

Dispersive analysis of $J/\psi \rightarrow \pi^0 e^+ e^-$

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$J/\psi \rightarrow \pi^0 e^+ e^-$: motivation

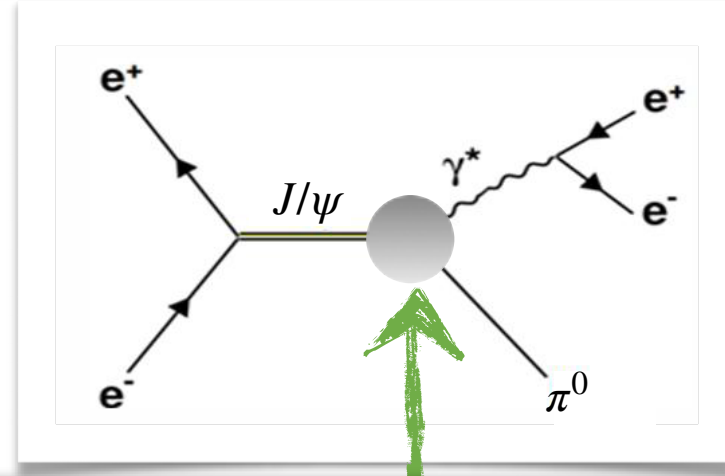
- Sensitive to the transition form factor (TFF) of $J/\psi \rightarrow \pi^0 \gamma^*$

- The branching fractions of $J/\psi \rightarrow P e^+ e^-$ ($P = \pi^0, \eta, \eta'$)
BESIII, PRD 89, 092008 (2014)

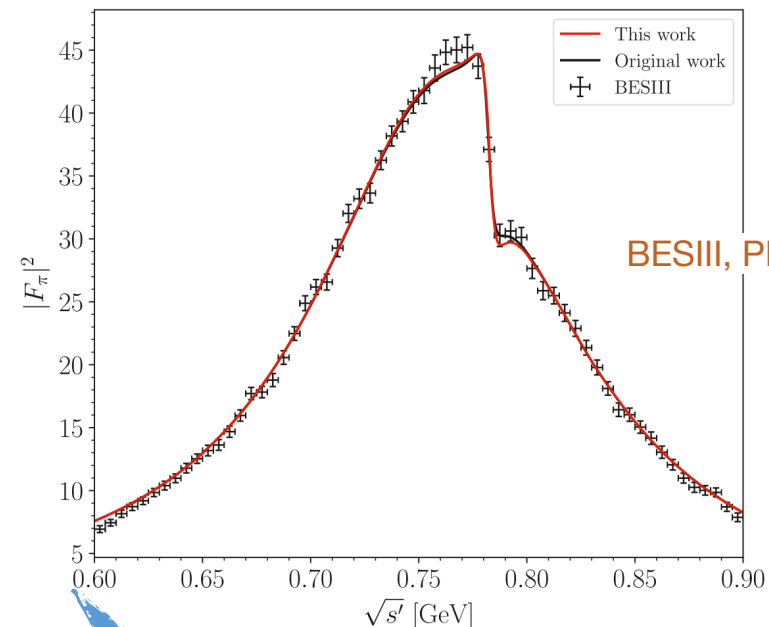
- Exp v.s. VMD

η and η' $\checkmark$; π^0 $\times$

- VMD estimate:
charmonium dominance,
isospin-breaking transition
- **Isospin-conserving manner**,
dominated by light-quark degrees
of freedom ($2\pi \dots$)



Transition form factor for $J/\psi \rightarrow \pi^0 \gamma^*$

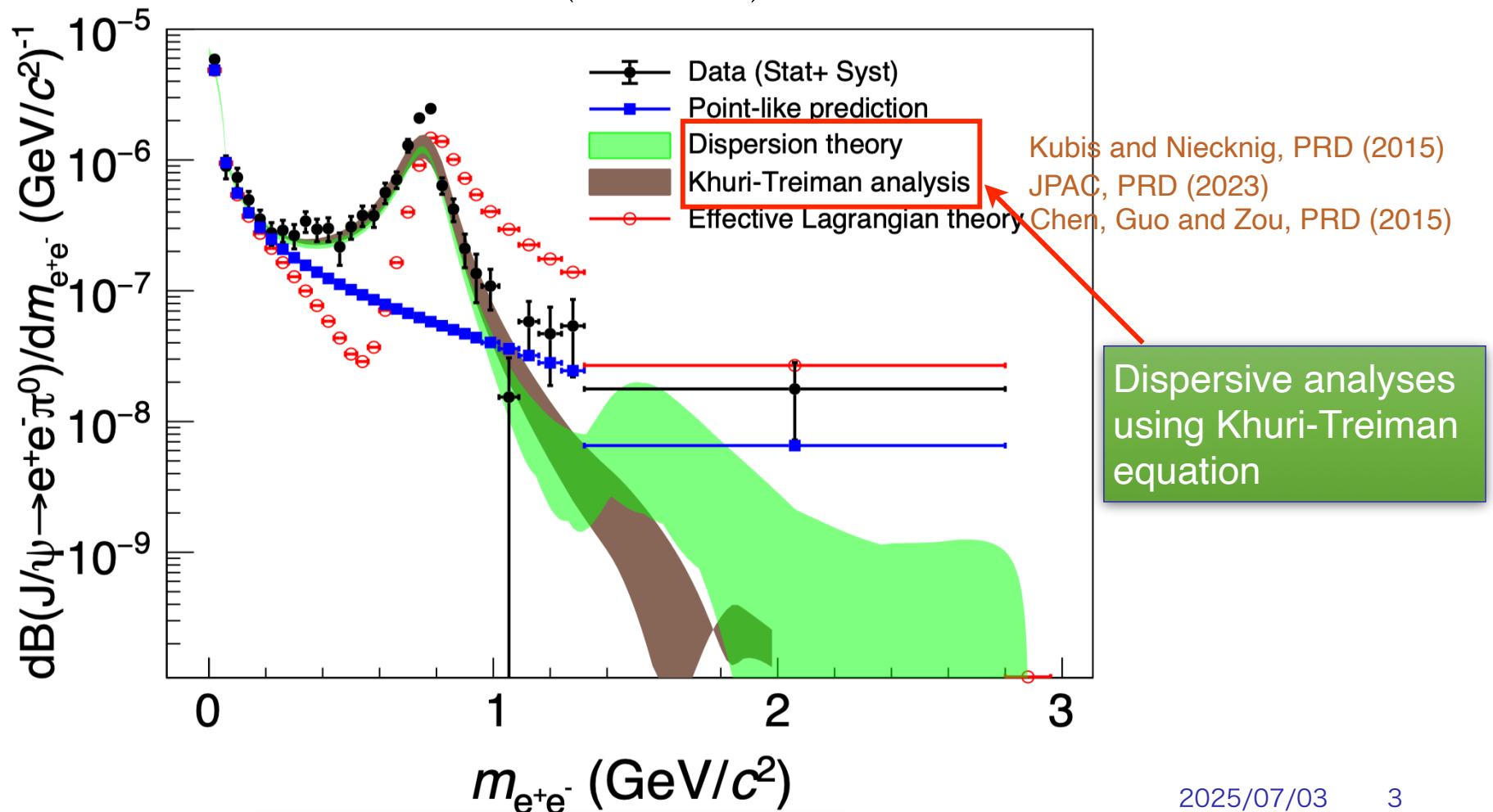


BESIII, PLB 753 (2016)

