SOUTH CHINA NORMAL UNIVERSITY

Machine Learning Unveils the Power Law of Finite-Volume Energy Shifts

Zhenyu Zhang (张振宇)

第八届强子谱和强子结构研讨会

桂林，广西师范大学，2025.7.13

Wei-Jie Zhang, Zhenyu Zhang, Jifeng Hu, Bing-Nan Lu, Jin-Yi Pang and Qian Wang, Chin. Phys. Lett. 42(2025)7, 070202.

Table of Contents

华南师范大学 SOUTH CHINA NORMAL UNIVERSITY

01 Introduction to finite-volume extrapolation (FVE)

02 Study FVE using machine learning

03 Summary

01 Introduction

Introduction to FVE

- Lattice QCD performs the calculations in a given **finite volume** discretized into a cubic lattice

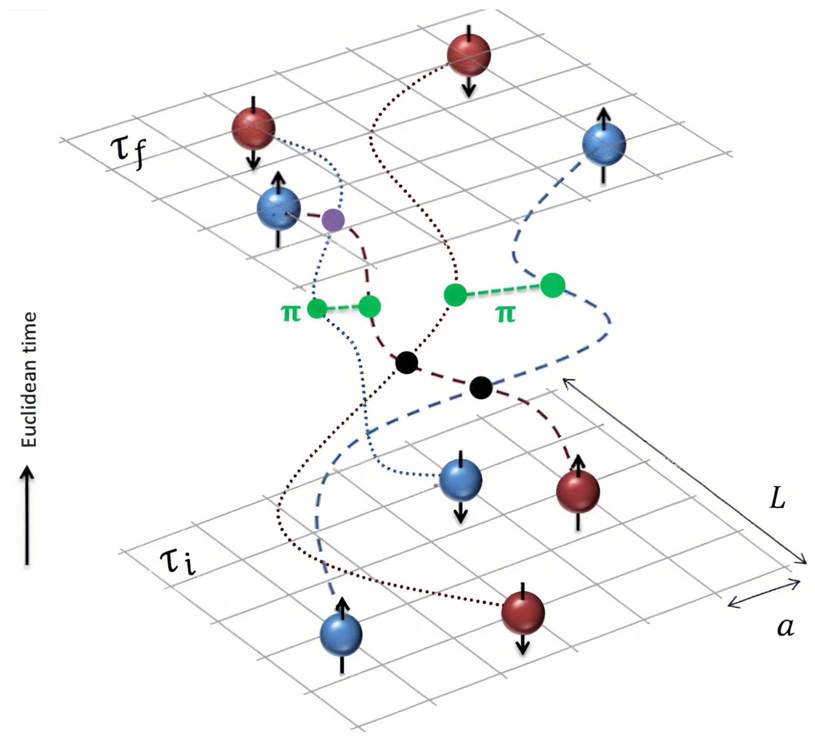
- **Finite-volume extrapolation** is essential to extract physical observables from lattice calculations

- Short-range case (Success in Lüscher formula[1]) ✓

- Long-range case (e.g. OPE in nucleon scattering, meson exchange in DD^* system[2]) ✗

- Solving methods
  (1) modifying the Lüscher formula numerically[3]
  (2) quantization conditions using plane-wave basis[4]
  (3) block-diagonalize the finite volume effective Hamiltonian[5]
  … …

Without explicit formula



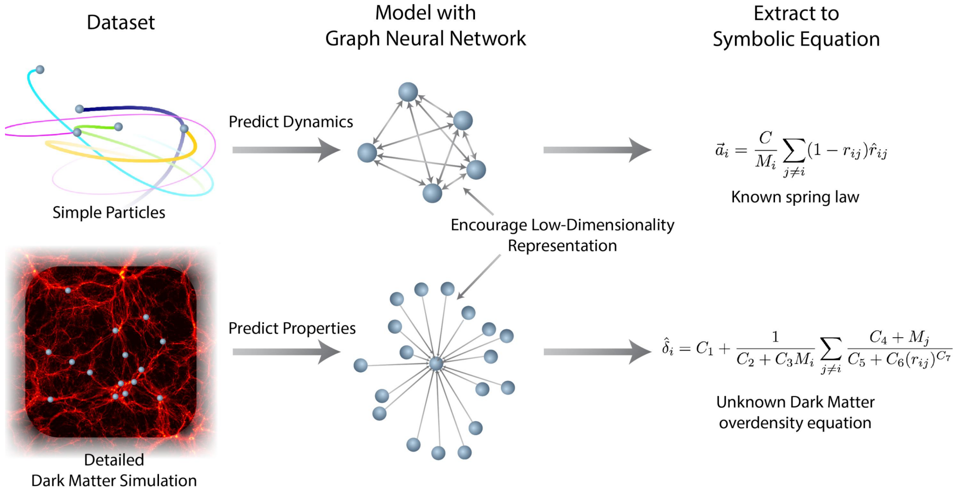
[1] Lüscher M, Commun. Math. Phys. 104, 177(1986).
[2] Meng L, et al., Phys. Rev. D 109, L071506(2024).
[3] Bubna R, et al., JHEP 05, 168(2024).
[4] Meng L, et al., JHEP 10, 051(2021).
[5] Yu K, et al., JHEP 04, 108(2025).

