中南大學
CENTRAL SOUTH UNIVERSITY

Investigation of the three-body decay for D_s^+ and D^0

Speaker: Wei Liang

Collaborator: Chu-wen Xiao, Yu Lu, De-liang Yao, Jing-yu Yi, Yu-wen Peng, Sara Rahmani

Based on Phys.Rev.D 108 (2023) 11, 116001, Phys.Rev.D 110 (2024) 3, 036013 and arXiv: 2503.02224

2025/07/13 GuiLin

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Background and Motivations.

- Experiments：

① $D_s^+(D^+) \to K^+K^-\pi^+, \pi^+\pi^0\eta, \pi^+\pi^-\pi^+, K_s^0K_s^0\pi^+, K_s^0K^+\pi^0$:

P. L. Frabetti et al. [E687], Phys. Lett. B 351, 591-600 (1995).

R. E. Mitchell et al. [CLEO], Phys. Rev. D 79, 072008 (2009).

M. Ablikim et al. [BESIII], Phys. Rev. D 104, 012016 (2021).

M. Ablikim et al. [BESIII], Phys. Rev. Lett. 123, 112001 (2019)

B. Aubert et al. [BaBar], Phys. Rev. D 79, 032003 (2009).

M. Ablikim et al. [BESIII], Phys. Rev. D 106, 112006 (2022).

M. Ablikim et al. [BESIII], Phys. Rev. D 105, L051103 (2022).

M. Ablikim et al. [BESIII], Phys. Rev. Lett. 129, 182001 (2022).

② $D^0 \to K^+K^-\eta, K_s^0K_s^0\eta, \pi^+\pi^-\eta, K^-\pi^+\eta, \pi^0\pi^0\pi^0, \eta\eta\eta$:

L. K. Li et al. [Belle], JHEP 09, 075 (2021).

M. Ablikim et al. [BESIII], Phys. Rev. D 101, 052009 (2020).

Y. Q. Chen et al. [Belle], Phys. Rev. D 102, 012002 (2020).

P. Rubin et al. [CLEO], Phys. Rev. Lett. 96,081802 (2006).

M.Ablikim et al. [BESIII], Phys. Lett. B 781,368 (2018).

- Theories：

① $D_s^+ \to K^+K^-\pi^+, \pi^+\pi^0\eta, \pi^+\pi^-\pi^+, K_s^0K_s^0\pi^+, K_s^0K^+\pi^0$:

J. Y. Wang et al. Phys. Lett. B 821, 136617 (2021).

Z. Y. Wang et al. Phys. Rev. D 105, 016025 (2022).

R. Escribano et al. arXiv:2302.03312 [hep-ph].

R. Molina et al. Phys. Lett. B 803, 135279 (2020).

J. M. Dias et al. Phys. Rev. D 94, 096002 (2016).

N. N. Achasov et al. Phys. Rev. D 107, 056009 (2023).

L. R. Dai et al. Eur. Phys. J. C 82, 225 (2022).

X. Zhu et al. Phys. Rev. D 107, 034001 (2023).

② $D^0 \to K^-\pi^+\eta, \pi^0\pi^0\pi^0, \eta\eta\eta, \pi^0\pi^0\bar{K}^0$:

Genaro Toledo et al. Eur.Phys.J.C 81, 268 (2021).

Z. Y. Wang et al. Phys. Rev D 105, 016030 (2022).

Xiao-Hui Zhang et al. Phys.Rev.D 110, 114050 (2024).

Background and Motivations.

- $D_s^+ \to \pi^+\pi^-K^+$:

J.M. Link et al. [FOCUS Collaboration], Phys. Lett. B 601, 10-19 (2004).

Decay channel	Fit fraction (%)	Phase ϕ_j (degrees)	Amplitude coefficient
$\rho(770)K^+$	$38.83 \pm 5.31 \pm 2.61$	0 (fixed)	1 (fixed)
$K^*(892)\pi^+$	$21.64 \pm 3.21 \pm 1.14$	$161.7 \pm 8.6 \pm 2.2$	$0.747 \pm 0.080 \pm 0.031$
NR	$15.88 \pm 4.92 \pm 1.53$	$43.1 \pm 10.4 \pm 4.4$	$0.640 \pm 0.118 \pm 0.026$
$K^*(1410)\pi^+$	$18.82 \pm 4.03 \pm 1.22$	$-34.8 \pm 12.1 \pm 4.3$	$0.696 \pm 0.097 \pm 0.025$
$K_0^*(1430)\pi^+$	$7.65 \pm 5.0 \pm 1.70$	$59.3 \pm 19.5 \pm 13.2$	$0.444 \pm 0.141 \pm 0.060$
$\rho(1450)K^+$	$10.62 \pm 3.51 \pm 1.04$	$-151.7 \pm 11.1 \pm 4.4$	$0.523 \pm 0.091 \pm 0.020$
C.L. = 5.5%	$\chi^2 = 38.5$	d.o.f. = 43 (#bins) - 17 (#free parameters)	

$$\frac{\Gamma(D_s^+ \to K^+\pi^+\pi^-)}{\Gamma(D_s^+ \to K^+K^-\pi^+)} = 0.127 \pm 0.007 \pm 0.014\,.$$

Medina Ablikim et al. [BESIII Collaboration], JHEP 08, 196 (2022).

Amplitude	Phase ϕ_n (rad)	FF(%)	Statistical significance(σ)
$D_s^+ \to K^+\rho^0$	0.0 (fixed)	$32.5 \pm 3.1 \pm 3.6$	>10
$D_s^+ \to K^+\rho(1450)^0$	$2.72 \pm 0.14 \pm 0.24$	$12.7 \pm 3.2 \pm 2.7$	>10
$D_s^+ \to K^+f_0(500)$	$0.98 \pm 0.17 \pm 0.19$	$7.0 \pm 2.2 \pm 4.0$	6.8
$D_s^+ \to K^+f_0(980)$	$5.02 \pm 0.15 \pm 0.15$	$4.4 \pm 1.3 \pm 1.1$	6.9
$D_s^+ \to K^+f_0(1370)$	$6.03 \pm 0.14 \pm 0.26$	$19.9 \pm 3.1 \pm 2.9$	>10
$D_s^+ \to K^*(892)^0\pi^+$	$3.03 \pm 0.09 \pm 0.04$	$30.3 \pm 1.9 \pm 1.8$	>10
$D_s^+ \to K^*(1410)^0\pi^+$	$5.62 \pm 0.14 \pm 0.09$	$4.7 \pm 2.2 \pm 2.1$	5.2
$D_s^+ \to K_0^*(1430)^0\pi^+$	$1.89 \pm 0.19 \pm 0.18$	$18.9 \pm 2.5 \pm 2.4$	8.6

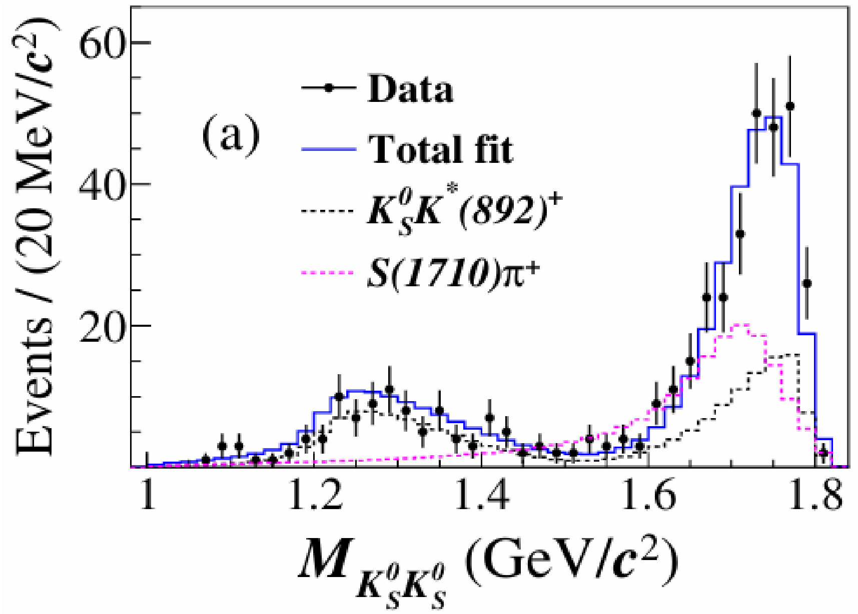
Intermediate process	BF(10^{-3})	PDG(10^{-3})
$D_s^+ \to K^+\rho^0$	$1.99 \pm 0.20 \pm 0.22$	2.5 ± 0.4
$D_s^+ \to K^+\rho(1450)^0$	$0.78 \pm 0.20 \pm 0.17$	0.69 ± 0.64
$D_s^+ \to K^*(892)^0\pi^+$	$1.85 \pm 0.13 \pm 0.11$	1.41 ± 0.24
$D_s^+ \to K^*(1410)^0\pi^+$	$0.29 \pm 0.13 \pm 0.13$	1.23 ± 0.28
$D_s^+ \to K_0^*(1430)^0\pi^+$	$1.15 \pm 0.16 \pm 0.15$	0.50 ± 0.35
$D_s^+ \to K^+f_0(500)$	$0.43 \pm 0.14 \pm 0.24$	-
$D_s^+ \to K^+f_0(980)$	$0.27 \pm 0.08 \pm 0.07$	-
$D_s^+ \to K^+f_0(1370)$	$1.22 \pm 0.19 \pm 0.18$	-
$D_s^+ \to (K^+\pi^+\pi^-)_{NR}$	-	1.03 ± 0.34

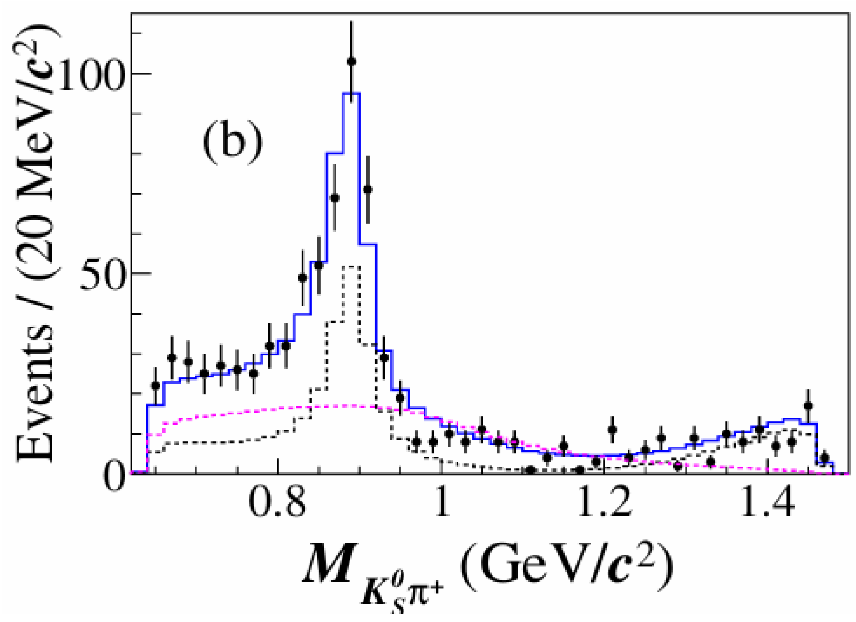
$$\mathcal{B}(D_s^+ \to K^+\pi^+\pi^-) = (6.11 \pm 0.18_{\text{stat.}} \pm 0.11_{\text{syst.}}) \times 10^{-3}$$

Background and Motivations.