$\rho - \omega$ interference

- A significant resonant structure corresponding to ρ, ω (100 billion J/ψ events)

$$\frac{d\text{BR}_{\psi \rightarrow \pi^0 \ell^+ \ell^-}}{\text{BR}_{\psi \rightarrow \pi^0 \gamma} ds} = \frac{16\alpha}{3\pi} \left(1 + \frac{2m_\ell^2}{s} \right) \frac{q_\ell(s) q_{\psi\pi^0}^3(s)}{(M_\psi^2 - M_{\pi^0}^2)^3} \left| F_{\psi\pi^0}(s) \right|^2, \quad F_{\psi\pi^0}(s) = \frac{f_{\psi\pi^0}(s)}{f_{\psi\pi^0}(0)}$$



$J/\psi \rightarrow \pi^0 \gamma^*$ transition form factor

- The $J/\psi \rightarrow \pi^0 \gamma^*$ transition form factor: Landsberg, Phys. Rept. (1985)

$$\langle \pi^0(p) | j_\mu(0) | \psi(p_V, \lambda) \rangle = -i \epsilon_{\mu\nu\alpha\beta} \epsilon^\nu(p_V, \lambda) p^\alpha q^\beta f_{\psi\pi^0}(s)$$

- (P-wave) $\pi\pi$ intermediate state:

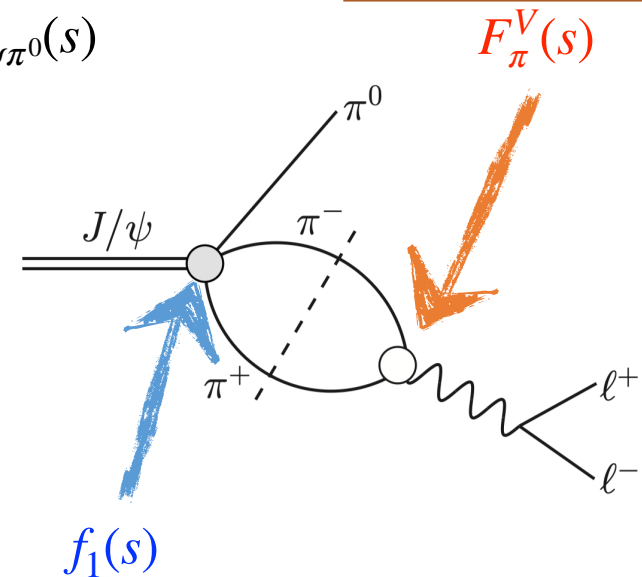
$$\frac{\text{disc}}{2i} f_{\psi\pi^0}^{(2\pi)}(s) = \frac{s \sigma_\pi^3(s)}{96\pi} F_\pi^{V*}(s) f_1(s)$$

Koepp, PRD (1974)

- Dispersion relation** (unsubtracted):

$$f_{\psi\pi^0}(s) = \frac{1}{2\pi i} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{disc } f_{\psi\pi^0}(s')}{s' - s - i\epsilon},$$

π vector form factor



P-wave $J/\psi \rightarrow 3\pi$ decay amplitude

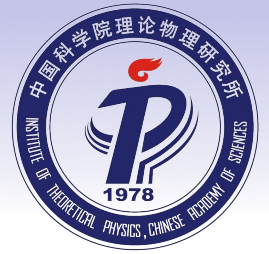
$\rho - \omega$ mixing

- F_π^V depends on $\rho - \omega$ mixing
- Naive approach leads to a spectral function in which the double discontinuities of 2π and 3π intermediate states may no longer cancel

$$F_\pi^V(s) \rightarrow \left(1 + \epsilon_{\rho\omega} \frac{s}{M_\omega^2 - s - iM_\omega\Gamma_\omega} \right) F_\pi^V(s) \rightarrow \text{Im} \left[\frac{\text{disc} f_{\psi\pi^0}(s) |_{2\pi}}{2i} \right] \neq 0$$

- Two-potential formalism + Dispersion relation
 - ✦ Three channel system: $2\pi, 3\pi, \ell^+\ell^-$
 - ✦ Generalize to a dispersion relation including a consistent treatment $\rho - \omega$ mixing
 - ✦ P-wave $J/\psi \rightarrow 3\pi$ decay amplitude is construct from the Khuri-Treiman (KT) formalism [Khuri and Treiman, Phys. Rev. \(1960\)](#)...
 - ✦ **Ensure unitarity, analyticity and crossing symmetry**

Dispersive representation



Modified dispersion relation (once sub.)

See also $\eta'\gamma^*\gamma^*$ TFF, Holz et.al., EPJC (2022)

$$\begin{aligned}
 f_{\psi\pi^0}^{(2\pi,3\pi)}(s) = & f_{\psi\pi^0}^{(2\pi)}(0) \\
 & + \frac{s}{96\pi^2} \int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s') F_\pi^{V*}(s') f_1(s')}{s' - s - i\epsilon} \left[1 + \frac{\tilde{\epsilon}_{\rho\omega}s}{M_\omega^2 - s - iM_\omega\Gamma_\omega} \right] \\
 & + \frac{w_{\psi\omega\pi}s}{M_\omega^2 - s - iM_\omega\Gamma_\omega} \left[1 + \frac{\tilde{\epsilon}_{\rho\omega}s}{48\pi^2 g_{\omega\gamma}^2} \int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s') |F_\pi^V(s')|^2}{s'(s' - s - i\epsilon)} \right] \\
 & + \frac{f_{\psi\pi^0}^{2\pi,3\pi}(0) w_{\psi\phi\pi}s}{M_\phi^2 - s - iM_\phi\Gamma_\phi},
 \end{aligned}$$

$I = 1$ “ $\rho - \omega$ ” mixing
 $I = 0$ “ $\omega - \rho$ ” mixing

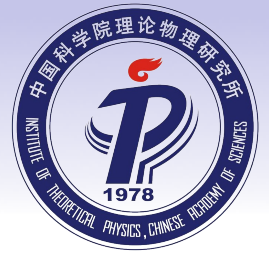
★ $\rho - \omega$ mixing parameter $\tilde{\epsilon}_{\rho\omega} = \epsilon_{\rho\omega} - e^2 g_{\omega\gamma}^2 = 1.65(2) \times 10^{-3}$

★ Weights $w_{\psi\omega(\phi)\pi}$

★ Coupling $g_{\omega\gamma} = \sqrt{\frac{3\Gamma(\omega \rightarrow e^+e^-)}{4\pi\alpha^2 M_\omega}} = 0.0606(9)$ from VMD model

★ P-wave $J/\psi \rightarrow 3\pi$ decay amplitude $f_1(s)$

Dispersive representation



● Modified dispersion relation (once sub.)

See also $\eta'\gamma^*\gamma^*$ TFF, Holz et.al., EPJC (2022)

$$\begin{aligned}
 f_{\psi\pi^0}^{(2\pi,3\pi)}(s) = & f_{\psi\pi^0}^{(2\pi)}(0) \\
 & + \frac{s}{96\pi^2} \int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s') F_\pi^{V*}(s') f_1(s')}{s' - s - i\epsilon} \left[1 + \frac{\tilde{\epsilon}_{\rho\omega}s}{M_\omega^2 - s - iM_\omega\Gamma_\omega} \right] \\
 & + \frac{w_{\psi\omega\pi}s}{M_\omega^2 - s - iM_\omega\Gamma_\omega} \left[1 + \frac{\tilde{\epsilon}_{\rho\omega}s}{48\pi^2 g_{\omega\gamma}^2} \int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s') |F_\pi^V(s')|^2}{s'(s' - s - i\epsilon)} \right] \\
 & + \frac{f_{\psi\pi^0}^{2\pi,3\pi}(0) w_{\psi\phi\pi}s}{M_\phi^2 - s - iM_\phi\Gamma_\phi},
 \end{aligned}$$