Introduction to SR

Symbolic regression(SR): deriving mathematical expressions directly from data

Applications

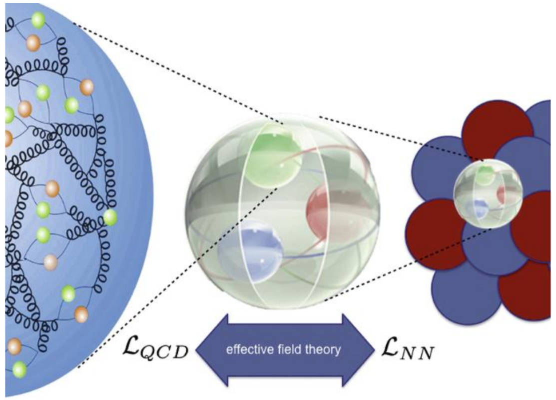
- AI Feynman algorithm[1] has derived 100 equations from the Feynman Lectures in Physics
- Graph Neural Networks + SR predict the concentration of dark matter from the mass distribution of nearby cosmic structures[2].
- SR for key low-energy observables[3], including the Higgs mass, muon anomalous magnetic moment, and dark matter relic density
- PySR + Deep neural network subtract background from jets in heavy ion collisions[4]
- … …

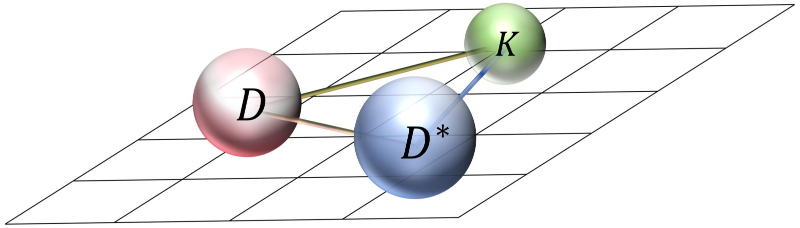


[1] Udrescu S-M, et al., Sci. Adv. 6, eaay2631(2020).
[2] Cranmer M, et al., arXiv:2006.11287(2020).
[3] AbdusSalam S, et al. Phys. Rev. D 111, 015022(2025).
[4] Mengel T, et al., Phys. Rev. C 108, L021901(2023).

Introduction to LEFT

Lattice EFT = Chiral EFT+ Lattice + Monte Carlo





(1) EFT description of hadron interactions
(contact terms + pion exchange potential)

(2) The degrees of freedom on the lattice are hadrons

(3) Lattice spacing $a \approx 1\text{fm}$
($\sim$ chiral symmetry breaking scale)

Solving low-energy many-body problems!

	LQCD	LEFT
degree of freedom	quarks & gluons	hadrons
lattice spacing	~0.1fm	~1fm
dispersion relation	relativistic	non-relativistic
continuum limit	✓	×
model	Lagrangian	Hamiltonian
solver	path integral	Schrödinger equation

[1] Dean Lee, Prog. Part. Nucl. Phys. 63, 117 (2009).
[2] Lähde, Meißner, "Nuclear Lattice Effective Field Theory", Springer (2019).

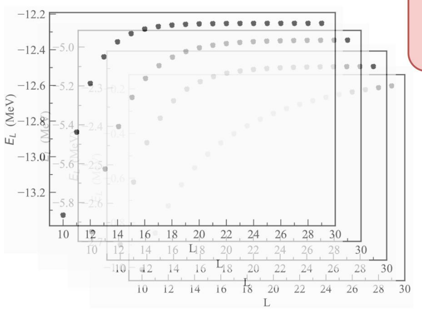
02

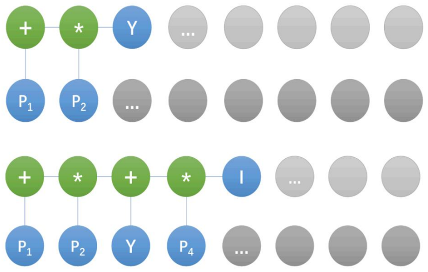
Framework and results

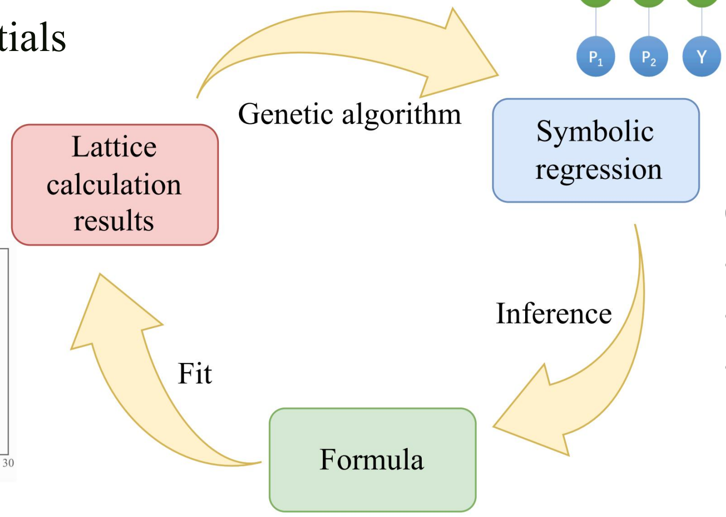
Framework

(1) **Samples generation**

- Calculate the binding energy of two-boson systems by LEFT[1]
- Consider different strength of potential
- Consider short-range and long-range potentials







(2) **Symbolic regression**

- Find optimal expression by PySR[2]
- Measure how well expression describes samples by Loss
- Estimate the form of the expression by Score

$$E_L = E_\infty + e^{-\kappa L}/L\ ?$$

(3) **Results analysis**

- Reproduce the Lüscher formula[3]
- Find the expression that unify short-range and long-range interactions

[1] Zhang Z, et al., Phys. Rev. D 111, 036002 (2025). [2] Cranmer M, arXiv:2305.01582 (2023). [3]Lüscher M, Commun. Math. Phys. 104, 177(1986).