- $D_s^+ \to K_s^0 K_s^0 \pi^+$:

M. Ablikim et al. [BESIII], Phys. Rev. D 105, L051103 (2022).





Amplitude	BF (10^{-3})
$D_s^+ \to K_S^0 K^*(892)^+ \to K_S^0 K_S^0 \pi^+$	$3.0 \pm 0.3 \pm 0.1$
$D_s^+ \to S(1710)\pi^+ \to K_S^0 K_S^0 \pi^+$	$3.1 \pm 0.3 \pm 0.1$

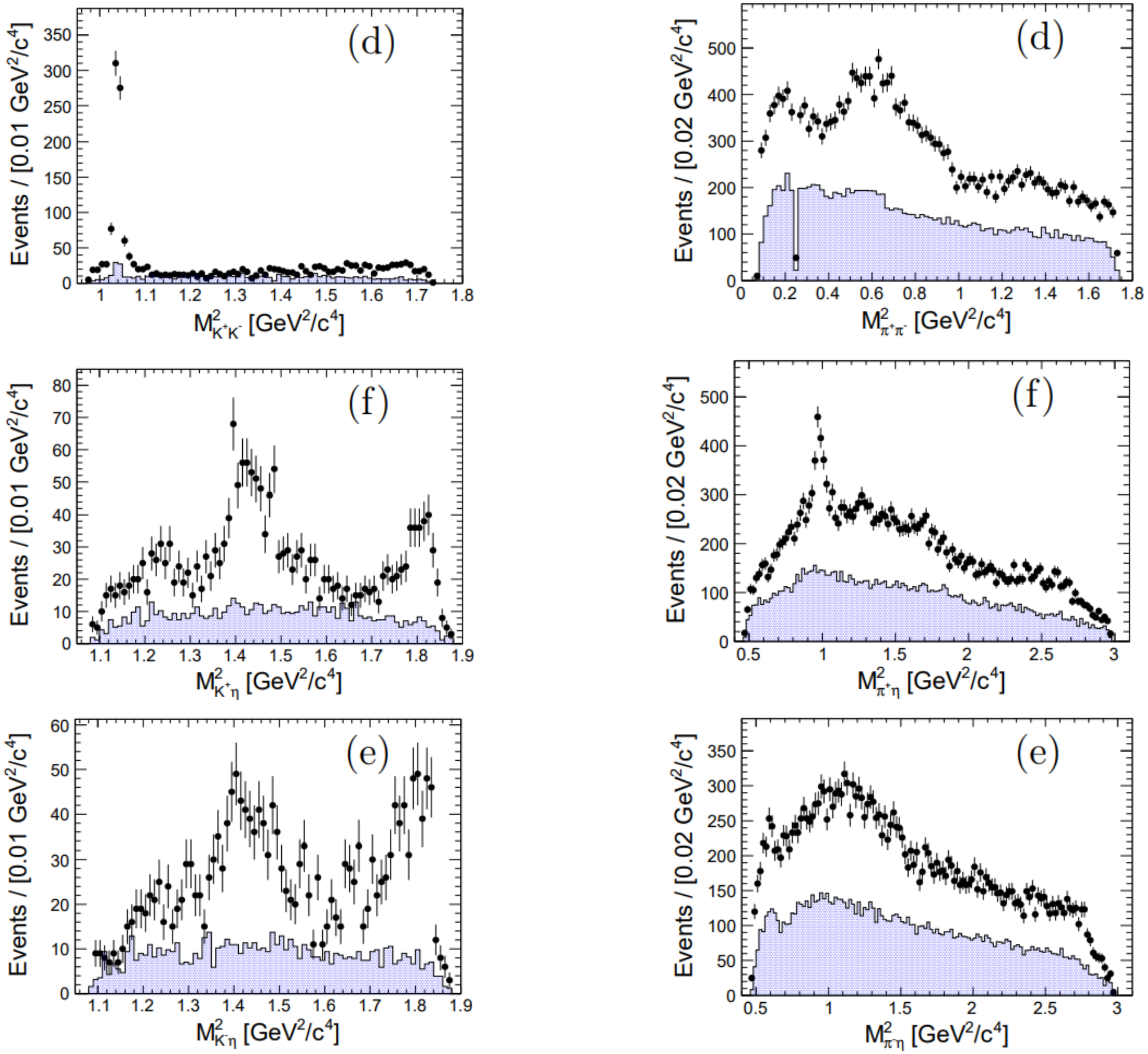
$$M_{S(1710)} = (1.723 \pm 0.011_{\rm stat} \pm 0.002_{syst}) GeV/c^2$$

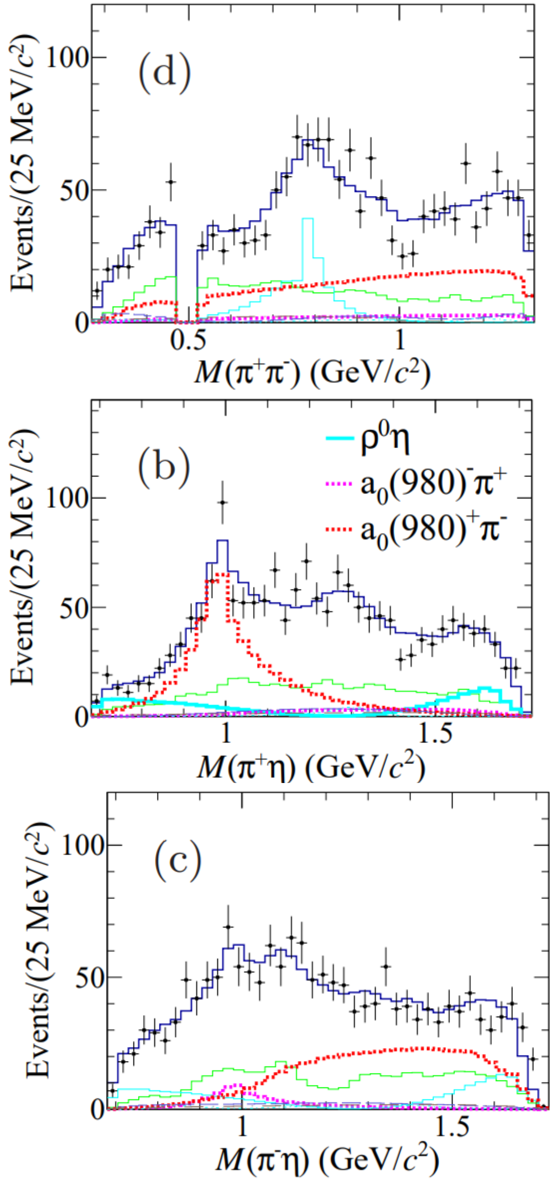
$$\Gamma_{S(1710)} = (0.140 \pm 0.014_{\rm stat} \pm 0.004_{syst}) GeV/c^2$$

Background and Motivations.

- $D^0 \to K^+K^-\eta,\ \ \pi^+\pi^-\eta$:

L. K. Li et al. [Belle], JHEP 09, 075 (2021).
M. Ablikim et al. [BESIII], Phys. Rev. D 110, L111102 (2024).





Formalism

Formalism

- The processes of three-body decay:

Feynman diagrams
- quark level

hadronize
- hadron level

S-wave
- propagators, two-body scattering amplitudes(Bethe-Salpeter equation)

other resonances
- relativistic amplitude

differential width distribution
- fitting experimental data

branching fractions

Propagators.

- The diagonal matrix G is two intermediate meson propagators:

$$G_{ii}(s) = i\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m_1^2+i\varepsilon}\frac{1}{(p_1+p_2-q)^2-m_2^2+i\varepsilon}.$$

- The integral is logarithmically divergent, there are two methods to solve this problem:

✓ the three-momentum cut off:

$$G_{ii}(s) = \int_0^{q_{\max}} \frac{q^2dq}{(2\pi)^2}\frac{\omega_1+\omega_2}{\omega_1\omega_2\left[s-(\omega_1+\omega_2)^2+i\varepsilon\right]}$$

$$\omega_i = \sqrt{\left(\vec{q}^{\,2}+m_i^2\right)} \qquad s=\left(p_1+p_2\right)^2$$

✓ the dimensional regularization method:

$$\begin{aligned} G_{ii}(s) = \frac{1}{16\pi^2}\{a_\mu + \ln\frac{m_1^2}{\mu^2} + \frac{m_2^2-m_1^2+s}{2s}\ln\frac{m_2^2}{m_1^2} \\ +\frac{q_{cm}(s)}{\sqrt{s}}[\ln\left(s-\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right)+\ln\left(s+\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right) \\ -\ln\left(-s-\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right)-\ln\left(-s+\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right)]\} \end{aligned}$$

- The value of the subtraction constant :

✓ a relationship between two regularization method :

$$a_\mu = 16\pi^2[G^{CO}(s_{thr},q_{max}) - G^{DR}(s_{thr},\mu)],$$

G. Montaña et al., Phys. Rev. D 107, 054014 (2023).

✓ a calculation which adopted by other references :

$$a_{PP'}(\mu) = -2\log\left(1+\sqrt{1+\frac{m_1^2}{\mu^2}}\right)+\cdots,$$

J. A. Oller et al., Phys. Lett. B 500, 263-272 (2001).

Two-body scattering amplitudes.

- T is the two-body scattering amplitudes， it can be evaluated by the coupled channel Bethe-Salpeter equation of ChUA:

$$T = [1 - VG]^{-1} V,$$

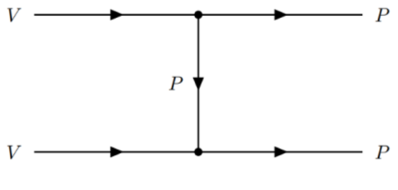
- The interaction potentials of each coupled channel for PP→PP processes:

➢ PP→PP：
- I = 0： $\pi^+\pi^-$， $\pi^0\pi^0$， K^+K^-， $K^0\overline{K}^0$， $\eta\eta$
- I = 1/2： $K^+\pi^-$， $K^0\pi^0$， $K^0\eta$
- I = 1： K^+K^-， $K^0\overline{K}^0$， $\pi^0\eta$

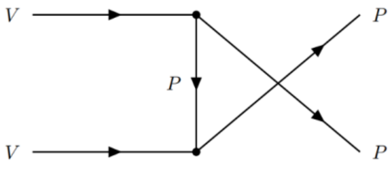
J. A. Oller and E. Oset, Nucl. Phys. A 620, 438-456 (1997).
L. S. Geng and E. Oset, Phys. Rev. D 79, 074009 (2009).
Z. L. Wang and B. S. Zou, Eur. Phys. J. C 82, 509 (2022).
M. Bando et al., Phys. Rept. 164, 217-314 (1988)

➢ VV→VV：
- Tree-level transition amplitudes of the four-vector-contact diagrams
- t(u)-channel vector-exchange diagrams

➢ VV→PP：
- t(u)-channel pseudoscalar-exchange diagrams



(a) The t-channel



(b) The u- channel

$$\mathcal{L}_{VPP} = -ig \langle V_\mu [P, \partial^\mu P] \rangle$$

$$F = \frac{\Lambda^2 - m_{ex}^2}{\Lambda^2 - q^2},$$

The decay of $D_s^+ \to \pi^+\pi^-K^+$

$D_s^+ \to \pi^+\pi^-K^+$

- The external and internal W-emission mechanism:

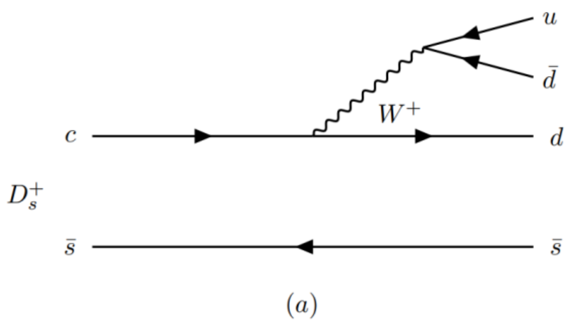


FIG. 1: W-external emission mechanism for the $D_s^+ \to K^+\pi^+\pi^-$ decay.

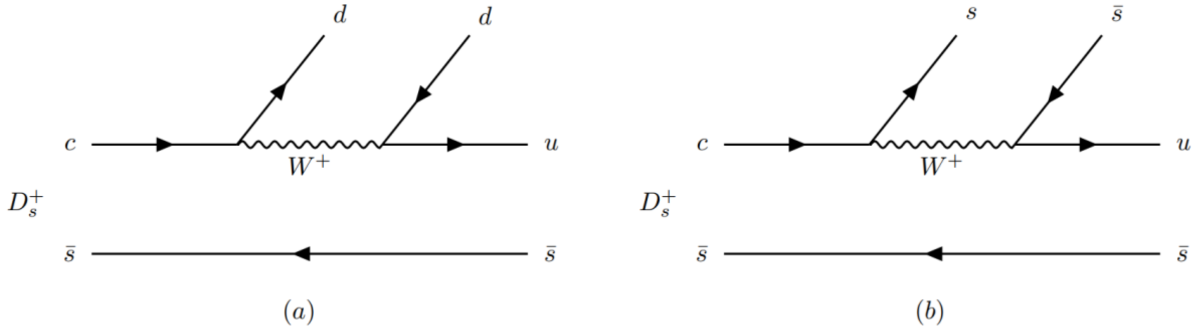
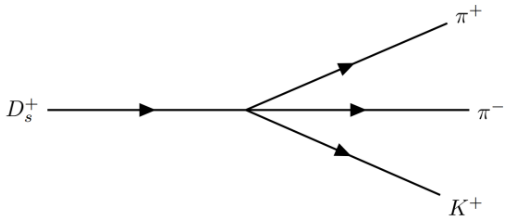


FIG. 2: W-internal emission mechanism for the $D_s^+ \to K^+\pi^+\pi^-$ decay.

