轻赝标介子的形状因子及其精细结构研究

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Overview

Form Factors

"In elementary particle physics and mathematical physics, in particular in effective field theory, a **form factor** is a function that encapsulates the properties of a certain interaction without including all of the underlying physics, but instead, providing the momentum dependence of a suitable matrix elements. Its further measured experimentally in confirmation or specification of a theory."

Momenta Redistribution

$\Downarrow$ QCD is believed to exhibit confinement

hadron structures $\otimes$ hard scattering

$\Downarrow$ decoupling of LD and SD interactions

factorisation theorem, EFT; g-2, CKM, B anomalies

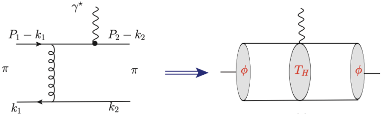
Form Factors

PION is the lightest Glodstone boson and one of the most simplest hadrons, hence the ideal probe to extremely rich QCD dynamics.

- (spacelike) electromagnetic form factor

$$\langle \pi^-(p_2) \big| J_\mu^{\rm em} \big| \pi^-(p_1) \rangle = e_q \, (p_1 + p_2)_\mu \, F_\pi(Q^2)$$

- the interaction distance of $J_\mu^{\rm em}$ is decided by the external reason Q^2
- Separate the hard partonic physics out of the hadronic physics (soft, nonperturbative objects) in exclusive processes **Factorization**



(a) (b)

$$\frac{d\sigma}{d\Omega} = \int_0^1 du_i \, \phi(u_1) \, T_H(u_i) \phi(u_2)$$

- The universal nonperturbative objects, studied by QCD-based analytical (QCDSRs, χPT, DSE, instanton) and numerical approaches (LQCD)
- also by data-driven method, CETQ, CT, MMHT, NNPDF, ABM, JAM, et.al. need precise QCD calculation of T_H as the inputs

The perturbative QCD approach

i Three-scale factorization

ii The soft-transversal dynamics

Three-scale Factorization

Exclusive Processes in Perturbative Quantum Chromodynamics #1

G.Peter Lepage (Cornell U., LNS), Stanley J. Brodsky (SLAC) (Mar, 1980)

Published in: *Phys.Rev.D* 22 (1980) 2157

pdf links DOI cite claim reference search 4,122 citations

Factorization and Asymptotical Behavior of Pion Form-Factor in QCD #1

A.V. Efremov (Dubna, JINR), A.V. Radyushkin (Dubna, JINR) (Nov, 1979)

Published in: *Phys.Lett.B* 94 (1980) 245-250

pdf DOI cite claim reference search 1,333 citations

- the first rigorous pQCD predictions to the entire domain of larger-momentum-transfer exclusive reactions

$$\mathcal{F}_\pi(Q^2) = \int_0^1 du_i\, \phi(u_1, \tilde{Q}_1)\, T_H(u_i, Q)\, \phi(u_2, \tilde{Q}_2)$$

‡ amplitudes are dominated by quark and gluon subprocesses at SDs

‡ evolution equations for process-independent hadron DAs $\psi(x_i, \tilde{Q})$

finding the constituents with light-cone momentum fraction x_i at small transversal separations

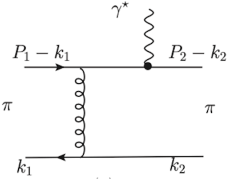
‡ leading twist DAs and α_s order calculation

prevents anomalous contributions from the end-point $x_i \sim 1$ integration regions

Three-scale Factorization

- End-point singularities appear at high twists

‡ $m_{1,2}^2 \ll Q^2$, $p_2 = (\frac{Q}{\sqrt{2}}, 0, 0_{\rm T})$, $p_3 = (0, \frac{Q}{\sqrt{2}}, 0_{\rm T})$, $k_2 = x_2 p_2$, $\bar{k}_2 = \bar{x}_2 p_2$



$\phi \propto u(1-u)$, $m_0^\pi \phi^{P,\sigma} \propto 1$

$$\propto \sum_t \int du_1 du_2 \kappa_t(u_i) \frac{\alpha_s(\mu)\phi_1^t(u_1)\phi_2^t(u_2)}{(u_1 u_2 Q^2)(u_2 Q^2)}$$