I = 1 “ $\rho - \omega$ ” mixing
I = 0 “ $\omega - \rho$ ” mixing

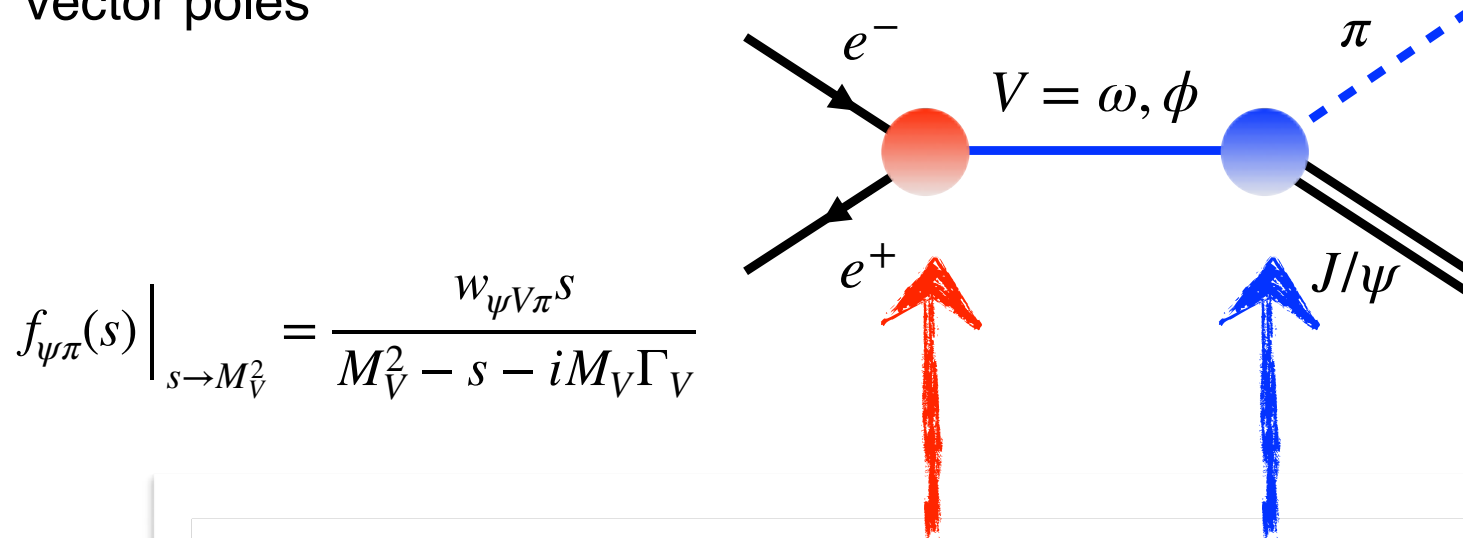
This nontrivial **consistency** condition is indeed satisfied

$$\text{Im} \left[\frac{1}{2i} \text{disc} f_{\psi\pi^0}(s) \Big|_{2\pi} + \frac{1}{2i} \text{disc} f_{\psi\pi^0}(s) \Big|_{3\pi} \right] = 0$$

!

Isoscalar-vector pole contribution

- Weights $w_{\psi\omega(\phi)\pi}$ are determined by the pole dominance model
- Matching the expressions of the $e^+e^- \rightarrow J/\psi\pi^0$ cross section at isoscalar-vector poles

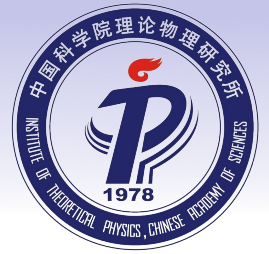


$$f_{\psi\pi}(s) \Big|_{s \rightarrow M_V^2} = \frac{w_{\psi V \pi} s}{M_V^2 - s - i M_V \Gamma_V}$$

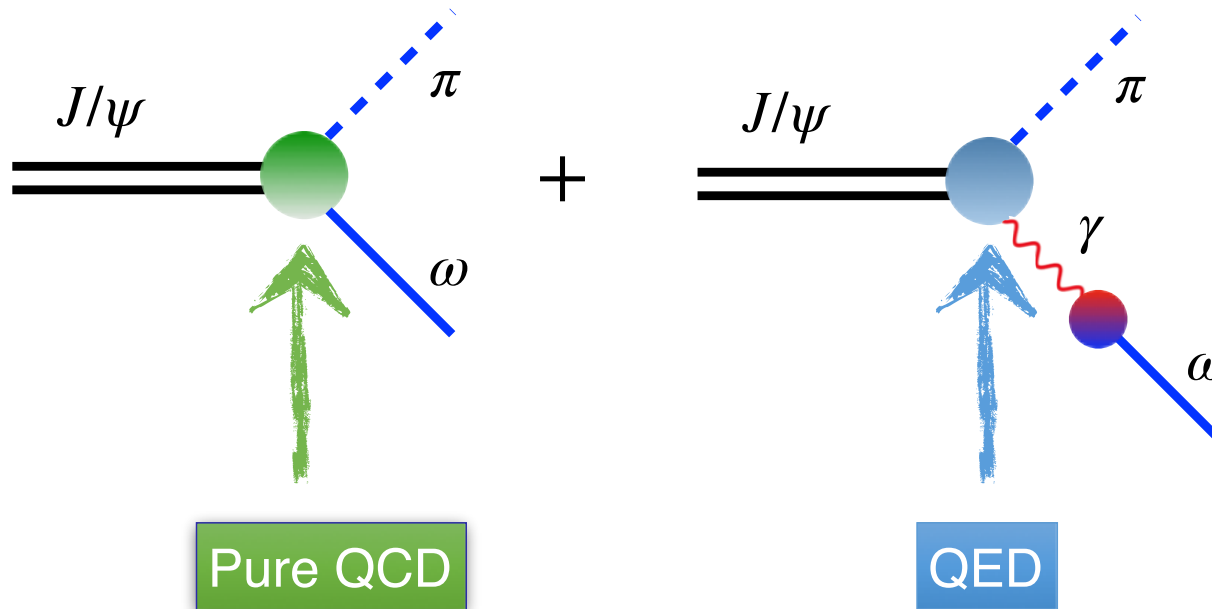
$$|w_{\psi V \pi}| = \sqrt{\frac{72 M_\psi^3 \Gamma_\psi \Gamma_\omega \mathcal{BR}(V \rightarrow e^+ e^-) \mathcal{BR}(J/\psi \rightarrow V \pi^0)}{\alpha^2 M_V \lambda^{\frac{3}{2}}(M_\psi^2, M_V^2, M_\pi^2)}}$$

$$= \begin{cases} 4.38(24) \times 10^{-5} \text{ GeV}^{-1}, & V = \omega \\ 4.71(3) \times 10^{-6} \text{ GeV}^{-1} \text{ or } 8.61(5) \times 10^{-7} \text{ GeV}^{-1}, & V = \phi \end{cases}$$

QED effects



- Weights $w_{\psi\omega(\phi)\pi}$ are complex due to the intermediate γ



- Phases of weights $w_{\psi\omega(\phi)\pi}$ are from the interference between $\omega(\phi)$ and γ

Khuri-Treiman representation

- Decay amplitude for $V(p_V) \rightarrow \pi^+(p_+) \pi^-(p_-) \pi^0(p_0)$:

$$\mathcal{M}(s, t, u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu p_+^\nu p_-^\alpha p_0^\beta \mathcal{F}(s, t, u)$$

- s-channel partial-wave expansion of the amplitude

$$\mathcal{F}(s, t, u) = \sum_{J=0}^{\infty} (q_{\pi\pi}(s)q_{\psi\pi}(s))^{J-1} P'_J(z_s) f_J(s)$$