Samples generation

Hamiltonian:

Short-range case

$$H = \sum_{i=1}^{2} \frac{\boldsymbol{p}_i^2}{2m_i} + f(\boldsymbol{p}_1^{(\prime)}, \boldsymbol{p}_2^{(\prime)}) V(\boldsymbol{q})$$

Interaction:

LO contact term

$$V(\boldsymbol{r}) = +C_0 \delta^3(\boldsymbol{r})$$

C_0 is the strength of the potential

Regulator[1] (equal to directly cutoff on lattice):

$$f(\boldsymbol{p}_1^{(\prime)}, \boldsymbol{p}_2^{(\prime)}) = \prod_{i=1}^{2} g_\Lambda(\boldsymbol{p}_i) g_\Lambda(\boldsymbol{p}_i')$$

$$g_\Lambda(\boldsymbol{p}) = \exp(-\boldsymbol{p}^6 / 2\Lambda^6)$$

Long-range case

$$H = \sum_{i=1}^{2} \frac{\boldsymbol{p}_i^2}{2m_i} + f(\boldsymbol{p}_1^{(\prime)}, \boldsymbol{p}_2^{(\prime)}) V_S(\boldsymbol{q}) + \hat{f}(\boldsymbol{q}) V_L(\boldsymbol{q})$$

Contact term and Yukawa potential

$$V(\boldsymbol{r}) = -C_{01} \delta^3(\boldsymbol{r}) - C_{02} \frac{e^{-\mu r}}{r}$$

$1/\mu$ reflecting the range of the force

$$\hat{f}(\boldsymbol{q}) = \exp(-(\boldsymbol{q}^2 + \mu^2)/\Lambda^2)$$

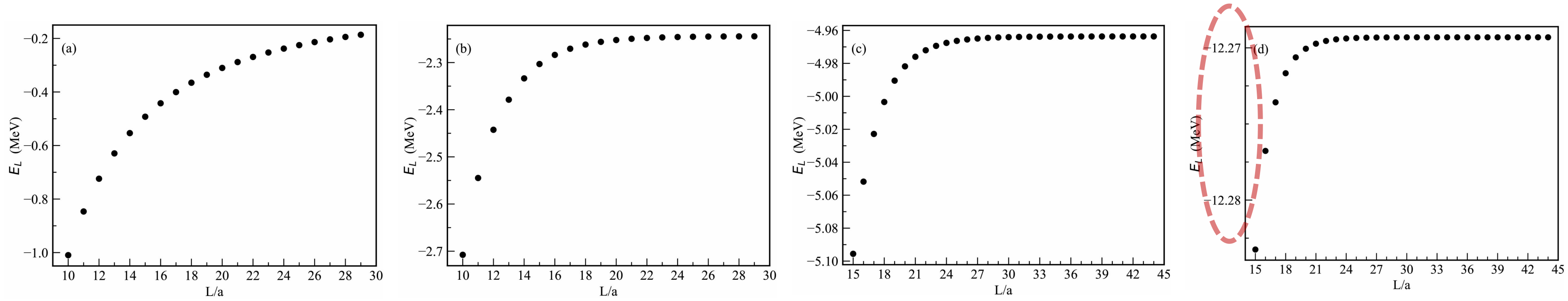
[1] B.-N. Lu et al., arXiv:2308.14559 (2023).

Samples generation

C_0 is set to 1.5, 2.0, 2.5 and 3.0 MeV^{-2}

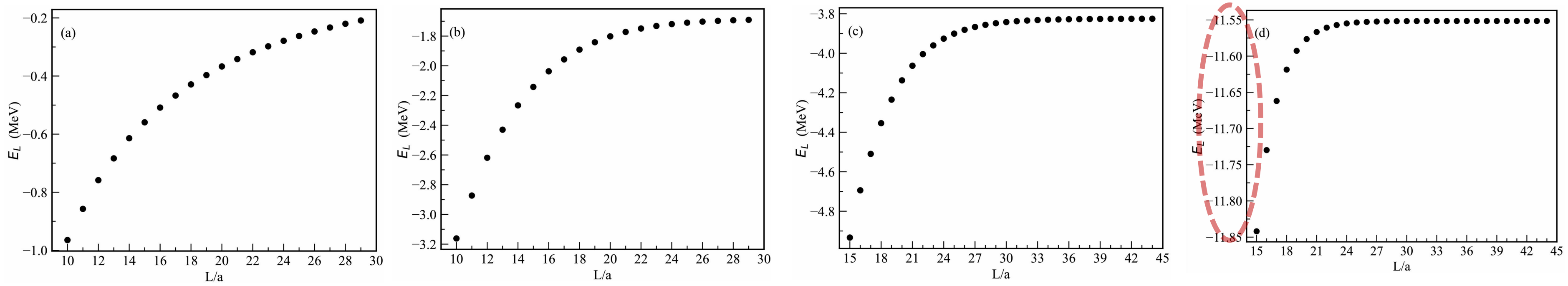
C_{01}=C_{02} is set to 0.03, 0.09, 0.12 and 0.21 MeV^{-2}