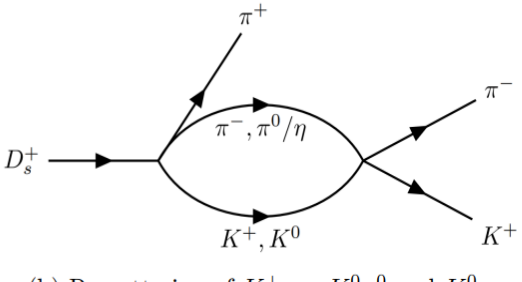
- The total contributions:

$$
\begin{aligned}
H =& H^{(a)} + H^{(b)} + H^{(2a)} + H^{(2b)} \\
=& V_{cd}V_{ud}(1+\beta)\Big[V_P\Big(\pi^+\pi^-K^+ - \frac{1}{\sqrt{2}}\pi^+\pi^0K^0 + \frac{1}{\sqrt{6}}\eta\pi^+K^0 + K^+K^0\bar{K}^0\Big) \\
& + V_P'\Big(-K^+K^+K^- - \eta\eta K^+ + \frac{2}{\sqrt{6}}\eta\pi^+K^0 - K^+K^0\bar{K}^0 + \frac{1}{\sqrt{3}}\eta\pi^0K^+\Big)\Big] \\
=& C_1\Big(\pi^+\pi^-K^+ - \frac{1}{\sqrt{2}}\pi^+\pi^0K^0 + \frac{1}{\sqrt{6}}\eta\pi^+K^0 + K^+K^0\bar{K}^0\Big) \\
& - C_2\Big(K^+K^+K^- + \eta\eta K^+ - \frac{2}{\sqrt{6}}\eta\pi^+K^0 + K^+K^0\bar{K}^0\Big).
\end{aligned}
$$

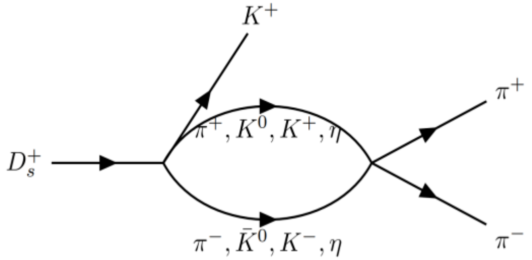
- Tree-level production and final state interactions via rescattering mechanism:



(a) Tree-level production.



(b) Rescattering of $K^+\pi^-$, $K^0\pi^0$ and $K^0\eta$.



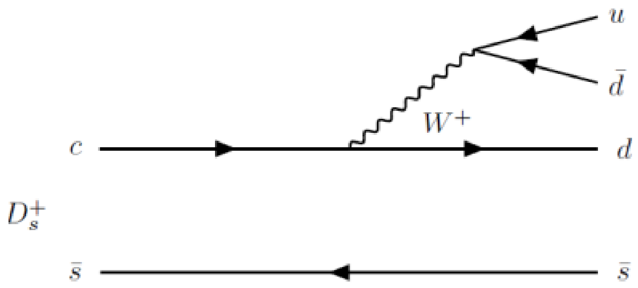
(c) Rescattering of $\pi^+\pi^-$, $\pi^0\eta$, $\eta\eta$ and K^+K^-.

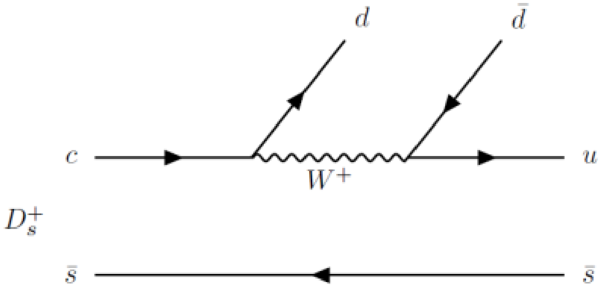
$D_s^+ \to \pi^+\pi^-K^+$

- The amplitudes for the decay $D_s^+ \to K^+\pi^+\pi^-$ in the S-wave:

$$\begin{aligned} t(s_{12}, s_{23}) = & C_1 \Big[1 + G_{\pi^-K^+}(s_{23}) T_{\pi^-K^+\to\pi^-K^+}(s_{23}) + G_{\pi^+\pi^-}(s_{12}) T_{\pi^+\pi^-\to\pi^+\pi^-}(s_{12}) \\ & - \frac{1}{\sqrt{2}} G_{\pi^0K^0}(s_{23}) T_{\pi^0K^0\to\pi^-K^+}(s_{23}) + \frac{1}{\sqrt{6}} G_{\eta K^0}(s_{23}) T_{\eta K^0\to\pi^-K^+}(s_{23}) \\ & + G_{K^0\bar{K}^0}(s_{12}) T_{K^0\bar{K}^0\to\pi^+\pi^-}(s_{12}) \Big] - C_2 \Big[G_{K^+K^-}(s_{12}) T_{K^+K^-\to\pi^+\pi^-}(s_{12}) \\ & + G_{\eta\eta}(s_{12}) T_{\eta\eta\to\pi^+\pi^-}(s_{12}) - \frac{2}{\sqrt{6}} G_{\eta K^0}(s_{23}) T_{\eta K^0\to\pi^-K^+}(s_{23}) \\ & + G_{K^0\bar{K}^0}(s_{12}) T_{K^0\bar{K}^0\to\pi^+\pi^-}(s_{12}) \Big] \end{aligned}$$

- The contribution of other intermediate states:





$$M_{K^*(892)}(s_{12}, s_{23}) = \frac{D_{K^*(892)} e^{i\alpha_{K^*(892)}}}{s_{23} - m_{K^*(892)}^2 + i m_{K^*(892)} \Gamma_{K^*(892)}} \left[(m_K^2 - m_\pi^2) \frac{m_{D_s^+}^2 - m_\pi^2}{m_{K^*(892)}^2} - s_{13} + s_{12} \right],$$

$$M_{K^*(1430)}(s_{12}, s_{23}) = \frac{D_{K^*(1430)} e^{i\alpha_{K^*(1430)}}}{s_{23} - m_{K^*(1430)}^2 + i m_{K^*(1430)} \Gamma_{K^*(1430)}} \left[(s_{23} - m_K^2 - m_\pi^2) \cdot (s_{13} + s_{12} - m_K^2 - m_\pi^2) \right],$$

$$M_\rho(s_{12}, s_{23}) = \frac{D_\rho e^{i\alpha_\rho}}{s_{12} - m_\rho^2 + i m_\rho \Gamma_\rho} (s_{23} - s_{13}),$$

$$M_{f_0(1370)}(s_{12}, s_{23}) = \frac{D_{f_0(1370)} e^{i\alpha_{f_0(1370)}}}{s_{12} - m_{f_0(1370)}^2 + i m_{f_0(1370)} \Gamma_{f_0(1370)}} \left[(s_{12} - 2m_\pi^2) \cdot (s_{13} + s_{23} - 2m_\pi^2) \right],$$

$$s_{12} + s_{23} + s_{13} = m_{D_s^+}^2 + m_K^2 + m_\pi^2 + m_\pi^2,$$

$D_s^+ \to \pi^+\pi^-K^+$

- The double differential width distribution of three-body decay:

$$\frac{d^2\Gamma}{ds_{12}ds_{23}} = \frac{1}{(2\pi)^3}\frac{1}{32m_{D_s^+}^3}\left(\left|t(s_{12},s_{23}) + M_{K^*(892)} + M_{K^*(1430)} + M_{f_0(1370)} + M_{\rho} + M_{\rho(1450)}\right|^2\right)$$

- The limits of integral variable for the invariant masses are higher than 1.2 GeV, we need to smoothly extrapolate $G(s)T(s)$ above the energy cut $\sqrt{s} \geq \sqrt{s_{cut}} = 1.1$ GeV :

$$G(s)T(s) = G(s_{\text{cut}})T(s_{\text{cut}})e^{-\alpha(\sqrt{s}-\sqrt{s_{\text{cut}}})}, \quad \text{for } \sqrt{s} > \sqrt{s_{\text{cut}}}$$

- The parameters need to be fitted:

S-wave: C_1, C_2, α

other resonances: $D_\rho, \alpha_\rho, D_{K^*(892)}, \alpha_{K^*(892)}, D_{K^*(1430)}, \alpha_{K^*(1430)}, D_{f_0(1370)}, \alpha_{f_0(1370)}, D_{\rho(1450)}, \alpha_{\rho(1450)},$

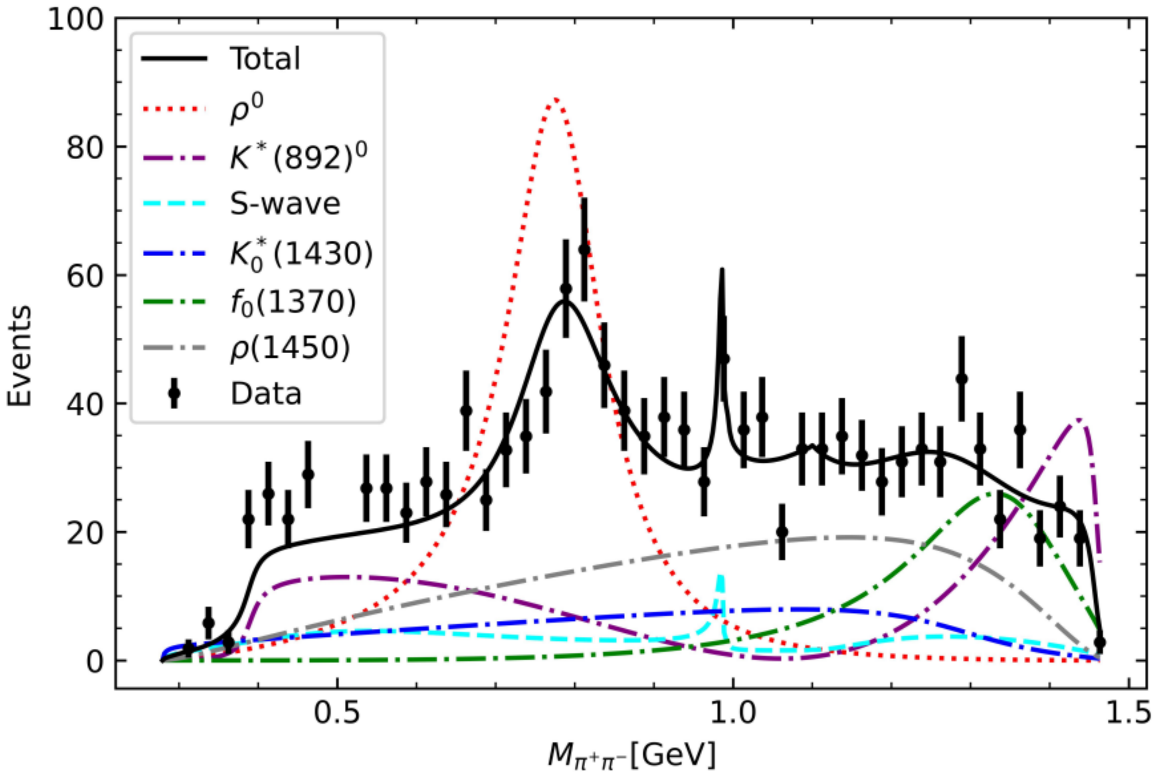
V. R. Debastiani, W. H. Liang, J. J. Xie, and E. Oset, Phys. Lett. B 766, 59 (2017).