‡ pick up k_T in the internal propagators

$$\mathcal{M} \propto \sum_{t=2,3,4} \int du_1 du_2 dk_{1T} dk_{2T} \kappa_t(u_i) \frac{\alpha_s(\mu)\phi_1^t(u_1)\phi_2^t(u_2)}{[u_1 u_2 Q^2 - (\triangle k_T)^2]\,(u_2 Q^2 - k_{2T}^2)}$$

‡ end-point singularity at leading and subleading powers

$$\mathcal{H} \propto \frac{\alpha_s(\mu)}{u_1 u_2 Q^2 - k_T^2} \sim \frac{\alpha_s(\mu)}{u_1 u_2 Q^2} - \alpha_s(\mu) \frac{k_T^2}{(u_1 u_2 Q^2)^2} + \cdots$$

‡ the power suppressed TMD terms becomes important at the end-points

Three-scale Factorization

A Study of the Applicability of Perturbative {QCD} to the Pion Form-factor #1

Tao Huang (CCAST World Lab, Beijing and Beijing, Inst. High Energy Phys.), Qi-xing Shen (CCAST World Lab, Beijing and Beijing, Inst. High Energy Phys.), Peter Kroll (Wuppertal U.) (Aug, 1989)

Published in: *Z.Phys.C* 50 (1991) 139-144 • Contribution to: Quarks 90

DOI cite claim reference search 75 citations

Analysis of the pion wave function in light cone formalism #1

Tao Huang (CCAST World Lab, Beijing and Beijing, Inst. High Energy Phys.), Bo-Qiang Ma (CCAST World Lab, Beijing and Beijing, Inst. High Energy Phys.), Qi-Xing Shen (CCAST World Lab, Beijing and Beijing, Inst. High Energy Phys.) (Mar 8, 1994)

Published in: *Phys.Rev.D* 49 (1994) 1490-1499 • e-Print: hep-ph/9402285 [hep-ph]

pdf DOI cite claim reference search 132 citations

微扰量子色动力学应用到遍举过程中的几个问题	曹俊	1998	黄涛	博士

- introduce k_T to regularize the end-point singularity

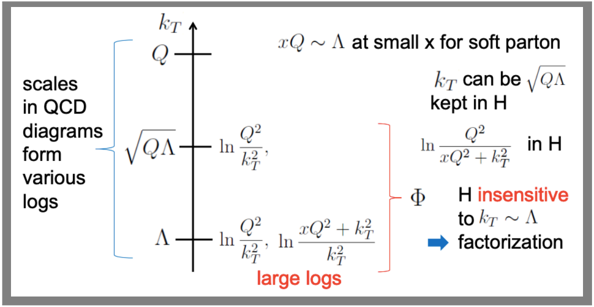
$$\mathcal{F}_\pi(Q^2) = \int_b^{\bar{b}} du_1 \int_{a/u_1}^{1-a/\bar{u}_1} du_2\, \phi(u_1, k_{1T})\, T_H(u_i, Q)\phi(u_2, k_{2T})$$

‡ constraints on the integration region $b = \left(1 - \sqrt{1-4a}\right)/2, \;\; a = \langle k_T^2 \rangle / Q^2, \;\; \langle k_T \rangle \sim 300$ MeV

‡ leading twist DAs within different b-dependent models, also at α_s order

Three-scale Factorization

- k_T varies within three scales [stolen from H.N Li]



‡ large single and double logarithms from QCD corrections, ie., $\alpha_s(\mu) \ln^2 \frac{k_T^2}{m_B^2}$

‡ k_T resummation for T to obtain $S(u_i, b_i, Q)$ suppresses the large transversal distances (small k_T) interactions by decreasing q^2 power in denominator

- integrating over k_T, $\ln^2(x_i)$ resides when the internal parton is on shell
- threshold resummation for ψ to obtain $S_t(x_i, Q)$ suppresses the small x_i regions, repairs the self-consistency between $\alpha_s(t)$ and hard log $\ln(u_1 u_2 Q^2/t^2)$

Three-scale Factorization

Hard Elastic Scattering in QCD: Leading Behavior #1

James Botts (SUNY, Stony Brook), George F. Sterman (SUNY, Stony Brook) (Mar 20, 1989)

Published in: *Nucl.Phys.B* 325 (1989) 62-100

DOI | cite | claim | reference search | 598 citations

The Perturbative pion form-factor with Sudakov suppression #1

Hsiang-nan Li (SUNY, Stony Brook), George F. Sterman (SUNY, Stony Brook) (Mar, 1992)