Short-range case



Long-range case

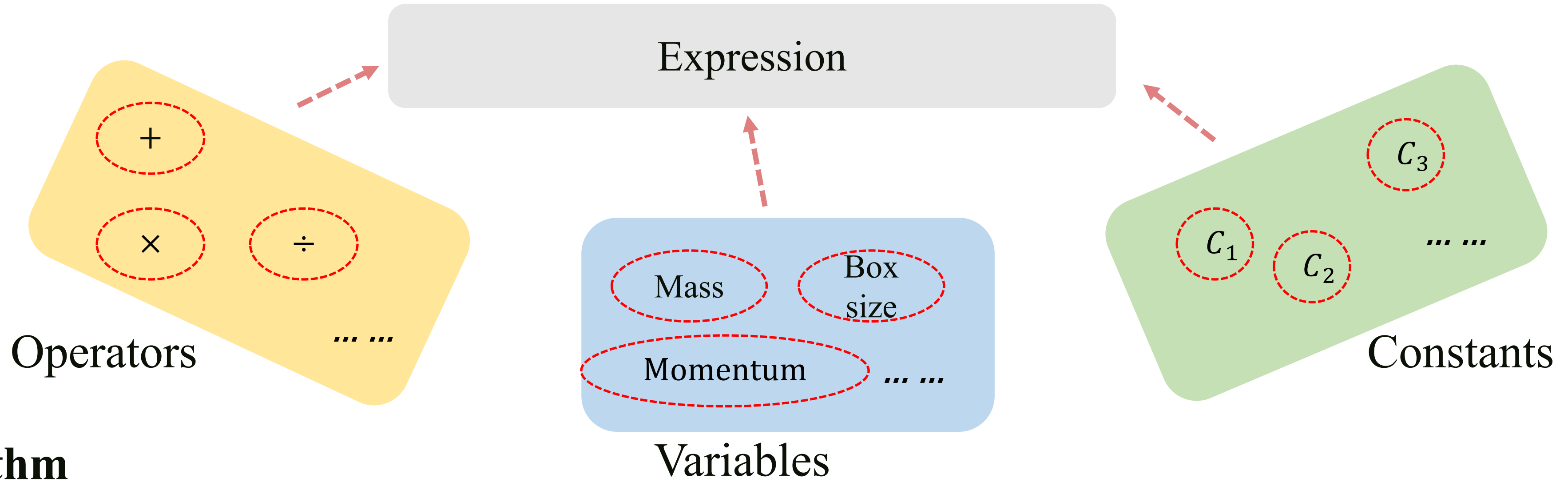
(μ=20 MeV)



- Calculate on box size L=10^3~30^3 fm^3 cubic lattice in periodic boundary conditions with N=2 bosons
- Set cutoff Λ =350(600) MeV, and lattice spacing a=1/200(1/300) MeV^{-1} ≈0.99(0.66) fm
- The convergence behavior exhibits slower in long-range case compared to short-range case

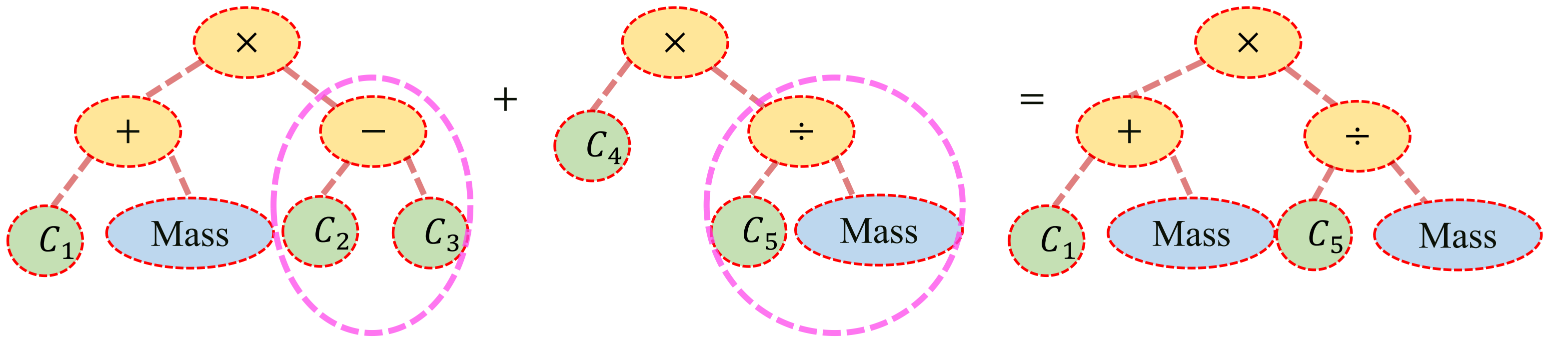
Symbolic regression

The PySR model samples the space of analytic expressions defined by the set of **operators**, input **variables**, and **constant** terms for minimization through **genetic algorithm**.



