θ_{1P}		$-57.8^\circ(32.2^\circ)$	22.4°	20.5°	18.7°	35.5°	$33.4 \pm 1.5^\circ$ [37]	
$1\ ^3P_2$	2^+	6762 (input)	6768	6761	6751	6787		
$2\ ^3P_0$	0^+	7112	7122	7094	7105	7107		
$2\ P_1$	1^+	7144	7145	7134	7136	7134		
$2\ P_1'$	1^+	7149	7150	7147	7144	7150		
θ_{2P}		$-59.1^\circ(30.9^\circ)$	18.9°	23.2°	21.2°	38.0°		

重强子谱：Bc介子

$2\ {}^3P_2$	2^+	7163	7164	7157	7155	7160
$3\ {}^3P_0$	0^+	7408		7474	7437	7420
$3\ P_1$	1^+	7440		7500	7465	7441
$3\ P_1'$	1^+	7442		7510	7474	7458
θ_{3P}		$-60.1^\circ(29.9^\circ)$				39.7°
$3\ {}^3P_2$	2^+	7456		7524	7483	7464
$n\ {}^{2S+1}L_J$	J^P	ours	[38]	[39, 40]	[41]	[42]
$1\ {}^3D_1$	1^-	7014 $(S-P-D)$	7028	7021	7007	7020
$2\ {}^3D_1$	1^-	7335 $(S-P-D)$		7392	7347	7336
$1\ {}^3F_2$	2^+	7239 $(P-D-F)$	7269	7232	7234	7235
$2\ {}^3F_2$	2^+	7508 $(P-D-F)$		7618		7518
$1\ {}^3D_3$	3^-	7035 (input)	7045	7029	7011	7030
$1\ D_2$	2^-	7025 (input)	7036	7025	7006	7024
$1\ D_2'$	2^-	7029	7041	7026	7016	7032
θ_{1D}		$-58.2^\circ(31.8^\circ)$	44.5°	-35.9°	-49.2°	45.0°
$2\ {}^3D_3$	3^-	7355		7405	7351	7348
$2\ D_2$	2^-	7345		7399	7339	7343
$2\ D_2'$	2^-	7349		7400	7359	7347
θ_{2D}		$-57.4^\circ(32.6^\circ)$			-40.3°	45.0°

Table 1. Mass spectra and mixing angles of the B_c system.

Summary

- J^P是更基本的量子数
- 依据J^P 给出普遍的波函数：相对论的；三大类；不是纯波，都含有其他分波
- 求解一个完整的J^P态的Salpter方程，得混合态：

（1）第一类混合：1-，2+，3-对应S-D，P-F，
以及D-G类型的混合

（2）第二类混合：1+，2-，3+对应$^3P_1$-$^1P_1$，
$^3D_2$-$^1D_2$，$^3F_3$-$^1F_3$类型的混合