Published in: *Nucl.Phys.B* 381 (1992) 129-140

DOI | cite | claim | reference search | 527 citations

Unification of the k(T) and threshold resummations #1

Hsiang-nan Li (Taiwan, Natl. Cheng Kung U.) (Dec, 1998)

Published in: *Phys.Lett.B* 454 (1999) 328-334 • e-Print: hep-ph/9812363 [hep-ph]

pdf | DOI | cite | claim | reference search | 89 citations

$$\mathcal{F}_\pi(Q^2) = \psi(u_1, \mu_{r_1})\, T_H(u_i, b_i, Q) S_t(u_i)\, e^{-S(u,b,Q)}\, \psi(u_2, \mu_{r_2})$$

- threshold-suppressed hard amplitude $T_H S_t$
- sudokov-multiplied light-cone distribution amplitudes ψe^{-S}
- leading twist & QCD leading order & resolution of endpoint singularities

Three-scale Factorization

‡ H.N. Li, Y.L .Shen, Y.M. Wang and H. Zou, PRD 83.054029 (2011)
twist 2@NLO+twist 3@LO

‡ **SC**, Y.Y. Fan and Z.J. Xiao, PRD 89.054015 (2014)
twist 2@NLO+twist 3@NLO

‡ **SC**, PRD 100.013007 (2019)
twist 2@NLO+twist 3@NLO+twist 4@LO, scale revolutions

‡ H.N. Li, Y.L. Shen and Y.M. Wang, JHEP 01(2014)004
joint resummation for $\ln(x_1 x_2 Q^2 b^2)$ for large b & small $x_1 x_2 Q^2$

‡ H.N. Li and Y.M. Wang, JHEP 06(2015)013
rapidity singularity and pinch singularity, non-dipolar Wilson links for TMDWFs

$$\mathcal{F}_\pi(Q^2) = \sum_{t_i} \psi^{t_1}(u_1, \mu_{r_1})\, T_H^{t_i,\mathrm{LO+NLO}}(u_i, b_i, Q)\, S_t(u_i)\, e^{-S(u_i,b,Q)}\, \psi^{t_2}(u_2, \mu_{r_2})$$

- high twist contributions and more fruitful hadron structures
- NLO QCD corrections in hard kernel and TMDWFs
- hard scale choice [*PMC*, Majaza, Brodsky and Wu, PRL 109.042002(2012), 110.192001(2013)]
- N^{2}LO from QCD collinear factorization leading twist
  [Chen^2, Feng and Jia, PRL 132. 201901(2024)], [Ji, Shi, Wang^3 and Yu, PRL 134. 221901(2025)]

Three-scale Factorization $\mathcal{F}_\pi(Q^2)$

$$\int_0^1 du_i \phi(u_1, \tilde{Q}_1)\, T_H(u_i, Q) \phi(u_2, \tilde{Q}_2) \quad \boxed{\text{1980s}}$$

$$\Downarrow$$

$$\int_b^{\bar{b}} du_1 \int_{a/u_1}^{1-a/\bar{u}_1} \phi(u_1, k_{1T})\, T_H(u_i, Q) \phi(u_2, k_{2T}) \quad \boxed{\text{1990s}}$$

$$\Downarrow$$

$$\psi(u_1, \mu_{r_1})\; T_H(u_i, b_i, Q)\; S_t(u_i)\; e^{-S(u_i, b_i, Q)}\; \psi(u_2, \mu_{r_2}) \quad \boxed{\text{2000s}}$$

$$\Downarrow$$

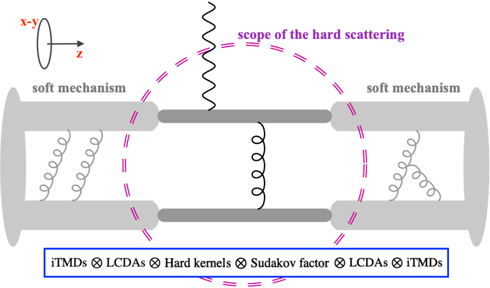
$$\sum_{t_i} \psi^{t_1}(u_1, \mu_{r_1})\; T_H^{t_i, \mathrm{LO+NLO}}(u_i, b_i, Q) S_t(u_i)\; e^{-S(u_i, b, Q)}\; \psi^{t_2}(u_2, \mu_{r_2}) \quad \boxed{\text{2010s}}$$