Genetic algorithm

Crossover



Mutation

… …

Symbolic regression

Metrics: two elements to measure the goodness of the output formulae in the PySR model, i.e., **Loss** and **Score**

- Loss is used to measure how well the output formula describes the samples, which is defined as mean square error:

$$\text{Loss} = \sum_{i=1}^{N} (E_{\text{PySR}}(L_i) - E_L(L_i))^2/N$$

- Score is used to estimate the form of the formula, which rewards minimal loss and penalizes the more complicated formula.

$$\text{Score} = -\frac{\Delta \ln(\text{Loss})}{\Delta C}$$

PySR model will consider both score, i.e. complexity, and loss to choose the best formula

Short-range case:

Complexity	Loss	Score	Equation
1	5.818	0.000	L
2	0.014	6.016	-2.312
4	0.010	0.180	$L - 2.410$
5	0.007	0.396	$-2.549\exp(L)$
6	0.005	0.220	$-3.144\exp(\sqrt{L})$
7	0.003	0.653	$(-0.028/L) - 1.992$
8	0.003	0.000	$(-0.028/L) - 1.992$
9	0.001	0.682	$-0.001/L^2 - 2.148$
10	3.679×10^{-4}	1.341	$(\sqrt{L} - 0.171)/L - 3.696$
11	1.831×10^{-6}	5.302	$-29.889\exp(-83.458L) - 2.245$
12	1.826×10^{-6}	0.003	$-\exp(-83.388L + 3.393) - 2.245$
13	6.333×10^{-7}	1.059	$\mathbf{-0.629\exp(-66.049\boldsymbol{L})/\boldsymbol{L} - 2.244}$
14	6.292×10^{-7}	0.006	$-0.628\exp(-66.022L)/L - 2.244$

Long-range case:

Complexity	Loss	Score	Equation
1	4.660	0.000	L
2	0.153	3.418	$\log(L)$
4	0.072	0.380	$-0.172/L$
5	0.021	1.237	$-0.613/\sqrt{L}$
6	0.008	1.006	$-\exp(0.058/L)$
8	0.002	0.676	$-\exp(0.056/L) - L$
9	0.002	0.253	$-0.004/L^2 - 1.413$
11	5.672×10^{-5}	1.646	$-15.311/\exp(46.272L) - 1.661$
13	6.928×10^{-6}	1.051	$\mathbf{-695.626\boldsymbol{L}/\exp(63.072\boldsymbol{L}) - 1.677}$
15	6.571×10^{-6}	0.026	$-681.508L/\exp(62.467L) + L^2 - 1.660$

Results

Short-range case:

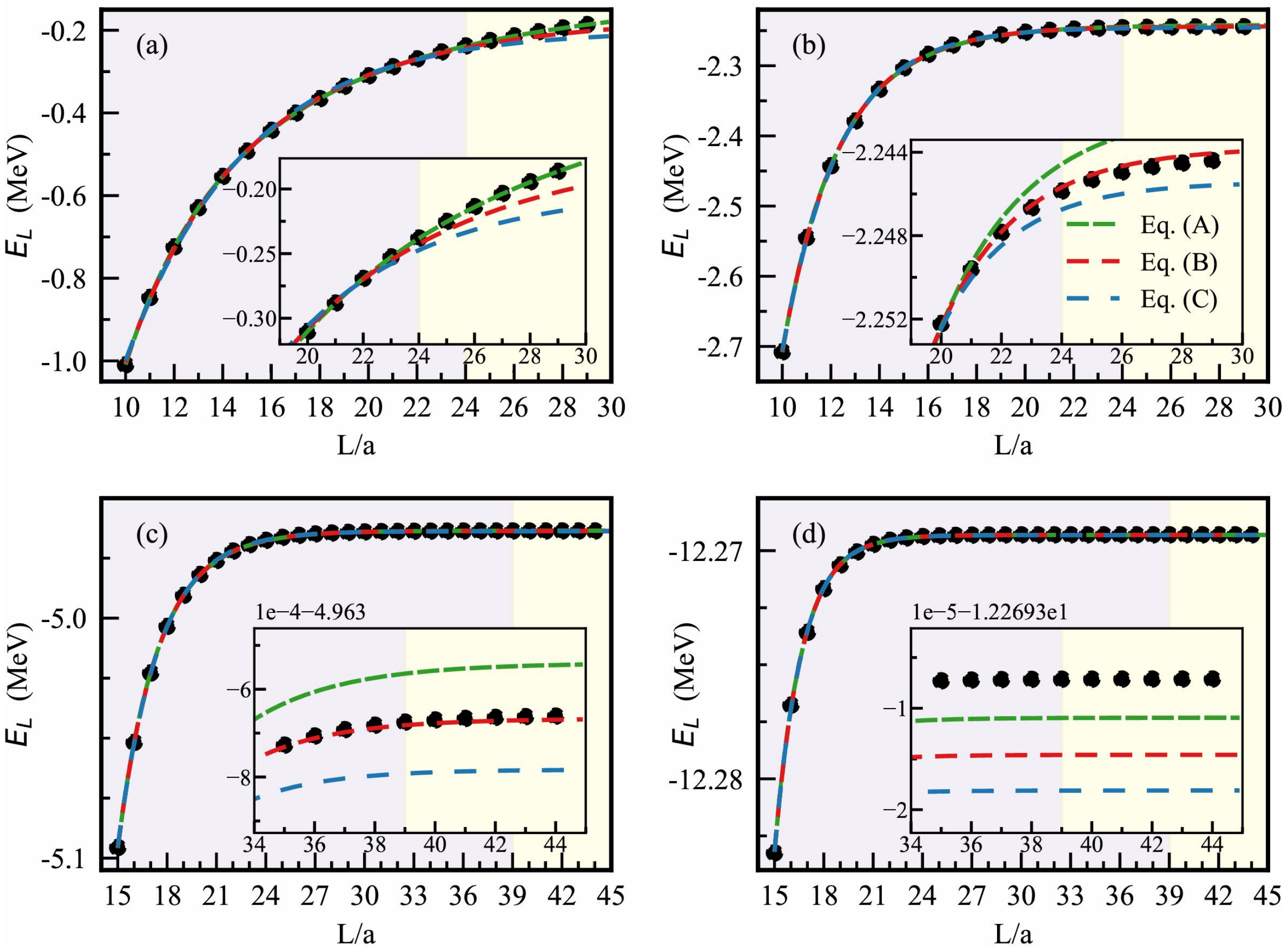
$$\boldsymbol{E_L = C_1 + C_2 e^{-C_3 L}/L^2} \qquad \textbf{(A)}$$