- KT decomposition of the amplitude via *reconstruction theorem*

Stern, Sazdjian and Fuchs, PRD (1993), for $\pi\pi$

$$\mathcal{F}(s, t, u) = \sum_{J=0}^{J_{\max}} (q_{\pi\pi}(s)q_{\psi\pi}(s))^{J-1} P'_J(z_s) \mathcal{F}_J(s) + (s \leftrightarrow t) + (s \leftrightarrow u)$$

- Consider only P-wave: $\mathcal{F}(s, t, u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$, $\mathcal{F} \equiv \mathcal{F}_1$

several complex variables $\rightarrow$ single complex variables

- Partial wave projection of the KT decomposition Khuri and Treiman, Phys. Rev. (1960)...

$$f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s), \quad \hat{\mathcal{F}}(s) = 3 \int_{-1}^1 \frac{dz_s}{2} (1 - z_s^2) \mathcal{F}(t(s, z_s)) \equiv 3 \langle (1 - z_s^2) \mathcal{F} \rangle$$

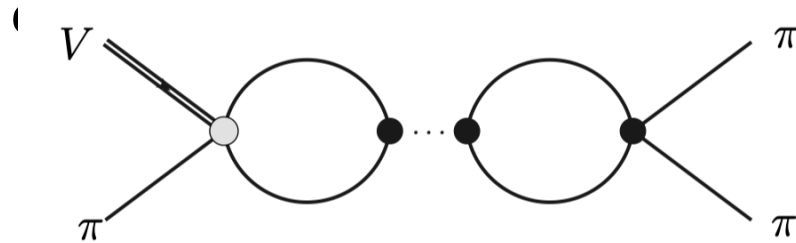
✳ $\mathcal{F}(s)$: right-hand cut (RHC)

✳ $\hat{\mathcal{F}}(s)$: left-hand cut (given by the RHC of the crossed channels $\mathcal{F}(t), \mathcal{F}(u)$)

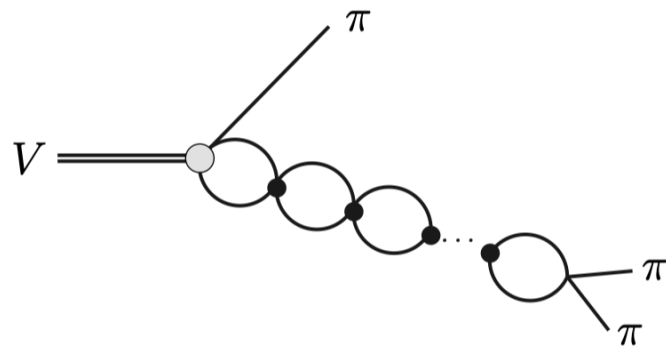
Three-particle decay dynamics



- In many decay processes one wants to take into account unitarity/FSI in the three possible

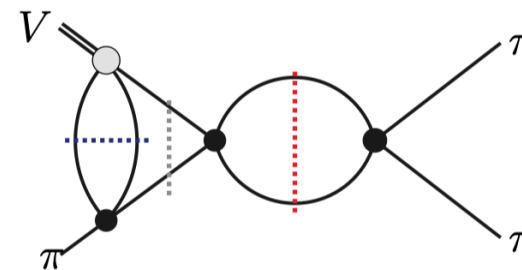


⇓ continuation

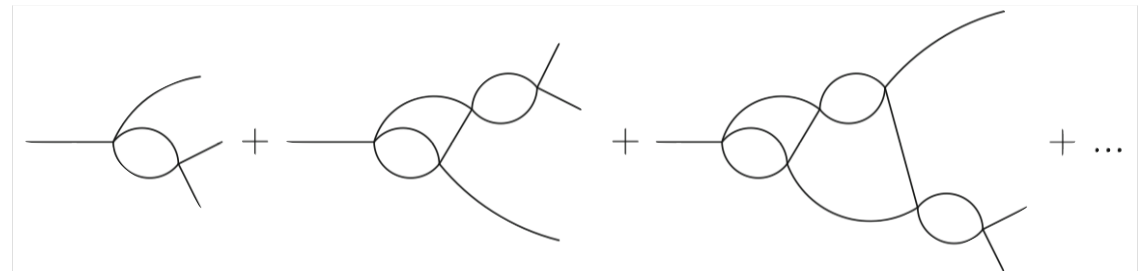
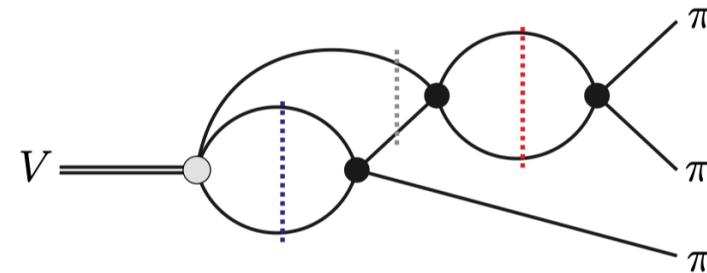


- Khuri-Treiman equations:

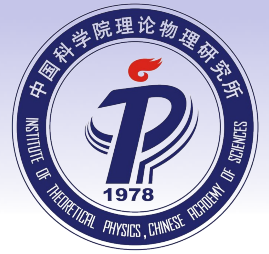
✦ Include full (direct+crossed) rescattering effects



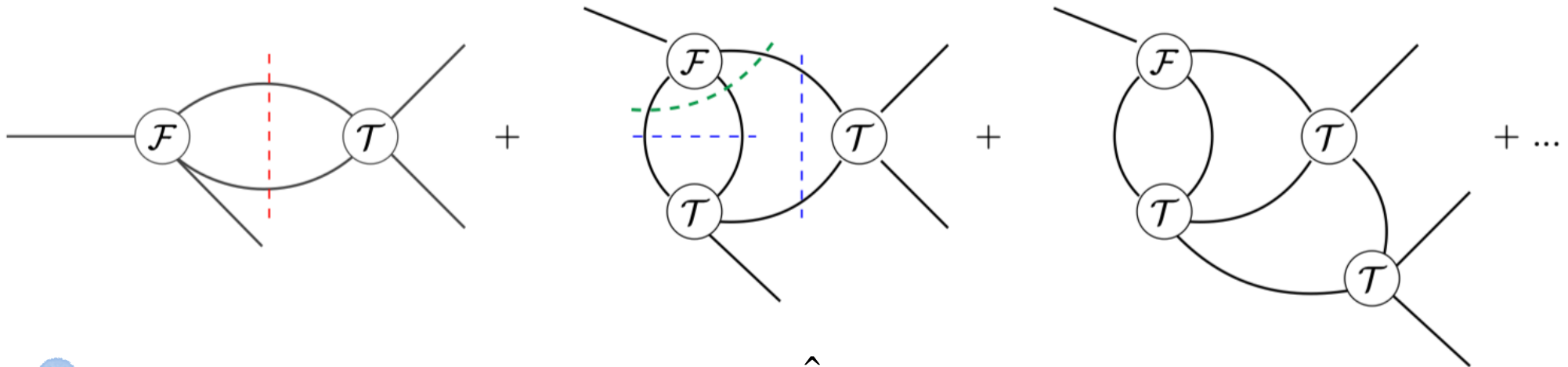
⇓ continuation



Unitarity and analyticity



- Unitarity relation for the single variable amplitude $\mathcal{F}(s)$



- **Complications:** integration contour for $\hat{\mathcal{F}}(s)$
- **“Pinocchio” method** (**triangle topology** continuation) Bronzan and Kacser, Phys. Rev. (1963)
 - ✦ deform path of **angular integral** to avoid crossing branch cuts
- Gasser-Rusetsky method Gasser and Rusetsky, EPJC (2018)
 - ✦ deform path of **dispersion integral**
- Pasquier inversion R. Pasquier and J.Y. Pasquier, Phys. Rev. (1968, 1969)
 - ✦ interchange of the order of integrations to obtain integral equations in one variable