- $T_H(u_i, b_i, Q) S_t(x_i, Q)$ threshold-suppressed hard scattering amplitude
  including both the longitudinal and transversal dynamics
- $e^{-S(u_i, b_i, Q)} \psi(u_i, \mu_r)$ sudokov-multiplied LCDAs wave functions at zero transversal separations $b_i \sim 0$, only the soft longitudinal dynamics, oversight of the soft transversal dynamics (intrinsic transverse momentum distributions) $\Downarrow$

$$\sum_{t_i} \psi^{t_1}(u_1, b_1, \mu_{r_1})\; T_H^{t_i, \mathrm{LO+NLO}}(u_i, b_i, Q)\; S_t(u_i)\; e^{-S(u_i, b, Q)}\; \psi^{t_2}(u_2, b_2, \mu_{r_2}) \quad \boxed{\text{2020s}}$$

The soft-transversal dynamics

the sketch map of pion electromagnetic form factor



[J. Chai and **SC**, PRD 111. L071902]

‡ central region of the e.m potential field
picks up the hard radiations of partons on the transversal plane

‡ outside the scope of hard scattering
energetic pions move fast along the z direction, accompanied by soft bremsstrahlung radiations, absorbed into the effects of high twist LCDAs

‡ in the exterior region, the soft radiations in the transversal plane are notably, but absent from the definition of LCDAs

- the soft pion wave function is generally to a product of LCDA and iTMDs

$$\langle 0|\bar{u}(x)\Gamma[x^-, x_\perp; 0, 0_\perp] d(0)|\pi^-(p)\rangle \propto \int du dk_\perp^2\, e^{iup^+x^- - ik_\perp\cdot x_\perp}\, \psi(u, k_T),$$

$$\psi(u, k_T) = \frac{f_{\mathcal{P}}}{2\sqrt{6}}\varphi(u, \mu_r)\Sigma(u, k_T), \quad \int_0^1 du\varphi(u, \mu_r) = 1, \int \frac{d^2k_\perp}{16\pi^3}\Sigma(u, k_T) = 1.$$

$$\mathcal{F}_\pi(Q^2) = \sum_{t_i} \psi^{t_1}(u_1, b_1, \mu_{r_1})\, T_H^{t_i,\mathrm{LO+NLO}}(u_i, b_i, Q)\, S_t(u_i)\, e^{-S(u_i,b,Q)}\, \psi^{t_2}(u_2, b_2, \mu_{r_2})$$

The soft-transversal dynamics

- a simple gaussian function with preserving rotational invariance

$$\Sigma(u,k_T) = 16\pi^2 \frac{\beta^2}{u(1-u)} e^{-\frac{\beta^2 k_T^2}{u(1-u)}} \Rightarrow \hat{\Sigma}(u,b_T) = 4\pi e^{-\frac{b_T^2 u(1-u)}{4\beta^2}}.$$

[Jakob, Kroll, PLB 315(1993)463]

- iTMDs associated to two-particle twist three LCDAs

$$\psi^{p,\sigma}(u,\mu) = \int \frac{d^2k_T}{16\pi^3}\varphi_{2p}^{p,\sigma}(u,\mu)\Sigma(u,k_T) + \int \frac{d^2k_{1T}d^2k_{2T}}{64\pi^5}\rho_+\varphi_{3p}^{p,\sigma}(u,\mu)\int \mathcal{D}\alpha_i \Sigma'(\alpha_i,k_{iT}),$$

$$\int \frac{d^2k_{1T}d^2k_{2T}}{64\pi^5}\int \mathcal{D}\alpha_i\Sigma'(\alpha_i,k_{iT}) = 1, \quad \int_0^1 du\,\varphi_{2p}^{p,\sigma}(u,\mu) = 1, \quad \int_0^1 du\,\varphi_{3p}^{p,\sigma}(u,\mu) = 0,$$

$$\hat{\Sigma}'(\alpha_i,b_1,b_2) = 4\pi e^{-\frac{2\alpha_3(b_1^2+b_2^2)+(\alpha_1+\alpha_2)(b_1-b_2)^2}{16\beta'^2}}.$$