$D_s^+ \to \pi^+\pi^-K^+$

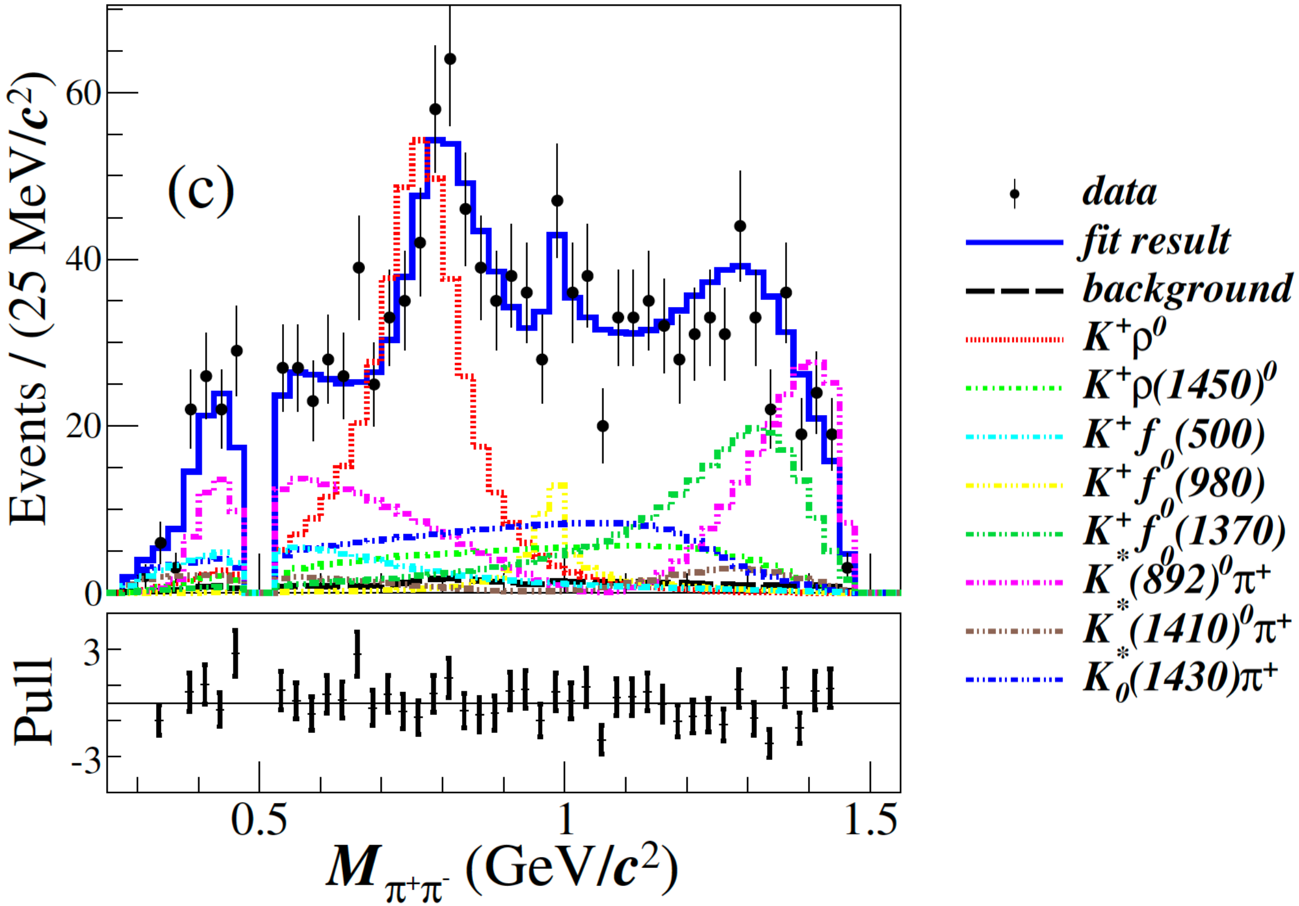
- $\boldsymbol{\pi^+\pi^-} \quad \boldsymbol{\chi^2/dof = 183.37/128 = 1.43}$

Medina Ablikim et al. [BESIII Collaboration], JHEP 08, 196 (2022).

Our Results:



BESIII Experiment:



Parameters	C_1	C_2	α	D_ρ	α_ρ	$D_{K^*(892)}$	$\alpha_{K^*(892)}$	$D_{K^*(1430)}$	$\alpha_{K^*(1430)}$	$D_{f_0(1370)}$	$\alpha_{f_0(1370)}$	$D_{\rho(1450)}$	$\alpha_{\rho(1450)}$	$\chi^2/dof.$
Fit	263.74	-63.08	12.34	80.77	0.18	62.99	3.54	-62.50	1.21	-60.24	3.07	-456.70	0.93	1.43

$D_s^+ \to \pi^+\pi^-K^+$

- $K^+\pi^+$ $\chi^2/dof = 183.37/128 = 1.43$

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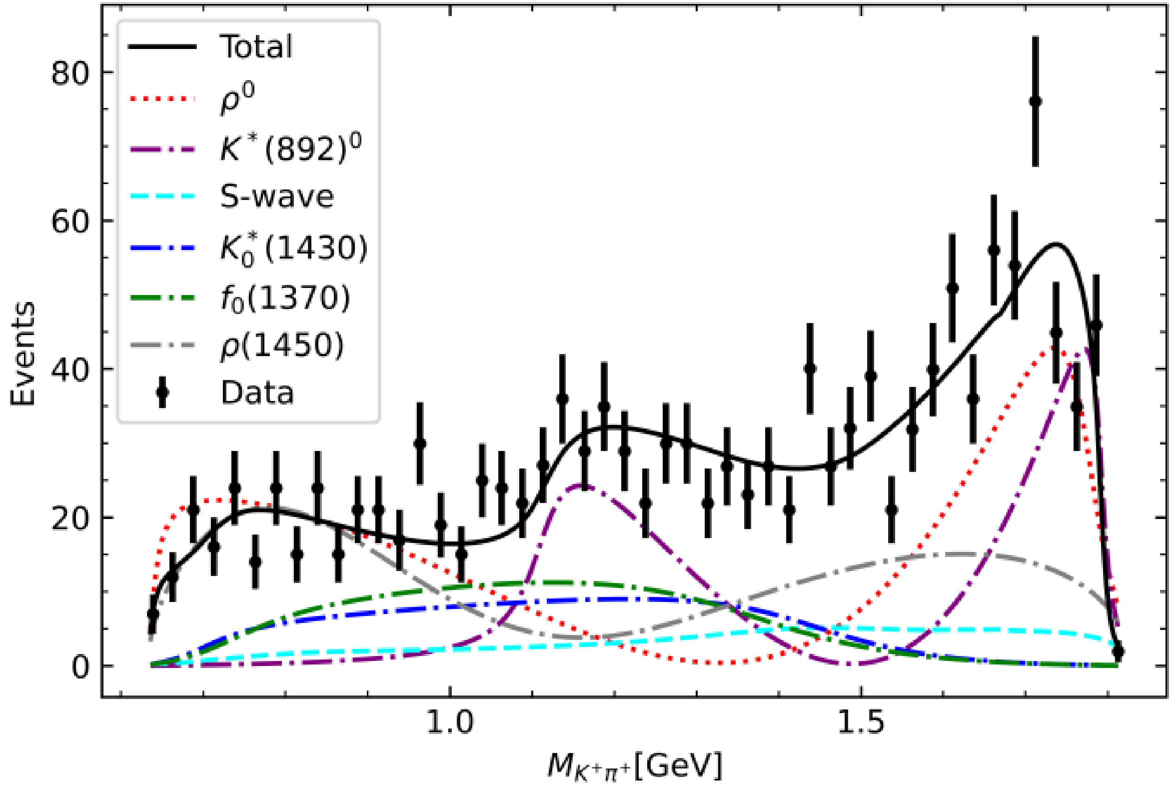

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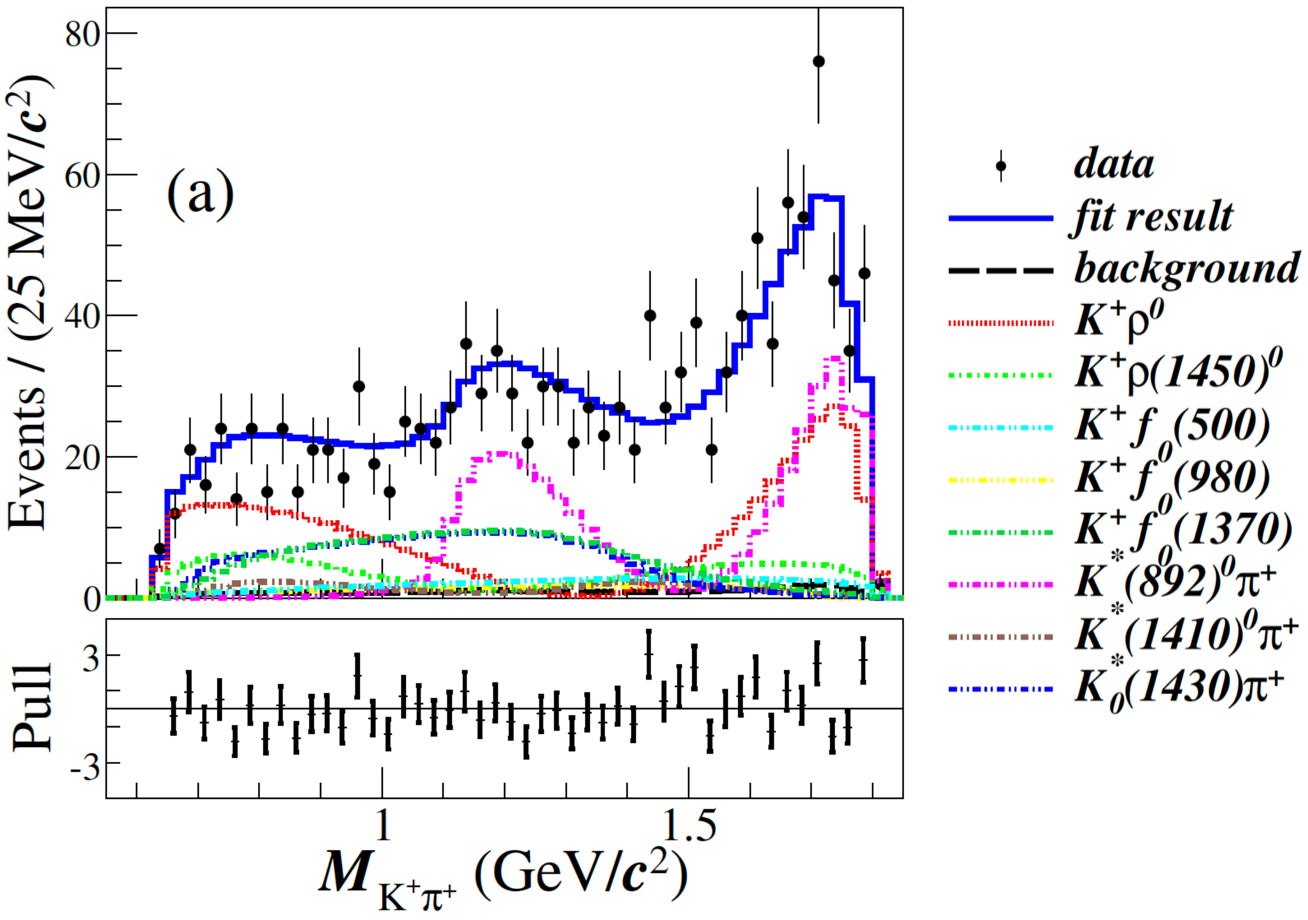

Parameters	C_1	C_2	α	D_ρ	α_ρ	$D_{K^*(892)}$	$\alpha_{K^*(892)}$	$D_{K^*(1430)}$	$\alpha_{K^*(1430)}$	$D_{f_0(1370)}$	$\alpha_{f_0(1370)}$	$D_{\rho(1450)}$	$\alpha_{\rho(1450)}$	$\chi^2/dof.$
Fit	263.74	-63.08	12.34	80.77	0.18	62.99	3.54	-62.50	1.21	-60.24	3.07	-456.70	0.93	1.43

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- $\boldsymbol{K^+\pi^-}$ $\quad \boldsymbol{\chi^2/dof = 183.37/128 = 1.43}$