Analytic continuation of the inhomogeneities

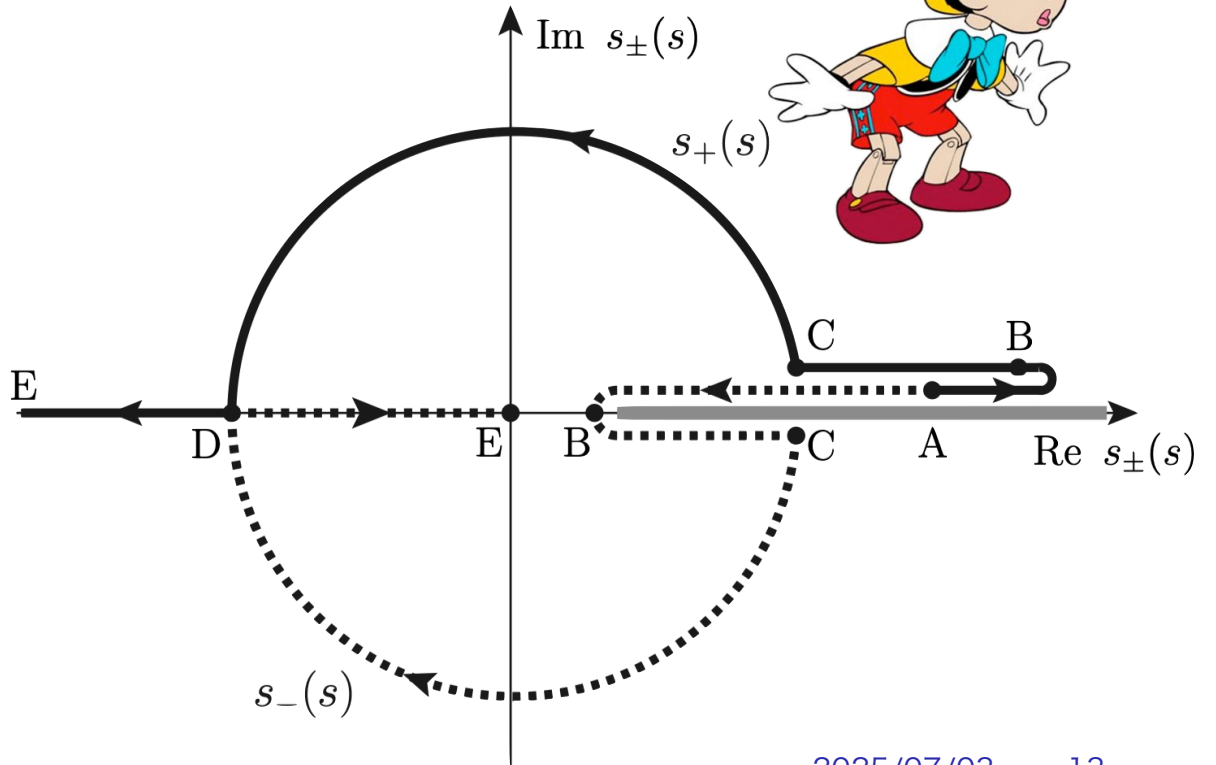
- $i\epsilon$ prescription: give the vector-meson mass an infinitesimal positive imaginary part, $M_V^2 + i\epsilon$ if $M_V > 3M_\pi$ Bronzan and Kacser, Phys. Rev. (1963)
- Angular average integral ($s + t + u = 3M_\pi^2 + M_V^2 \equiv 3s_0$)

$$\langle z^n \mathcal{F} \rangle \equiv \frac{1}{2} \int_{-1}^1 z^n \mathcal{F} \left(\frac{3s_0 - s + z\kappa(s)}{2} \right) dz = \frac{1}{\kappa(s)} \int_{s_-(s)}^{s_+(s)} ds' \left(\frac{2s' - 3s_0 + s}{\kappa(s)} \right)^n \mathcal{F}(s')$$

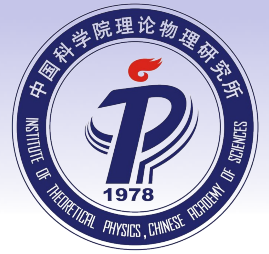
★ The trajectories of $s_\pm(s)$

$$2s_+(s) = \begin{cases} 3s_0 - s + |\kappa(s)| + i\epsilon, & s \in [4M_\pi^2, M_-^2], \\ 3s_0 - s + i|\kappa(s)|, & s \in [M_-^2, M_+^2], \\ 3s_0 - s - |\kappa(s)|, & s \in [M_+^2, \infty), \end{cases}$$

$$2s_-(s) = \begin{cases} 3s_0 - s - |\kappa(s)| + i\epsilon, & s \in \left[4M_\pi^2, \frac{M_V^2 - M_\pi^2}{2}\right], \\ 3s_0 - s - |\kappa(s)| - i\epsilon, & s \in \left[\frac{M_V^2 - M_\pi^2}{2}, M_-^2\right], \\ 3s_0 - s - i|\kappa(s)|, & s \in [M_-^2, M_+^2], \\ 3s_0 - s + |\kappa(s)|, & s \in [M_+^2, \infty), \end{cases}$$



KT equations: solutions

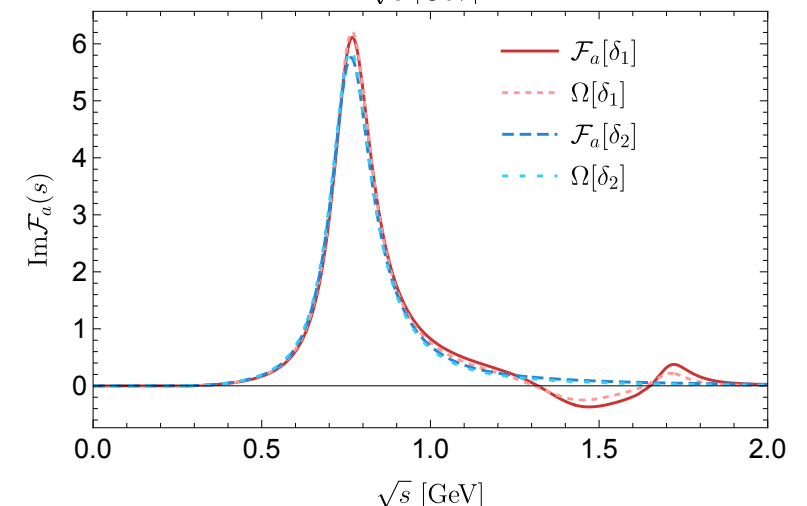
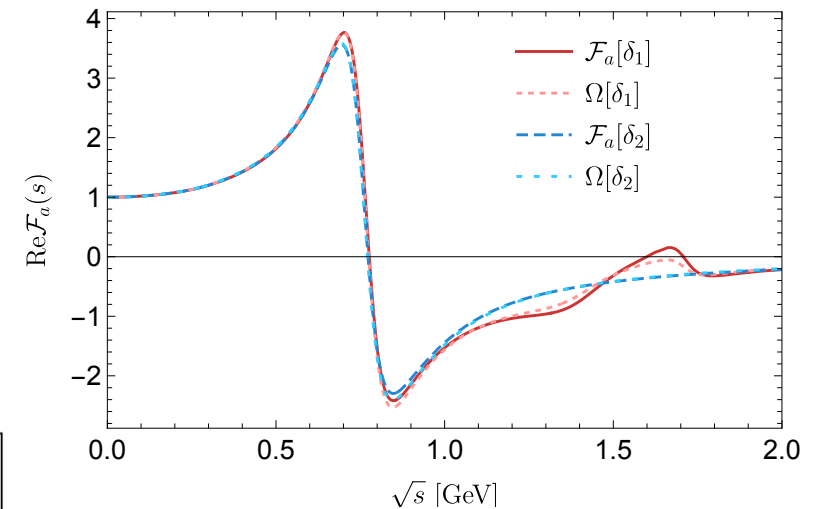
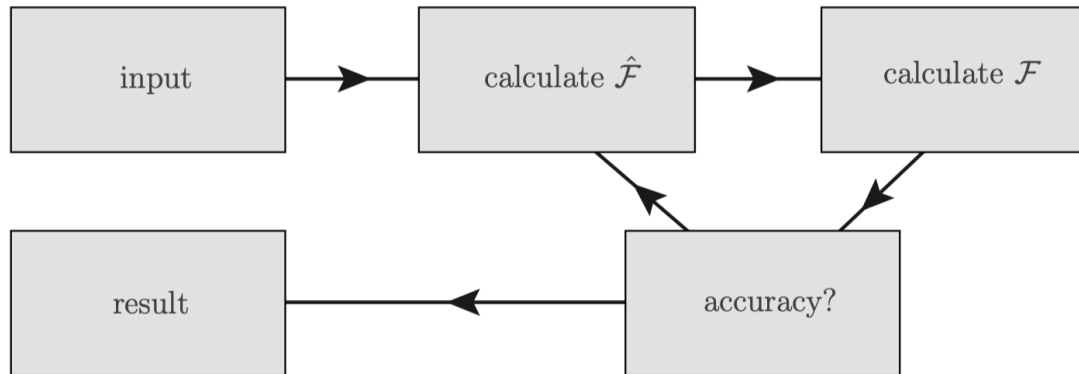