☆ $$\boldsymbol{E_L = C_1 + C_2 e^{-C_3 L}/L} \qquad \textbf{(B)}$$

$$\boldsymbol{E_L = C_1 + C_2 e^{-C_3 L}} \qquad \textbf{(C)}$$

Eq. (B) recurrents Lüscher's formula[1]

$$E_L = E_\infty + \frac{C'}{L} \exp(-\kappa L)$$



The results of fitting the formula to samples. (a) is the case of C_0=1.5 MeV^{-2}, (b) is the case of C_0=2.0 MeV^{-2}, (c) is the case of C_0=2.5 MeV^{-2} and (d) is the case of C_0= 3.0 MeV^{-2}.

[1] Lüscher M, Commun. Math. Phys. 104, 177(1986).

Results

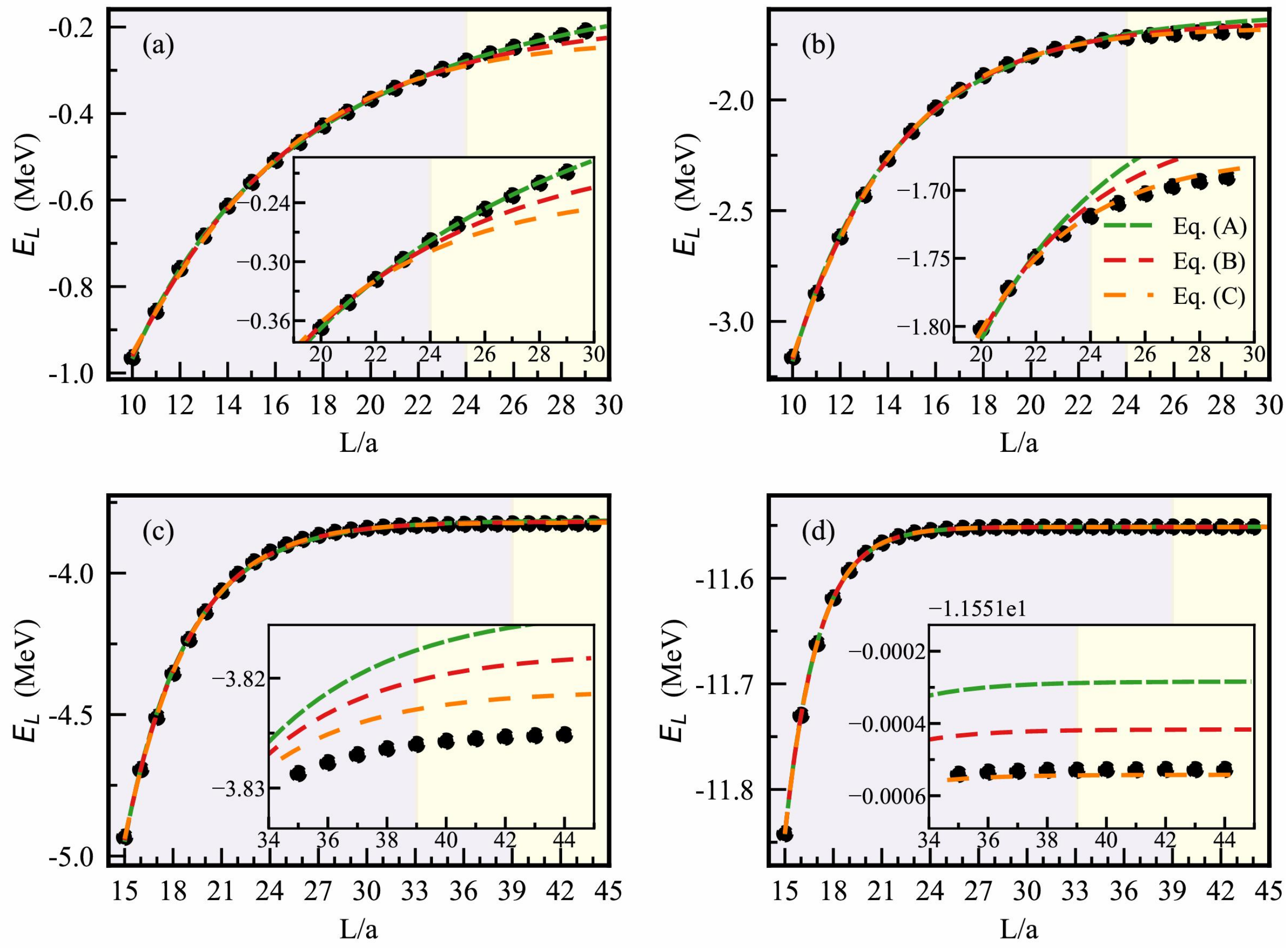
Long-range case:

$$E_L = C_1 + C_2 e^{-C_3 L}/L \qquad \text{(A)}$$

$$E_L = C_1 + C_2 e^{-C_3 L} \qquad \text{(B)}$$

$$\star \; E_L = C_1 + C_2 e^{-C_3 L} L \qquad \text{(C)}$$

- Set $\mu = 20$ MeV, the force range is about 10 fm.
- Compare with the formula for the short-range potential, the power of L becomes larger.
- This enlightens us that the power of L has a correlation with the force range.



Eq. (A), (B) and (C) fitting to the samples of the long-range potentials with C_{01}=C_{02}=0.03 MeV^{-2} (a), C_{01}=C_{02}=0.09 MeV^{-2} (b), C_{01}=C_{02}=0.12 MeV^{-2} (c), and C_{01}=C_{02}=0.21MeV^{-2} (d).

Results

Generalization:

- Samples with the long-range parameter μ=10, 20, ..., 120, 400, 600, 800 MeV are labeled from (a) to (o) in order

- For all the cases, the binding energy is around 3.83 MeV.

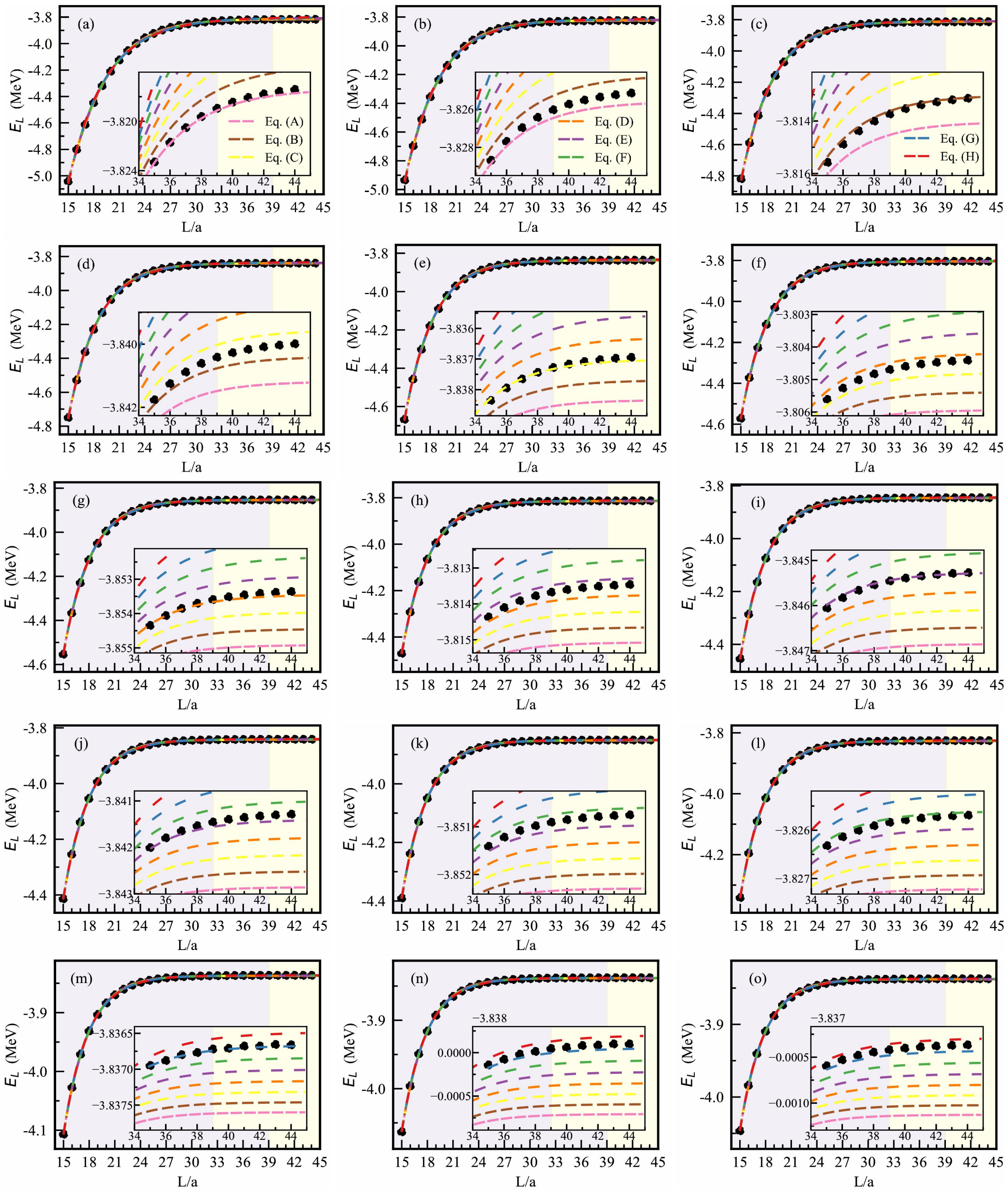
- The eight lines, i.e. Eq. (A), (B), (C), (D), (E), (F), (G) and (H), represent the power n=5/2, 2, 3/2, 1, 1/2, 0, −1/2, −1 of L from bottom to top in each figure.