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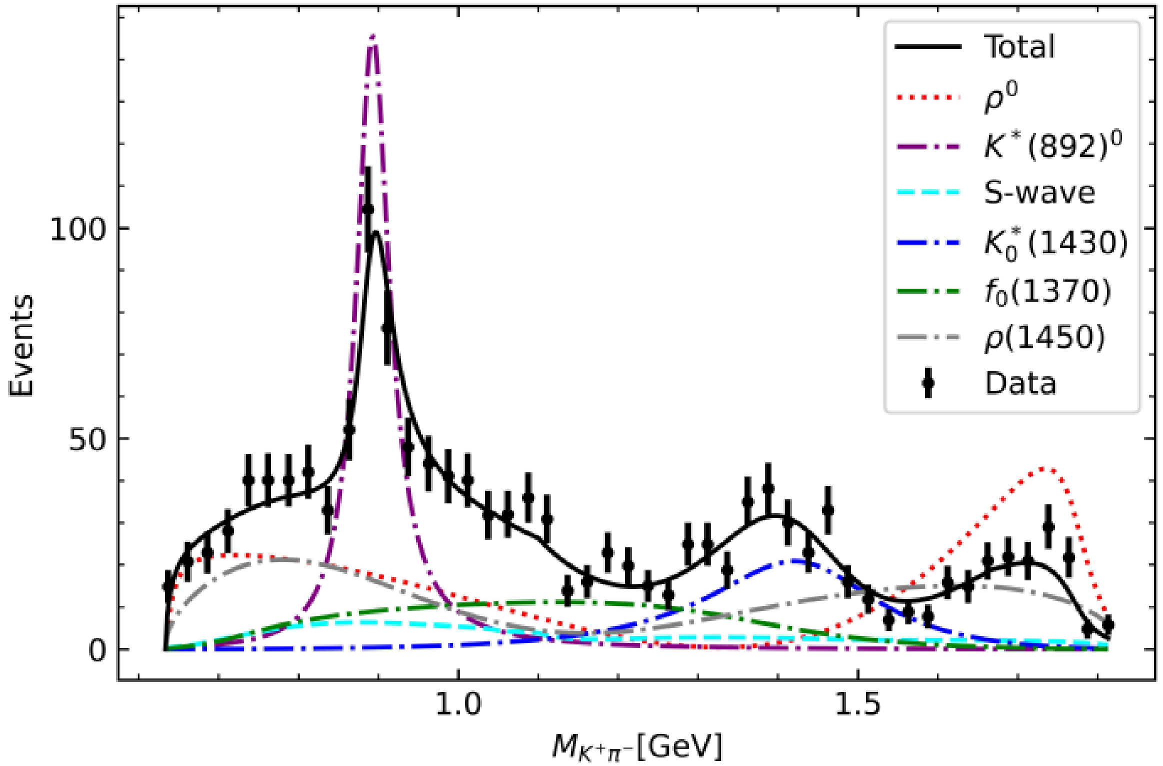
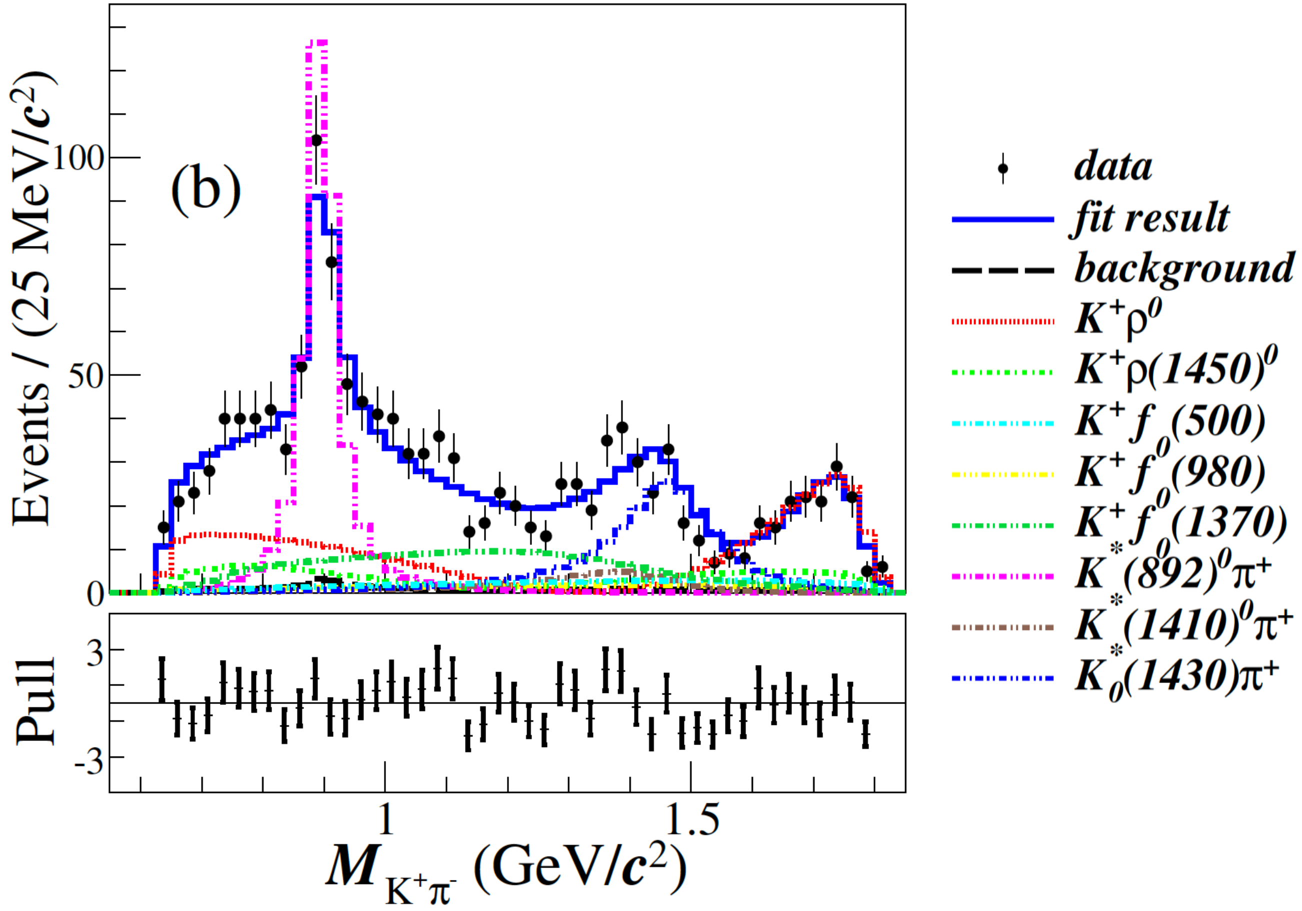


BESIII Experiment:



Parameters	C_1	C_2	α	D_ρ	α_ρ	$D_{K^*(892)}$	$\alpha_{K^*(892)}$	$D_{K^*(1430)}$	$\alpha_{K^*(1430)}$	$D_{f_0(1370)}$	$\alpha_{f_0(1370)}$	$D_{\rho(1450)}$	$\alpha_{\rho(1450)}$	$\chi^2/dof.$
Fit	263.74	-63.08	12.34	80.77	0.18	62.99	3.54	-62.50	1.21	-60.24	3.07	-456.70	0.93	1.43

Branching fractions

- The ratios of the branching fractions between different resonances :

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ f_0(500) \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 0.20^{+0.02}_{-0.02},$$

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ f_0(980) \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 0.06^{+0.02}_{-0.02},$$

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ \rho \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 1.59^{+0.02}_{-0.03},$$

$$\frac{\mathcal{B}\left[D_S^+ \to f_0(1370) K^+ \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 0.58^{+0.06}_{-0.11},$$

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ \rho(1450) \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 1.28^{+0.02}_{-0.05},$$

- The branching ratios for intermediate :

$$B(D_s^+ \to K^*(892)\pi^+, K^*(892) \to K^+\pi^-)$$
$$= (1.85 \pm 0.13 \pm 0.11) \times 10^{-3}$$

Decay process	Ours (10^{-3})	BESIII (10^{-3})	PDG (10^{-3})
$D_s^+ \to K^+ f_0(500)$	$0.38 \pm 0.03^{+0.03}_{-0.03}$	$0.43 \pm 0.14 \pm 0.24$	-
$D_s^+ \to K^+ f_0(980)$	$0.11 \pm 0.01^{+0.04}_{-0.04}$	$0.27 \pm 0.08 \pm 0.07$	-
$D_s^+ \to K^+ \rho^0$	$2.94 \pm 0.27^{+0.03}_{-0.05}$	$1.99 \pm 0.20 \pm 0.22$	2.5 ± 0.4
$D_s^+ \to K^+ f_0(1370)$	$1.07 \pm 0.10^{+0.11}_{-0.20}$	$1.22 \pm 0.19 \pm 0.18$	-
$D_s^+ \to K_0^*(1430)^0 \pi^+$	$1.06 \pm 0.10^{+0.01}_{-0.02}$	$1.15 \pm 0.16 \pm 0.15$	0.50 ± 0.35
$D_s^+ \to K^+ \rho(1450)^0$	$2.38 \pm 0.22^{+0.04}_{-0.09}$	$0.78 \pm 0.20 \pm 0.17$	0.69 ± 0.64

The decay of $D_s^+ \to K_S^0 K_S^0 \pi^+$

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The external and internal W-emission mechanism:

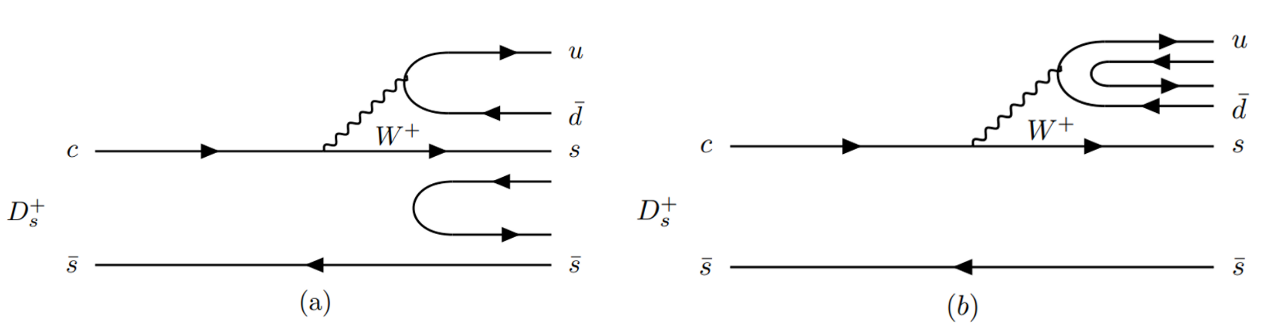


FIG. 1: W-external emission mechanism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay.

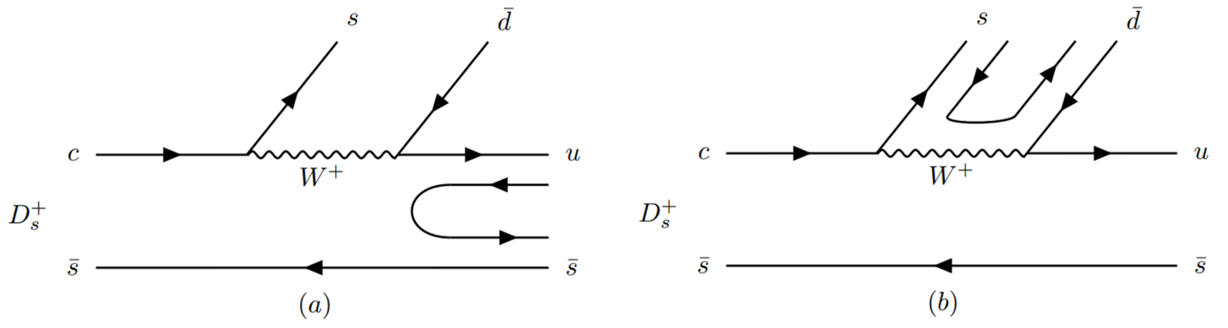


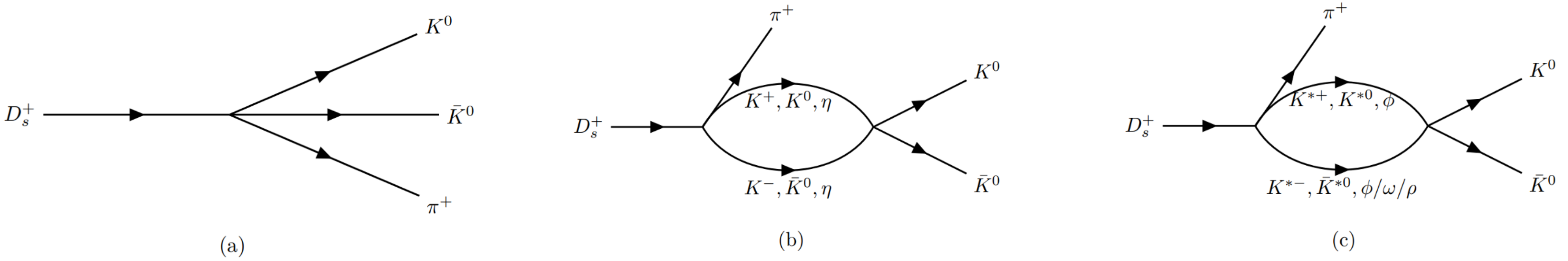
FIG. 2: W-external emission mechanism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay.