KT-type representation

$$\mathcal{F}(s) = a\Omega(s) \left\{ 1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s') \hat{\mathcal{F}}(s')}{|\Omega(s')| (s' - s)} \right\}, \quad \hat{\mathcal{F}}(s) = 3 \langle (1 - z_s^2) \mathcal{F} \rangle$$

Solution by numerical iteration

Initial input: $\mathcal{F}(s) = \Omega(s)$



KT equation: phase shifts

● KT-type representation

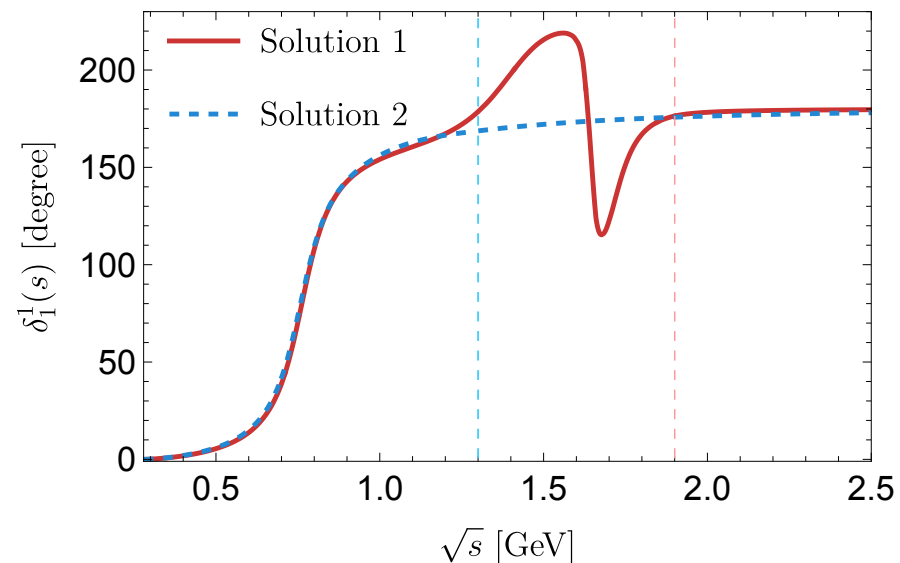
$$\mathcal{F}(s) = a\Omega(s) \left\{ 1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s') \hat{\mathcal{F}}(s')}{|\Omega(s')| (s' - s)} \right\}, \quad \hat{\mathcal{F}}(s) = 3 \langle (1 - z_s^2) \mathcal{F} \rangle$$

● $\pi\pi$ P-wave phase-shift δ taken as input

★ Solution 1: Schneider, Kubis and Niecknig, PRD (2012)

★ Solution 2: Pelaez, Raban and Ruiz de Elvira, PRD (2025) (fit-1)

★ The deviation in these solutions: **theoretical uncertainty**



$J/\psi \rightarrow \pi^0 \gamma^* \text{ TFF: } 2\pi$



- Dispersive representation (unsubtracted)

$$f_{\psi\pi^0}^{(2\pi)}(s) = \frac{1}{48\pi^2} \int_{4M_\pi^2}^{\infty} dx \frac{x\sigma_\pi^3(x)F_\pi^{V*}(x)f_1(x)}{x-s}$$

$$F_\pi^V(s) = \Omega(s)$$

- Sum rule

$$f_{\psi\pi^0}^{(2\pi)}(0) = \frac{1}{48\pi^2} \int_{4M_\pi^2}^{\infty} dx \sigma_\pi^3(x)F_\pi^{V*}(x)f_1(x)$$

✦ sum rule results in $\left| f_{\psi\pi^0}^{2\pi}(0) \right| = (4.8 \pm 0.2) \times 10^{-4} \text{GeV}^{-1}$

✦ $\left| f_{\psi\pi^0}(0) \right|$ from real photon width, $\Gamma_{\psi \rightarrow \pi^0 \gamma} = \frac{\alpha \left(M_\psi^2 - M_{\pi^0}^2 \right)^3}{24M_\psi^3} \left| f_{\psi\pi^0}(0) \right|^2$

$$\left| f_{\psi\pi^0}(0) \right| = (6.0 \pm 0.3) \times 10^{-4} \text{GeV}^{-1}$$

- ✦ two-pion intermediate state alone saturates the sum rule for the TFF normalization to about **80%**

$J/\psi \rightarrow \pi^0 \gamma^* \text{ TFF: } 4\pi (\rho')$

- The branching fractions of the J/ψ into multipion are actually larger
- From data on $e^+e^- \rightarrow [\text{hadrons}]_{I=1}$, the most important inelastic intermediate state of isospin $I = 1$ ought to be 4π , which is approximate to effective ρ' pole

$$\frac{\text{disc}}{2i} f_{\psi\pi^0}^{(\rho')}(s) = \frac{M_{\rho'}^2 g_{J/\psi \rightarrow \rho' \pi} g_{\rho' \gamma} \sqrt{s} \Gamma_{\rho'}(s)}{(M_{\rho'}^2 - s)^2 + s \Gamma_{\rho'}^2(s)}$$

$$\Gamma_{\rho'}^{(4\pi)}(s) = \theta(s - 16M_\pi^2) \frac{\gamma_{\rho' \rightarrow 4\pi}(s)}{\gamma_{\rho' \rightarrow 4\pi}(M_{\rho'}^2)} \Gamma_{\rho'}, \quad \gamma_{\rho' \rightarrow 4\pi}(s) = \frac{(s - 16M_\pi^2)^{\frac{9}{2}}}{s^2}$$

- $|g_{\rho' \gamma}| = 0.0752$ from VMD estimate Zanke, Hoferichter and Kubis, JHEP (2021)