- two transversal-size parameters β^2 and β'^2

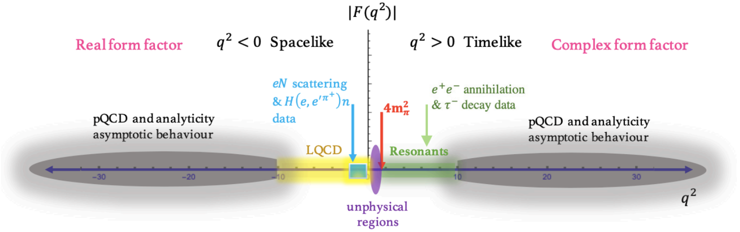
‡ asymptotic behavior of $F_{\pi\gamma\gamma^*}$: $\beta_\pi^2 = \frac{1}{8\pi^2 f_\pi^2\left(1+a_2^\pi+a_4^\pi+\cdots\right)} = 0.51 \pm 0.04$ GeV^{-2}

‡ corresponds to the mean transversal momentum $\left[\langle k_T^2\rangle\right]^{\frac{1}{2}} \equiv \left[\frac{\int dud^2k_T k_T^2|\psi(u,k_T)|^2}{\int dud^2k_T|\psi(u,k_T)|^2}\right]^{\frac{1}{2}} = 358 \pm 15$ MeV, revealing the soft transversal dynamics in the soft wave function.

‡ $\beta_K^2 = 0.30 \pm 0.05$ GeV^{-2} is obtained by fitting to the data of FFs, $\left[\langle k_T^2\rangle\right]_K^{\frac{1}{2}} = 0.55 \pm 0.07$ MeV

EMFFs and TFFs

Pion electromagnetic form factor



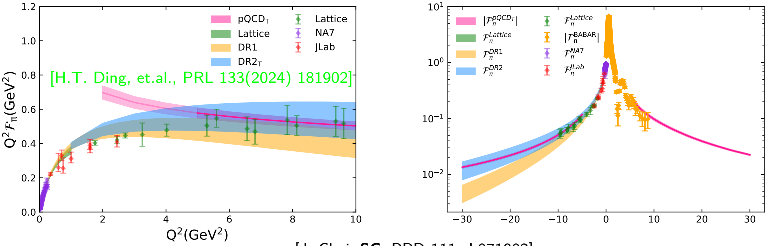
Kinematical clarification of pion electromagnetic form factor

- mismatch between the QCD based calculation and the available data
- could be restored by employing the dispersion relation
- The standard dispersion relation and The modulus representation

$$F_\pi(q^2) = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}F_\pi(s)}{s-q^2-i\epsilon}, \quad q^2 < s_0 \quad \Downarrow$$

$$F_\pi(q^2) = \exp\left[\frac{q^2\sqrt{s_0-q^2}}{2\pi}\int_{s_0}^{\infty}\frac{ds\,\ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}\right]$$ [**SC**, Khodjamirian, Rosov, PRD 102.074022 (2020)]

Pion electromagnetic form factor



[J. Chai, **SC**, PRD 111. L071902]

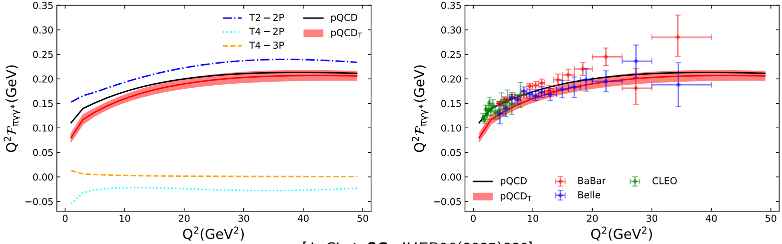
- take the modular DR to fit chiral mass, obtain $m_0^\pi(1\,\mathrm{GeV}) = 1.84 \pm 0.07$ GeV

  larger than the previous pQCD result ~ 1.37 GeV [J. Chai, **SC** and J. Hua, EPJC 83. 556(2023)]
  consists with the ChPT ~ 1.79 GeV [H. Leutwyler PLB 378(1996)313-318]
  a significant decrease of the FF due to the soft transversal dynamics in the small and intermediate q^2.