- The total contributions for the decay $D_s^+ \to K_s^0 K_s^0 \pi^+$:

$$|H\rangle = |H^{(1a)}\rangle + |H^{(1b)}\rangle + |H^{(2a)}\rangle + |H^{(2b)}\rangle$$

$$= C_1 \pi^+ K^+ K^- + C_2 \pi^+ K^0 \bar{K}^0 + \frac{2}{3} C_3 \pi^+ \eta\eta + C_4 \pi^+ K^{*+} K^{*-} + C_5 \pi^+ K^{*0} \bar{K}^{*0} + C_6 \pi^+ \phi\phi + \frac{1}{\sqrt{2}} C_7 \pi^+ \omega\phi + \frac{1}{\sqrt{2}} C_8 \pi^+ \rho^0 \phi,$$

- Tree-level production and final state interactions via rescattering mechanism:



$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The amplitudes for the decay $D_s^+ \to K_s^0 K_s^0 \pi^+$ in the S-wave:

$$
\begin{aligned}
t(M_{12})|_{K^0\bar{K}^0\pi^+} =& C_1 G_{K^+K^-}(M_{12}) T_{K^+K^-\to K^0\bar{K}^0}(M_{12}) + C_2 + C_2 G_{K^0\bar{K}^0}(M_{12}) T_{K^0\bar{K}^0\to K^0\bar{K}^0}(M_{12}) \\
&+ \frac{2}{3} C_3 G_{\eta\eta}(M_{12}) T_{\eta\eta\to K^0\bar{K}^0}(M_{12}) + C_4 G_{K^{*+}K^{*-}}(M_{12}) T_{K^{*+}K^{*-}\to K^0\bar{K}^0}(M_{12}) \\
&+ C_5 G_{K^{*0}\bar{K}^{*0}}(M_{12}) T_{K^{*0}\bar{K}^{*0}\to K^0\bar{K}^0}(M_{12}) + C_6 G_{\phi\phi}(M_{12}) T_{\phi\phi\to K^0\bar{K}^0}(M_{12}) \\
&+ \frac{1}{\sqrt{2}} C_7 G_{\omega\phi}(M_{12}) T_{\omega\phi\to K^0\bar{K}^0}(M_{12}) + \frac{1}{\sqrt{2}} C_8 G_{\rho^0\phi}(M_{12}) T_{\rho^0\phi\to K^0\bar{K}^0}(M_{12}),
\end{aligned}
$$

$$
|K_S^0\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right)
$$

$$
\begin{aligned}
t(M_{12})|_{K_S^0K_S^0\pi^+} =& -\frac{1}{2} C_1 G_{K^+K^-}(M_{12}) T_{K^+K^-\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_2 - \frac{1}{2} C_2 G_{K^0\bar{K}^0}(M_{12}) T_{K^0\bar{K}^0\to K^0\bar{K}^0}(M_{12}) \\
&- \frac{1}{3} C_3 G_{\eta\eta}(M_{12}) T_{\eta\eta\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_4 G_{K^{*+}K^{*-}}(M_{12}) T_{K^{*+}K^{*-}\to K^0\bar{K}^0}(M_{12}) \\
&- \frac{1}{2} C_5 G_{K^{*0}\bar{K}^{*0}}(M_{12}) T_{K^{*0}\bar{K}^{*0}\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_6 G_{\phi\phi}(M_{12}) T_{\phi\phi\to K^0\bar{K}^0}(M_{12}) \\
&- \frac{1}{2\sqrt{2}} C_7 G_{\omega\phi}(M_{12}) T_{\omega\phi\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2\sqrt{2}} C_8 G_{\rho^0\phi}(M_{12}) T_{\rho^0\phi\to K^0\bar{K}^0}(M_{12}),
\end{aligned}
$$

L. R. Dai et al. Eur. Phys. J. C 82, 225 (2022).

- The diagonal matrix G is two intermediate meson propagators(dimensional regularization method):

$$
\begin{aligned}
G_{ii}(s) =& \frac{1}{16\pi^2}\{a_\mu + \ln\frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + s}{2s}\ln\frac{m_2^2}{m_1^2} + \frac{q_{cm}(s)}{\sqrt{s}}[\ln\left(s - \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right) + \ln\left(s + \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right) \\
&- \ln\left(-s - \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right) - \ln\left(-s + \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right)]\}
\end{aligned}
$$

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The value of the subtraction constant : J. A. Oller and U.-G. Meißner, Phys. Lett. B 500, 263-272 (2001)

$$a_{PP'}(\mu) = -2\log\left(1+\sqrt{1+\frac{m_1^2}{\mu^2}}\right)+\cdots,$$

✓ the pseudoscalar-pseudoscalar interaction:

$\mu = 0.6$ GeV

✓ the vector-vector meson interaction:

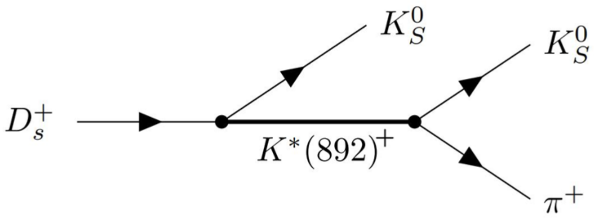
$\mu = 1.0$ GeV

✓ In our formalism: the pseudoscalar-vector interactions

μ: a free parameter

J. A. Oller and E. Oset, Nucl. Phys. A 620, 438-456 (1997) .
L. S. Geng and E. Oset, Phys. Rev. D 79, 074009 (2009).

- The contribution of the vector resonance generated in the P-wave:



$$t_{K^*(892)^+}(M_{12},M_{23}) = \frac{\mathcal{D}e^{i\alpha_{K^*(892)^+}}}{M_{23}^2 - M_{K^*(892)^+}^2 + iM_{K^*(892)^+}\Gamma_{K^*(892)^+}}\left[\frac{(m_{D_s^+}^2 - m_{K_s^0}^2)(m_{K_s^0}^2 - m_{\pi^+}^2)}{M_{K^*(892)^+}^2} - M_{12}^2 + M_{13}^2\right],$$

- The double differential width distribution:

$$\frac{d^2\Gamma}{dM_{12}dM_{23}} = \frac{1}{(2\pi)^3}\frac{M_{12}M_{23}}{8m_{D_s^+}^3}\frac{1}{2}\,|\,\mathcal{M}\,|^2, \qquad \mathcal{M} = t(M_{12})|_{K_S^0K_S^0\pi^+} + t_{K^*(892)^+}(M_{12},M_{23}) + (1\leftrightarrow 2),$$

$D_s^+ \to K_s^0 K_s^0 \pi^+$

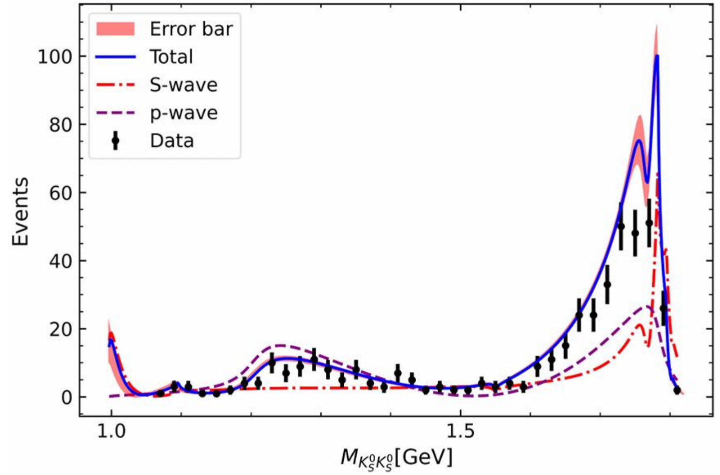
- The parameters need to be fitted:

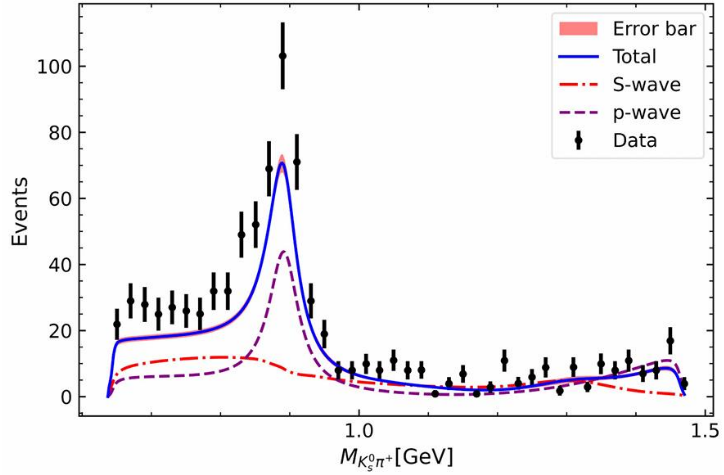
S-wave: $\mu, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ P-wave: $D_{K^*(892)}, \alpha_{K^*(892)},$

- Combined fitting results:

Parameters	μ	C_1	C_2	C_3
Fit	0.648 ± 0.01 GeV	8640.90 ± 1115.80	2980.71 ± 638.37	-1902.86 ± 293.27
Parameters	C_4	C_5	C_6	C_7
Fit	56906.35 ± 10869.67	-13433.15 ± 5017.76	-58284.22 ± 7319.04	102835.76 ± 23333.56
Parameters	C_8	D	$\alpha_{K^*(892)^+}$	χ^2/d.o.f.
Fit	202807.71 ± 30750.45	54.8 ± 2.0	0.0024 ± 4.30	2.55

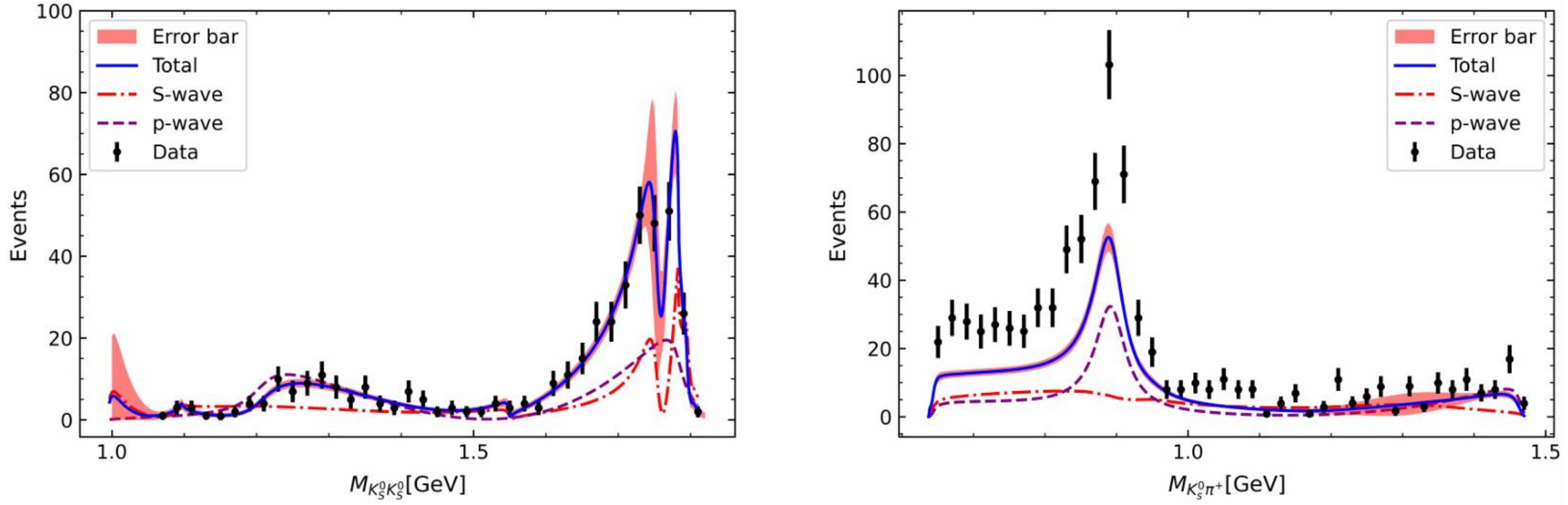
	This work	Ref. [68]	Ref. [100]	Ref. [44]	Ref. [66]	Ref. [45]
Parameters	$\mu = 0.648$	$\mu = 0.716$	$q_{\max} = 0.931$	$\mu = 1.0$	$q_{\max} = 1.0$	$q_{\max} = 1.0$
$a_0(980)$	$1.0598 + 0.024i$	$1.0419 + 0.0345i$	$1.0029 + 0.0567i$	...	...	...
$f_0(980)$	$0.9912 + 0.003i$	...	$0.9912 + 0.0135i$	...	...	...
$a_0(1710)$	$1.7981 + 0.0018i$	$1.7936 + 0.0094i$	...	$1.780 - 0.066i$	$1.72 - 0.010i$	$1.76 \pm 0.03i$
$f_0(1710)$	$1.7676 + 0.0093i$	...	...	$1.726 - 0.014i$	...	...