- Effective coupling:

$$\text{BR}(J/\psi \rightarrow \rho' \pi) \text{BR}(\rho' \rightarrow 2\pi) = 2.2(1.2) \times 10^{-4} \text{ PDG}$$

$$\text{BR}(\rho' \rightarrow 2\pi) = 6\%$$

Zanke, Hoferichter and Kubis, JHEP (2021)



$$\text{BR}(J/\psi \rightarrow \rho' \pi) = 3.7(1.8) \times 10^{-3}$$

$$|g_{J/\psi \rightarrow \rho' \pi}| = 2.73_{-0.80}^{+0.61} \times 10^{-3} \text{ GeV}^{-1}$$

$$|f_{\psi\pi^0}^{(\rho')}(0)| = 1.3_{-0.4}^{+0.3} \times 10^{-4} \text{ GeV}^{-1} \quad 20\%?$$

Destructive interference implied by large N_c QCD

$J/\psi \rightarrow \pi^0 \gamma^*$ TFF: charmonium

- We adopt the simple monopole ansatz Fu et al., Mod.Phys.Lett.A (2012)

$$f_{\psi\pi^0}^{c\bar{c}}(s) = \frac{\left| f_{\psi\pi^0}^{(c\bar{c})}(0) \right| e^{i\delta^{c\bar{c}}}}{1 - s/\Lambda^2}$$

- The effective pole mass Λ is fixed to the mass of the lowest 1^{--} charmonium J/ψ

- Constraint: $\left| f_{\psi\pi^0}(0) \right| = (6.0 \pm 0.3) \times 10^{-4} \text{ GeV}^{-1}$

$$\left| f_{\psi\pi^0}^{(c\bar{c})}(0) \right| = -f_{\psi\pi^0}^{(2\pi,\rho')}(0) \cos \delta^{c\bar{c}} + \sqrt{\left| f_{\psi\pi^0}(0) \right|^2 - \left(f_{\psi\pi^0}^{(2\pi,\rho')}(0) \right)^2} \sin^2 \delta^{c\bar{c}}$$

Bin averaged fits

- We fix the sign of the weight factor for ϕ : $w_{\psi\phi\pi} = -|w_{\psi\phi\pi}|$
- Subtracted constant $f_{\psi\pi^0}^{(2\pi)}(0)$ is fixed by the sum rule
- Only **three** fit parameters

$$|w_{\psi\omega\pi}|, \delta_{\psi\omega\pi} \text{ and } \delta^{c\bar{c}}$$

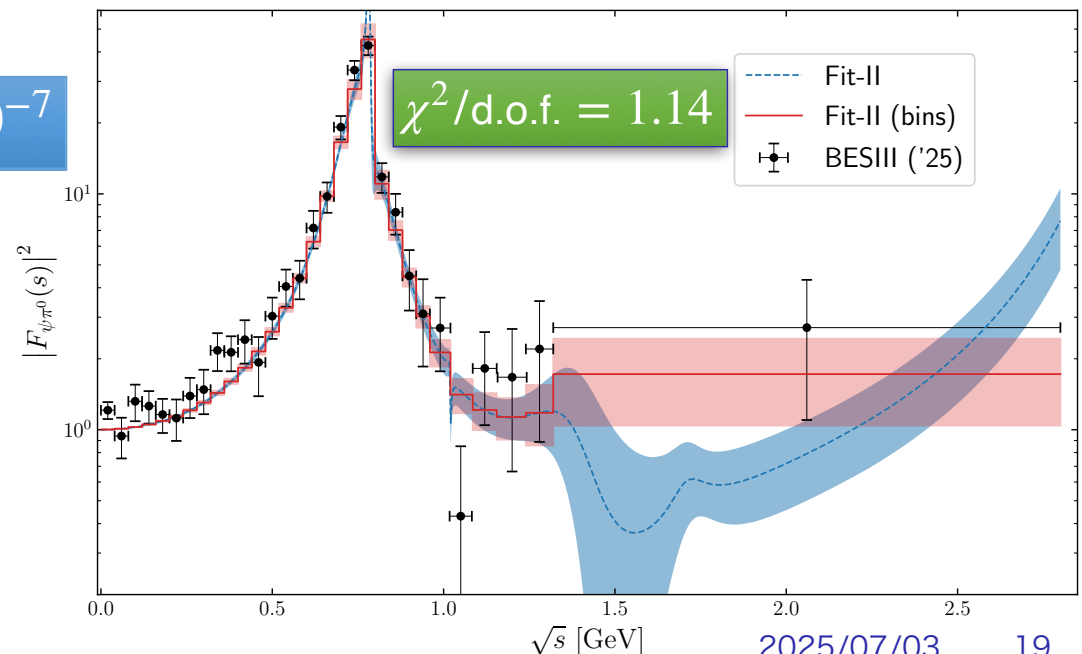
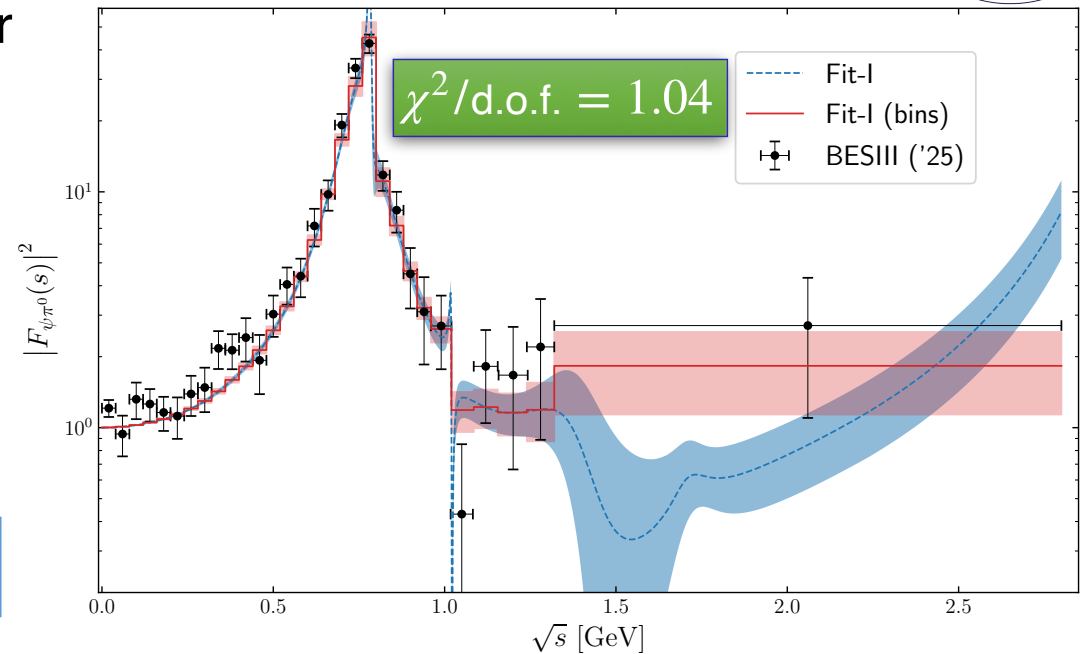
Fit-I & Fit-II

PDG

$$\text{BR}(J/\psi \rightarrow \phi\pi^0) = 3 \times 10^{-6} \text{ or } 1 \times 10^{-7}$$