- a mean-square transverse charge radius $\langle b_\pi^2 \rangle = 0.30 \pm 0.03$ fm^2, exhibits excellent agreement with the purelly dimensional relation $\langle b_\pi^2 \rangle = 2/3 \langle r_\pi^2 \rangle$
  $\langle r_\pi^2 \rangle = 0.45 \pm 0.01$ fm^2 is the mean-square charge radius of the pion [PDG 2024]

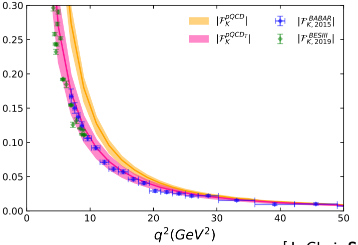
Pion transition form factor

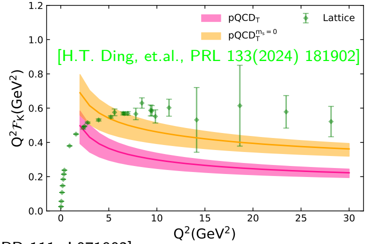
- $F_{\pi\gamma\gamma^*}$ is the theoretically most clean observable $\propto a_n^\pi$
- Hadronic light-by-light scattering (HLbL) contribution to $a_\mu^{HLbL;\pi^0}$
- In 2009, BaBar collaboration reported the measurement exceeding the asymptotic QCD prediction $Q^2\mathcal{F}_{\pi\gamma\gamma^*}(Q^2) = \sqrt{2}f_\pi$ in $q^2 \leqslant 10$ GeV^2
- flat DAs ? fruitful structures (polynomials) in leading twist LCDAs ?
- The attractive pion TFF is heat off with the measurement from Belle collaboration in 2012, which shows a consistent with the asymptotic QCD limit
- settle down the "fat pion" issue at Belle II, BESIII, JLab and future colliders ?



[J. Chai, **SC**, JHEP06(2025)229]

Kaon electromagnetic form factor

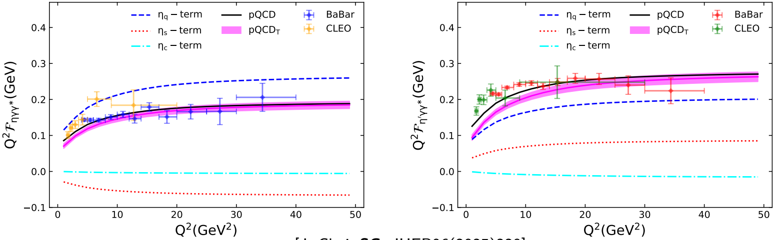




[J. Chai, **SC**, PRD 111. L071902]

- $m_0^K(1\,\mathrm{GeV}) = 1.90 \pm 0.09$ GeV, CHPT relation without involving light quark masses
- fit the transversal-size parameter $\beta_K^2 = 0.30 \pm 0.05$ GeV^{-2} from timelike data
  settle for the second best and take $\beta_K^2 = \beta_K'^2$
- $\langle b_K^2 \rangle = 0.14 \pm 0.03$ fm^2, $\langle b_K^2 \rangle \sim 1/2 \langle r_K^2 \rangle$, $\langle r_K^2 \rangle = 0.31 \pm 0.03$ fm^2 [PDG 2024]
- the iTMDs is indispensable to explain the data in the intermediate q^2
- iTMDs-improved pQCD result of spacelike FF is small than the lattice data
  agrees with results obtained from the DSE approach and the collinear QCD factorization
  large $SU(3)$ flavor breaking emerges an additional term proportional to m_s in the twist-three LCDAs

η, η' transition form factors



[J. Chai, **SC**, JHEP06(2025)229]

- up to three-particle twist-four accuracy of η_q, η_s, η_c TFFs

$$\mathcal{F}_{\eta\gamma\gamma^*} = \cos\phi \, \frac{e_u^2 + e_d^2}{\sqrt{2}} \mathcal{F}_{\eta_q\gamma\gamma^*} - \sin\phi \, e_s^2 \, \mathcal{F}_{\eta_s\gamma\gamma^*} - 0.006 \, e_c^2 \, \mathcal{F}_{\eta_c\gamma\gamma^*} \, ,$$

$$\mathcal{F}_{\eta'\gamma\gamma^*} = \sin\phi \, \frac{e_u^2 + e_d^2}{\sqrt{2}} \mathcal{F}_{\eta_q\gamma\gamma^*} + \cos\phi \, e_s^2 \, \mathcal{F}_{\eta_s\gamma\gamma^*} - 0.016 \, e_c^2 \, \mathcal{F}_{\eta_c\gamma\gamma^*} \, .$$