$D_s^+ \to K_s^0 K_s^0 \pi^+$

- Fitting results: (Fit only for $K_S^0 K_S^0$ spectrum)



- The ratios of the branching fractions between different resonances :

$$\frac{\mathcal{B}(D_s^+ \to S(980)\pi^+, S(980) \to K_S^0 K_S^0)}{\mathcal{B}(D_s^+ \to K_s^0 K^*(892)^+, K^*(892)^+ \to K_S^0 \pi^+)} = 0.122^{+0.032}_{-0.023}, \quad \frac{\mathcal{B}(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0 K_S^0)}{\mathcal{B}(D_s^+ \to K_s^0 K^*(892)^+, K^*(892)^+ \to K_S^0 \pi^+)} = 0.552^{+0.460}_{-0.297},$$

- The branching ratios for intermediate :

$$\mathcal{B}(D_s^+ \to S(980)\pi^+, S(980) \to K_S^0 K_S^0) = (0.36 \pm 0.04^{+0.10}_{-0.06}) \times 10^{-3},$$
$$\mathcal{B}(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0 K_S^0) = (1.66 \pm 0.17^{+1.38}_{-0.89}) \times 10^{-3},$$

$$B\left(D_s^+ \to K^*(892) K_S^0 \to K_S^0 K_S^0 \pi^+\right) = (3.0 \pm 0.3 \pm 0.1) \times 10^{-3};$$
$$B\left(D_s^+ \to S(1710)\pi^+ \to K_S^0 K_S^0 \pi^+\right) = (3.1 \pm 0.3 \pm 0.1) \times 10^{-3}.$$

The decay of $D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

$D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

- The external and internal W-emission mechanism:

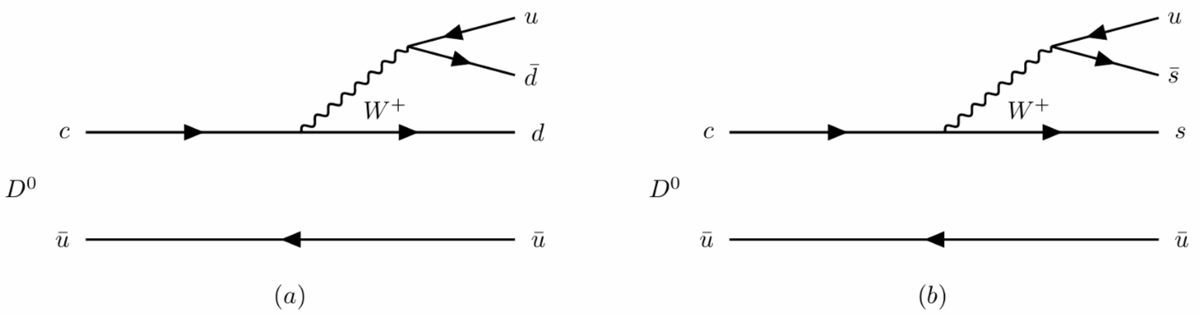


FIG. 1: External W-emission mechanism for the processes: (a) $c \to W^+d$; (b) $c \to W^+s$.

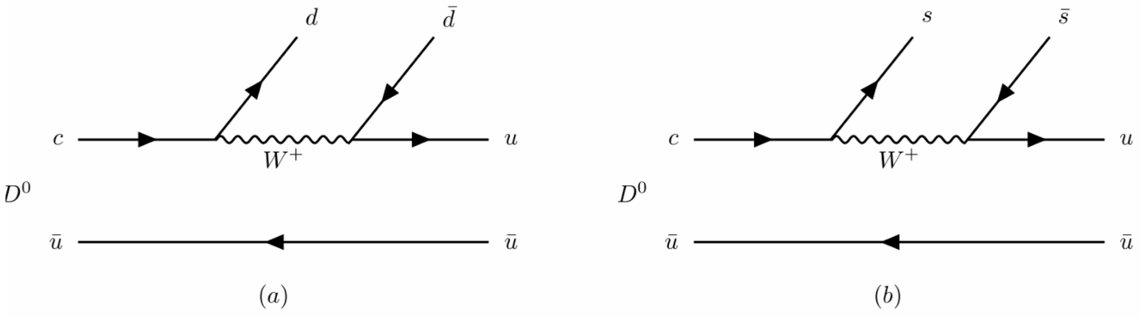


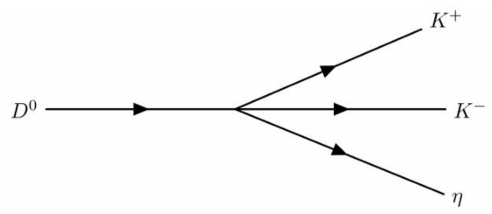
FIG. 2: Internal W-emission mechanism for the processes: (a) $c \to W^+d$; (b) $c \to W^+s$.

- The total contributions for the decay $D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$:

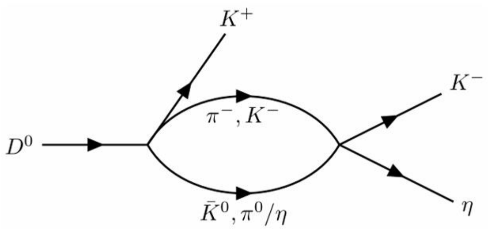
$$
\begin{aligned}
|H\rangle &= \left|H^{(1a)}\right\rangle + \left|H^{(1b)}\right\rangle + \left|H^{(2a)}\right\rangle + \left|H^{(2b)}\right\rangle \\
&= \frac{2}{\sqrt{6}}(2C_1+\beta C_1+\beta C_2)\pi^+\pi^-\eta + (C_1-C_2)\pi^+K^0K^- + (C_1-C_2)\pi^-K^+\bar{K}^0 - \frac{1}{\sqrt{2}}(2C_2+\beta C_1+\beta C_2)K^+K^-\pi^0 + \frac{1}{\sqrt{6}}\beta(C_1-C_2)K^0\bar{K}^0\eta, \\
&+ \frac{1}{\sqrt{6}}(2C_2+\beta C_1+\beta C_2)K^+K^-\eta + \frac{1}{3\sqrt{6}}\beta(C_1-C_2)\eta\eta\eta + \frac{1}{\sqrt{6}}\beta(C_2-C_1)\pi^0\pi^0\eta + \frac{1}{\sqrt{2}}\beta(C_1-C_2)\pi^0K^0\bar{K}^0
\end{aligned}
$$

$D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

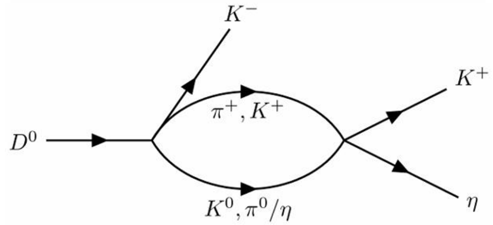
- Tree-level production and final state interactions via rescattering mechanism:



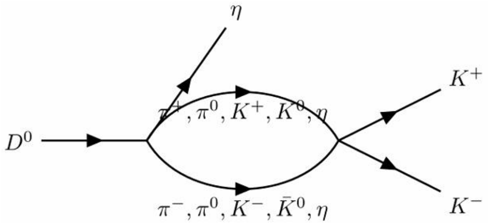
(a) Tree-level production.



(b) Rescattering of $\pi^-\bar{K}^0$, $K^-\pi^0$ and $K^-\eta$.

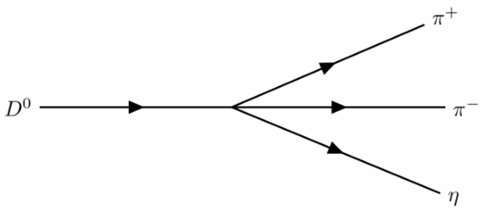


(c) Rescattering of π^+K^0, $K^+\pi^0$ and $K^+\eta$.

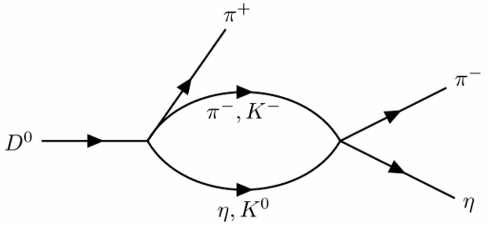


(d) Rescattering of $\pi^+\pi^-$, $\pi^0\pi^0$, K^+K^-, $K^0\bar{K}^0$, and $\eta\eta$.

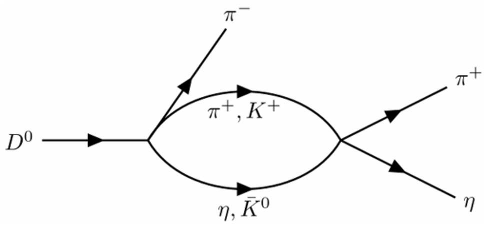
$$
\begin{aligned}
t_{D^0\to K^+K^-\eta}(s_{12},s_{23}) =\ & \frac{2}{\sqrt{6}}(2C_1+\beta C_1+\beta C_2)G_{\pi^+\pi^-}(s_{12})\,T_{\pi^+\pi^-\to K^+K^-}(s_{12}) \\
& + \frac{1}{\sqrt{6}}(2C_2+\beta C_1+\beta C_2) + \frac{1}{\sqrt{6}}(2C_2+\beta C_1+\beta C_2)\,G_{K^+K^-}(s_{12})\,T_{K^+K^-\to K^+K^-}(s_{12}) \\
& + \frac{1}{3\sqrt{6}}\beta(C_1-C_2)G_{\eta\eta}(s_{12})\,T_{\eta\eta\to K^+K^-}(s_{12}) - \frac{1}{\sqrt{6}}\beta(C_1-C_2)G_{\pi^0\pi^0}(s_{12})\,T_{\pi^0\pi^0\to K^+K^-}(s_{12}) \\
& + \frac{1}{\sqrt{6}}\beta(C_1-C_2)G_{K^0\bar{K}^0}(s_{12})\,T_{K^0\bar{K}^0\to K^+K^-}(s_{12}) + (C_1-C_2)G_{\pi^-\bar{K}^0}(s_{23})\,T_{\pi^-\bar{K}^0\to K^-\eta}(s_{23}) \\
& - \frac{1}{\sqrt{2}}(2C_2+\beta C_1+\beta C_2)\,G_{K^-\pi^0}(s_{23})\,T_{K^-\pi^0\to K^-\eta}(s_{23}) + \frac{1}{\sqrt{6}}(2C_2+\beta C_1+\beta C_2)\,G_{K^-\eta}(s_{23})\,T_{K^-\eta\to K^-\eta}(s_{23}) \\
& + (C_1-C_2)G_{\pi^+K^0}(s_{13})\,T_{\pi^+K^0\to K^+\eta}(s_{13}) - \frac{1}{\sqrt{2}}(2C_2+\beta C_1+\beta C_2)\,G_{K^+\pi^0}(s_{13})\,T_{K^+\pi^0\to K^+\eta}(s_{13}) \\
& + \frac{1}{\sqrt{6}}(2C_2+\beta C_1+\beta C_2)\,G_{K^+\eta}(s_{13})\,T_{K^+\eta\to K^+\eta}(s_{13}),
\end{aligned}
$$



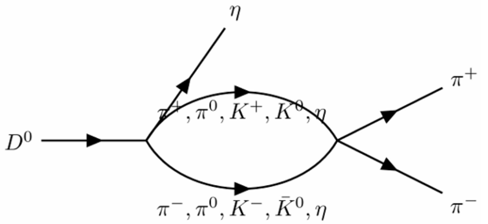
(a) Tree-level production.



(b) Rescattering of $\pi^-\eta$ and K^-K^0.



(c) Rescattering of $\pi^+\eta$ and $K^+\bar{K}^0$.