Multi-solution ambiguity

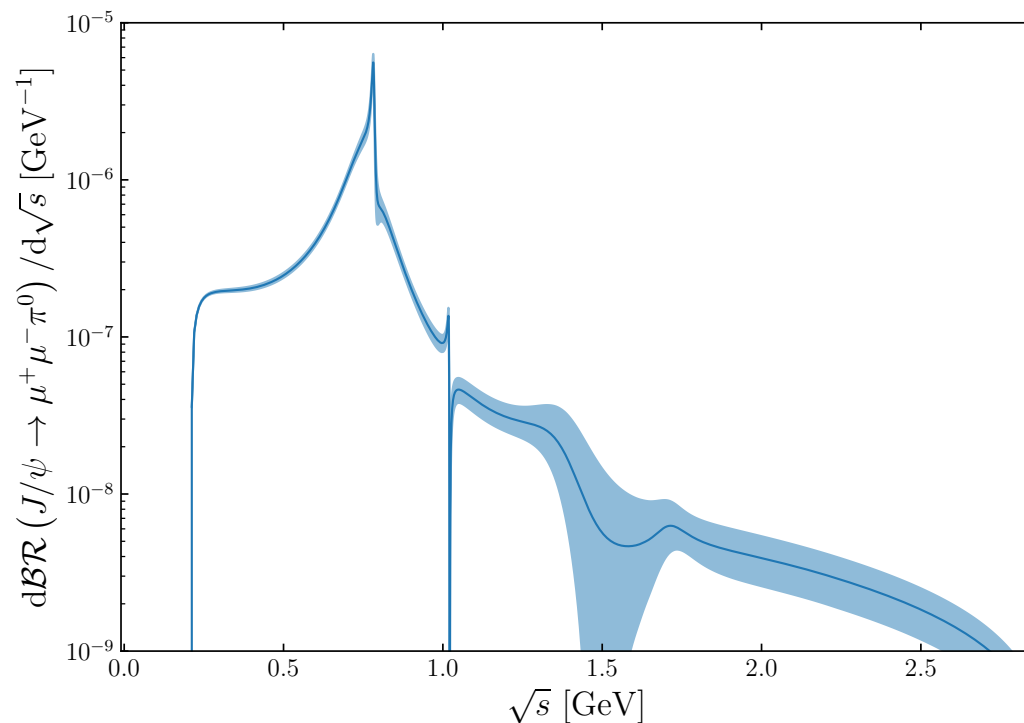
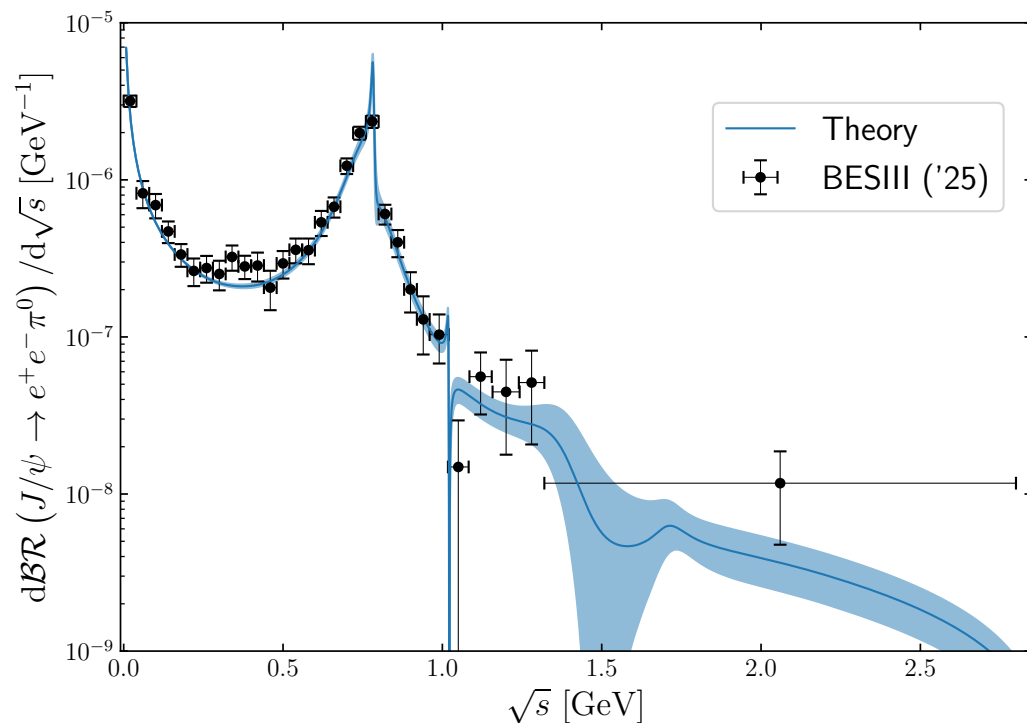
- ✿ Significant cusp structure corresponding to ω (ϕ ?)
- ✿ Nontrivial dip structure corresponding to $\rho'(1450)$



Branching fractions



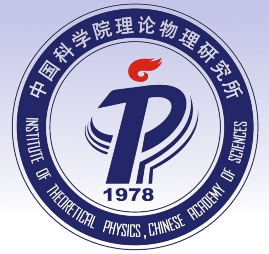
Differential branching fractions



Branching fractions ($\times 10^{-7}$)

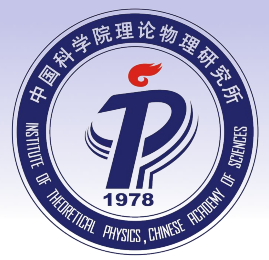
	Exp [10]	This Work	DR [11]	RChT [69]	RChT [4]	VMD [2]
$J/\psi \rightarrow \pi^0 e^+ e^-$	$8.06 \pm 0.31(\text{stat}) \pm 0.38(\text{syst})$	$7.00^{+0.60}_{-0.58}$	(5.5 ... 6.4)	12.94 ± 0.44	11.91 ± 1.38	$3.89^{+0.37}_{-0.33}$
$J/\psi \rightarrow \pi^0 \mu^+ \mu^-$	-	$4.15^{+0.54}_{-0.54}$	(2.7 ... 3.3)	3.04 ± 0.10	2.80 ± 0.32	$1.01^{+0.10}_{-0.09}$

Summary and outlook



- We update the results for $J/\psi \rightarrow e^+e^-\pi^0$ by incorporating $\rho - \omega$ interference with the dispersive method ensuring unitarity, analyticity and crossing symmetry
- $\text{BR}(J/\psi \rightarrow \pi^0 e^+ e^-) = 7.00_{-0.58}^{+0.60} \times 10^{-7}$, $\text{BR}(J/\psi \rightarrow \pi^0 \mu^+ \mu^-) = 4.15_{-0.54}^{+0.54} \times 10^{-7}$
- Extend to $\omega, \phi \rightarrow e^+e^-\pi^0$, where the $\rho - \omega$ mixing effect was ignored in previous dispersive analyses [JPAC, 2505.15309](#)

Thank you for your attention!



Backup

Analytic continuation of the inhomogeneities

$$\langle z^n \mathcal{F} \rangle \equiv \frac{1}{2} \int_{-1}^1 z^n \mathcal{F} \left(\frac{3s_0 - s + z\kappa(s)}{2} \right) dz = \frac{1}{\kappa(s)} \int_{s_-(s)}^{s_+(s)} ds' \left(\frac{2s' - 3s_0 + s}{\kappa(s)} \right)^n \mathcal{F}(s')$$

a singularity-free function $\tilde{\mathcal{F}}(s)$

$$\tilde{\mathcal{F}}(s) \equiv \kappa^3(s) \hat{\mathcal{F}}(s) = 3 \int_{s_-(s)}^{s_+(s)} ds' \left(\kappa^2(s) - (2s' - 3s_0 + s)^2 \right) \mathcal{F}(s')$$

$$\mathcal{F}(s) = a\Omega(s) \left\{ 1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s') \tilde{\mathcal{F}}(s')}{|\Omega(s')| \kappa^3(s') (s' - s)} \right\}$$