$$E_L = C_1 + C_2 e^{-C_3 L} L^n$$

- When the **force range decreases to short-range** (μ goes to infinity), the power n comes back to the formula

$$E_L = C_1 + C_2 e^{-C_3 L} L^{-1}$$

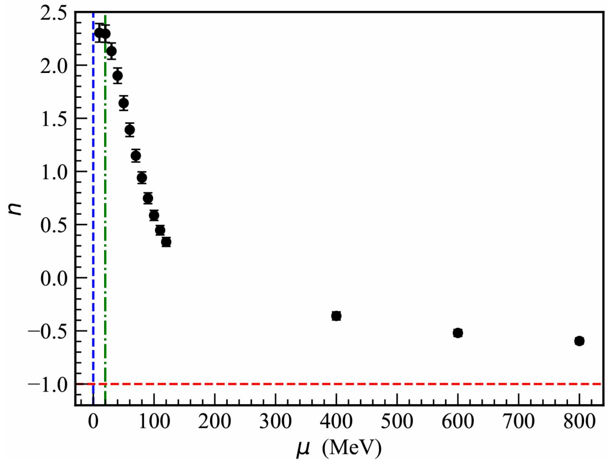
deduced from the **Lüscher formula**.



Results

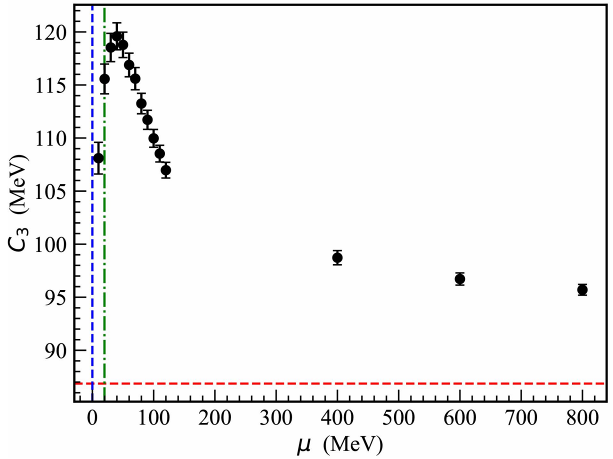
$$E_L = C_1 + C_2 e^{-C_3 L} L^n \quad \Leftrightarrow \quad E_L = E_\infty + \frac{C'}{L}\exp(-\kappa L)$$ Lüscher formula

The dependence of the power n on the range parameter μ.



- Blue dashed line is the infinity long-range limit($\mu = 0$)
- Green dotdashed line corresponds to the lower limit of box size($\hbar c/10$ fm ~ 20 MeV)
- Red dashed line is the short-range value -1 from Lüscher formula($n = -1$)

The dependence of the parameter C_3 on the range parameter μ.



- Blue dashed line is the infinity long-range limit($\mu = 0$)
- Green dotdashed line corresponds to the lower limit of box size($\hbar c/10$ fm ~ 20 MeV)
- Red dashed line is the binding momentum($\kappa_B = \sqrt{mE_\infty}$)

The formula recovers the **short-range limit** and expresses the **long-range trend**

03

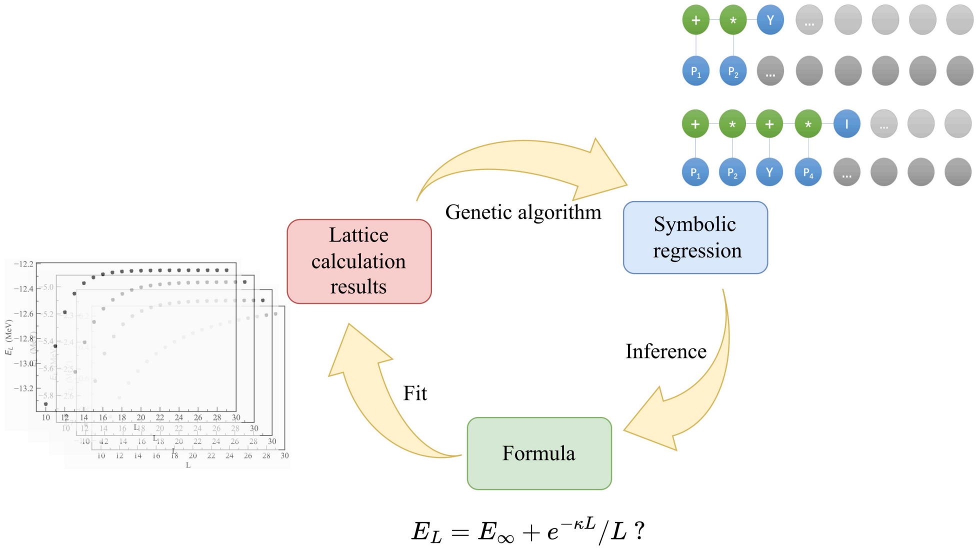
Summary

Summary

(1) Successful application of symbolic regression: the finite volume formula is successfully derived, which reproduces the Lüscher formula.

(2) Universal formula for long-range case: a modified extrapolation formula is proposed for long-range case.

(3) Extensions to many-body systems: finite volume effect in three- or many-body systems can be explored.



華南師範大學
SOUTH CHINA NORMAL UNIVERSITY

THANKS !!!