- predominantly η_q component in $F_{\eta\gamma\gamma^*}$, dominantly η_q and important η_s components in $F_{\eta^*\gamma\gamma^*}$, η_c component contribution is negligible in magnitude
- The iTMD-improved pQCD predictions of η TFF favors the smaller mixing angles $37.7^\circ \pm 0.7^\circ$

η, η' transition form factors

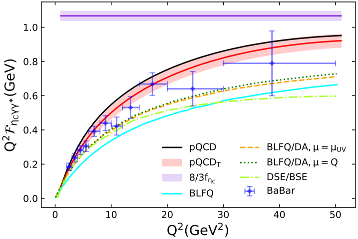
- $\mathcal{F}_{\eta_q\gamma\gamma^*} = \mathcal{F}_{\eta_s\gamma\gamma^*} = \mathcal{F}_{\eta_c\gamma\gamma^*} = \mathcal{F}_{\pi\gamma\gamma^*}$ in the pQCD limit, hence

$$\delta\mathcal{F} \equiv \mathcal{F}_{\eta\gamma\gamma^*} - \mathcal{F}_{\eta'\gamma\gamma^*} \overset{Q^2\to\infty}{\longrightarrow} (0.071 \pm 0.032)\sqrt{2}f_\pi = 0.013\pm0.006$$

errors from $\phi = 39.6^\circ \pm 2.6^\circ$

- compable with the BABAR measurement at $Q^2 = 112\,\mathrm{GeV}^2$:

$$\delta\mathcal{F} = 0.25^{+0.02}_{-0.02} - 0.23^{+0.03}_{-0.03} = 0.02 \pm 0.02$$



[J. Chai, **SC**, JHEP06(2025)229]

a larger value $\beta^2_{\eta_c} = 0.60$ GeV^{-2}

along with $\mathcal{F}_{\eta_c\gamma\gamma^*}(0) = 0.092$ GeV^{-1}

yields $\langle b^2_{\eta_c}\rangle = 0.058$ fm^2

$\sim \langle r^2_{\eta_c}\rangle$ obtained from the LQCD [J. Delaney et al., 2024]

- the large value of $\mathcal{F}_{\eta_c\gamma\gamma^*}(0) = 0.092$ GeV^{-1} is also confirmed by the recent BESIII measurement of $\Gamma(\eta_c \to \gamma\gamma) = 11.3 \pm 1.47$ keV [PRL 134.181901(2025)]
which deviates from the corresponding world-average value by 3.4σ.

Conclusion

- pQCD is a powerful approach to study an exclusive QCD process
- the LCDAs description of hadron oversights the soft transversal dynamics
- the universal soft function is actually a product of LCDAs and iTMDs

- we study the EMFFs and TFFs of light pseudoscalar mesons in the iTMDs-improved pQCD approach
- find the better agreements with the data and improve the prediction power down to a few GeV^2

- highly precise measurements are highly anticipated to reveal more information

Thank you for your patience.

- Introducing an auxiliary function $g_\pi(q^2) \equiv \frac{\ln F_\pi(q^2)}{q^2\sqrt{s_0-q^2}}$ [Geshkenbein 1998]
- Cauchy theorem and Schwartz reflection principle

$$g_\pi(q^2) = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}\, g_\pi(s)}{s-q^2-i\epsilon}$$

- At $s > s_0$ on the real axis, the imaginary part of g_π reads as

$$\mathrm{Im}\, g_\pi(s) = \mathrm{Im}\left[\frac{\ln(|F_\pi(s)|e^{i\delta_\pi(s)})}{-is\sqrt{s-s_0}}\right] = \frac{\ln|F_\pi(s)|}{s\sqrt{s-s_0}},$$

- Substituting $g_\pi(q^2)$ and $\mathrm{Im}\, g_\pi(s)$ into the dispersion relation, for $q^2 < s_0$

$$\frac{\ln F_\pi(q^2)}{q^2\sqrt{s_0-q^2}} = \frac{1}{2\pi}\int_{s_0}^{\infty} \frac{ds\, \ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}$$

- The modulus representation [**SC**, Khodjamirian, Rosov PRD 102.074022 (2020)]

$$F_\pi(q^2 < s_0) = \exp\left[\frac{q^2\sqrt{s_0-q^2}}{2\pi}\int_{s_0}^{\infty} \frac{ds\, \ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}\right]$$