(d) Rescattering of $\pi^+\pi^-$, $\pi^0\pi^0$, K^+K^-, $K^0\bar{K}^0$, and $\eta\eta$.

$$
\begin{aligned}
t_{D^0\to \pi^+\pi^-\eta}(s_{12},s_{23}) =\ & \frac{2}{\sqrt{6}}(2C_1+\beta C_1+\beta C_2) + \frac{2}{\sqrt{6}}(2C_1+\beta C_1+\beta C_2)G_{\pi^+\pi^-}(s_{12})\,T_{\pi^+\pi^-\to\pi^+\pi^-}(s_{12}) \\
& + \frac{1}{\sqrt{6}}(2C_2+\beta C_1+\beta C_2)\,G_{K^+K^-}(s_{12})\,T_{K^+K^-\to\pi^+\pi^-}(s_{12}) + \frac{1}{3\sqrt{6}}\beta(C_1-C_2)G_{\eta\eta}(s_{12})\,T_{\eta\eta\to\pi^+\pi^-}(s_{12}) \\
& - \frac{1}{\sqrt{6}}\beta(C_1-C_2)G_{\pi^0\pi^0}(s_{12})\,T_{\pi^0\pi^0\to\pi^+\pi^-}(s_{12}) + \frac{1}{\sqrt{6}}\beta(C_1-C_2)G_{K^0\bar{K}^0}(s_{12})\,T_{K^0\bar{K}^0\to\pi^+\pi^-}(s_{12}) \\
& + \frac{2}{\sqrt{6}}(2C_1+\beta C_1+\beta C_2)G_{\pi^-\eta}(s_{23})\,T_{\pi^-\eta\to\pi^-\eta}(s_{23}) + \frac{2}{\sqrt{6}}(2C_1+\beta C_1+\beta C_2)G_{\pi^+\eta}(s_{13})\,T_{\pi^+\eta\to\pi^+\eta}(s_{13}) \\
& + (C_1-C_2)G_{K^0K^-}(s_{23})\,T_{K^0K^-\to\pi^-\eta}(s_{23}) + (C_1-C_2)G_{K^+\bar{K}^0}(s_{13})\,T_{K^+\bar{K}^0\to\pi^+\eta}(s_{13}),
\end{aligned}
$$

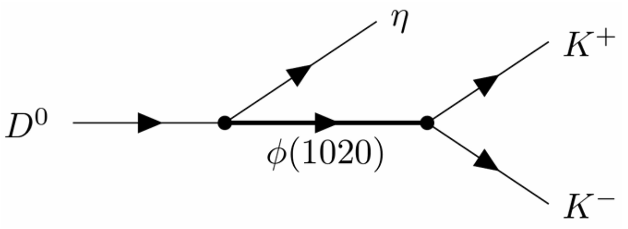
$D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

- G is two intermediate meson propagators(three-momentumcut-offmethod):

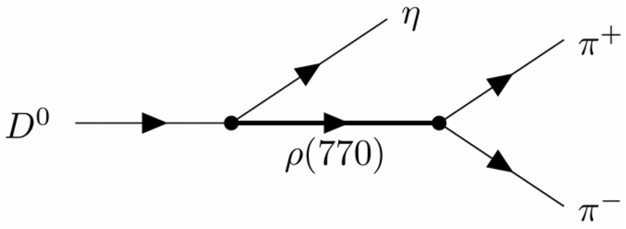
$$G(s)=\frac{1}{16\pi^2 s}\left\{\sigma\left(\arctan\frac{s+\Delta}{\sigma\lambda_1}+\arctan\frac{s-\Delta}{\sigma\lambda_2}\right)-\left[(s+\Delta)\ln\frac{(1+\lambda_1)\,q_{\max}}{m_1}+(s-\Delta)\ln\frac{(1+\lambda_2)\,q_{\max}}{m_2}\right]\right\},$$

$$\sigma=\left[-\left(s-(m_1+m_2)^2\right)\left(s-(m_1-m_2)^2\right)\right]^{1/2},\quad \Delta=m_1^2-m_2^2,\quad \lambda_i=\sqrt{1+m_i^2/q_{\max}^2}\ (i=1,2).$$

- The contribution of the vector resonance generated in the P-wave:



(a) Mechanism via the $\phi(1020)$.



(b) Mechanism via the $\rho(770)$.

$$M_\phi(s_{12},s_{23})=\frac{D_\phi e^{i\alpha_\phi}}{s_{12}-m_\phi^2+im_\phi\Gamma_\phi}(s_{23}-s_{13}),$$

$$M_\rho(s_{12},s_{23})=\frac{D_\rho e^{i\alpha_\rho}}{s_{12}-m_\rho^2+im_\rho\Gamma_\rho}(s_{23}-s_{13}),$$

- The double differential width distribution:

$$\frac{d^2\Gamma}{ds_{12}ds_{23}}=\frac{1}{(2\pi)^3}\frac{1}{32m_{D^0}^3}\left|t_{D^0\to\pi^+\pi^-\eta}(s_{12},s_{23})+M_\rho(s_{12},s_{23})\right|^2,\qquad \frac{d^2\Gamma}{ds_{12}ds_{23}}=\frac{1}{(2\pi)^3}\frac{1}{32m_{D^0}^3}\left|t_{D^0\to K^+K^-\eta}(s_{12},s_{23})+M_\phi(s_{12},s_{23})\right|^2,$$

- The parameters need to be fitted:

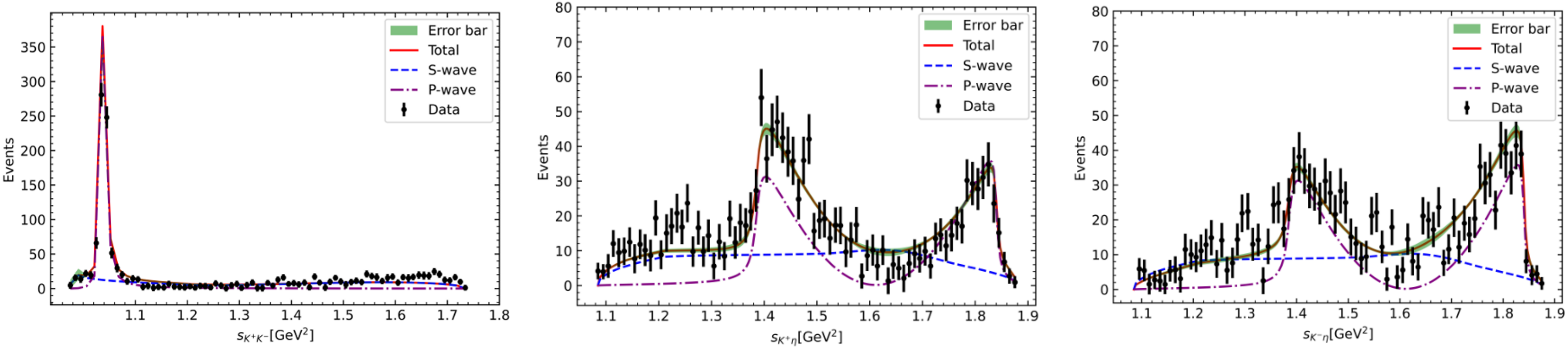
S-wave: C_1, C_2, β, α P-wave: D_ρ, α_ρ, D_ϕ, α_ϕ,

$D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

L. K. Li et al. [Belle], JHEP 09, 075 (2021).

- $\boldsymbol{K^+K^-\eta}$

Parameters	C_1	C_2	β	α	$D_{\phi(1020)}$	$\alpha_{\phi(1020)}$	$\chi^2/dof.$
Fit	-3448.24±221.07	99.07±91.70	0.77±0.04	6.28±0.05	123.75±2.02	2.89±0.12	1.48

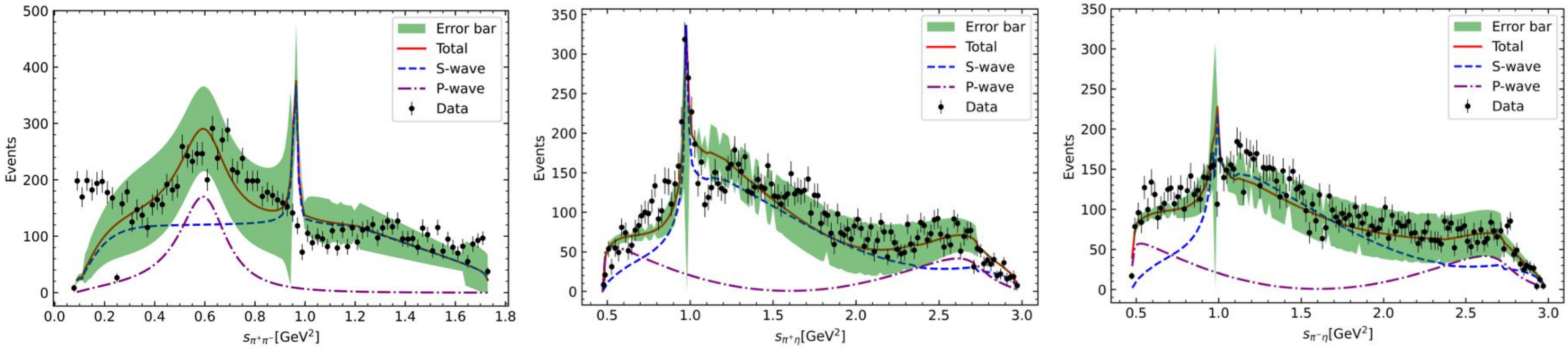


$D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

L. K. Li et al. [Belle], JHEP 09, 075 (2021).

- $\boldsymbol{\pi^+\pi^-\eta}$

Parameters	C_1	C_2	β	α	$D_{\rho(770)}$	$\alpha_{\rho(770)}$	$\chi^2/dof.$
Fit	419.79± 23.61	2196.24±57.25	1.00±0.006	0.38±0.10	-170.97±2.90	2.20±0.03	5.23

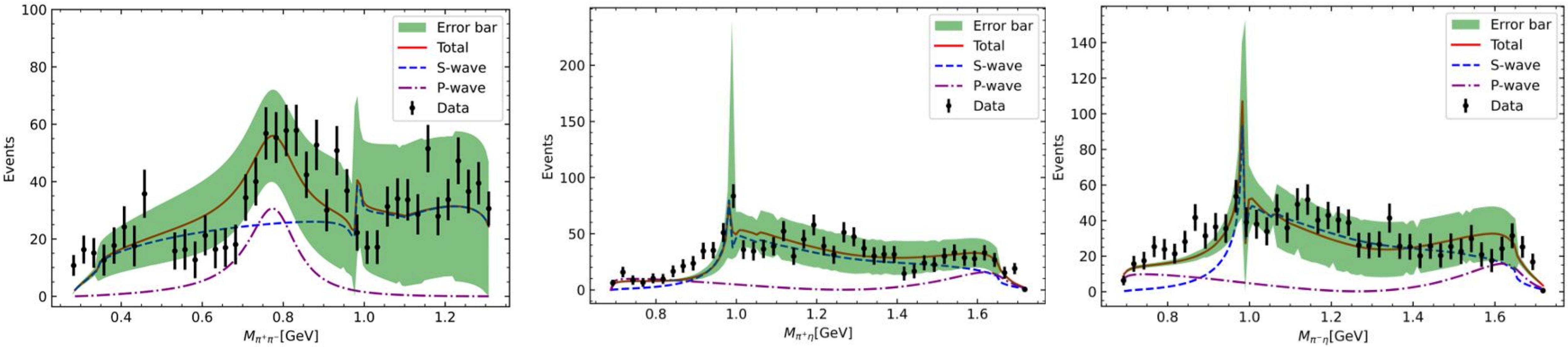


$D^0 \to K^+K^-\eta$ and $\pi^+\pi^-\eta$

M. Ablikim et al. [BESIII], Phys. Rev. D 110, L111102 (2024).

- $\boldsymbol{\pi^+\pi^-\eta}$

Parameters	C_1	C_2	β	α	$D_{\rho(770)}$	$\alpha_{\rho(770)}$	$\chi^2/dof.$
Fit	282.74±15.25	-516.33±26.32	1.00±0.03	2.47±0.44	58.41±2.98	4.36±0.13	2.52



Conclusions

Summary

- Based on the measurements for the decay $D_s^+ \to \pi^+\pi^-K^+/ K_S^0K_S^0\pi^+$ and $D^0 \to K^+K^-\eta/ \pi^+\pi^-\eta$, we adopt the chiral unitary approach to investigate these processes theoretically via considering the contributions of the W external and internal emission mechanisms. Besides, the contributions of the other intermediate resonances are also take into account.

- $D_s^+ \to \pi^+\pi^-K^+$: we reproduce the $\pi^+\pi^-$, $K^+\pi^-$ and $K^+\pi^+$ invariant mass distributions by considering the coherent effects between the S and P waves. Besides, the branching fractions of the dominant decay channels are almost in good agreement with the experimental measurements and PDG within the uncertainties .

- $D_s^+ \to K_S^0K_S^0\pi^+$: the fitted results show that the enhancement around 1.7 GeV in $K_S^0K_S^0$ mass spectrum is overlapped with two visible peaks, indicating the mixing signal originated from the resonances $a_0(1710)$ and $f_0(1710)$ due to their different poles (masses).

- $D^0 \to K^+K^-\eta/ \pi^+\pi^-\eta$: We make a combined fit of the invariant mass spectra measured by the Belle and BESIII Collaborations, where the results are in good agreement with the experiments, and the signal of the $a_0(980)$ shows great significance. Besides, the antisymmetry data for the production of the $a_0(980)^+$and $a_0(980)^-$ is described well in the combined fit.

Thank you!