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Bridging B-meson Shape Functions in QCD and HQET

Shuai Zhao

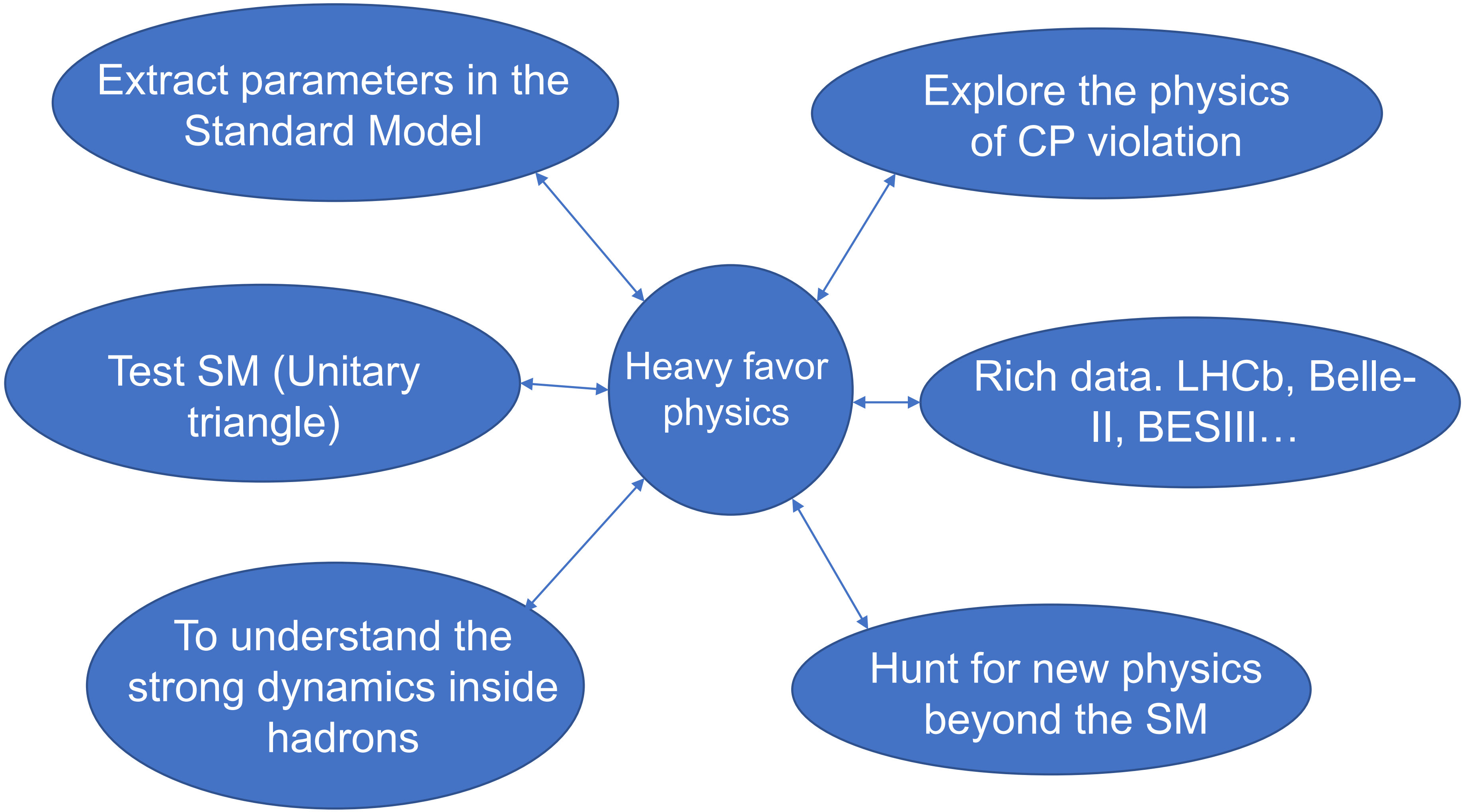
zhaos@tju.edu.cn

Based on W. Wang, J. Xu, Q.-A. Zhang, **SZ**, 2504.18018

Outline

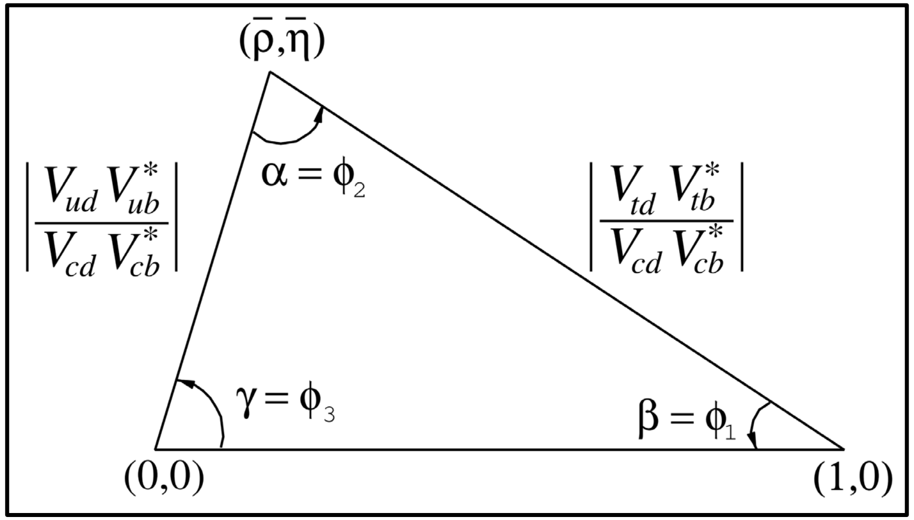
- Heavy meson inclusive and exclusive decays
- Heavy meson shape function
- Two-step matching
- Matching relation between shape functions in QCD and HQET

Heavy flavor physics



The unitary CKM triangle

- Over constraining the CKM triangle is a crucial test of CP violation and the Standard model



- $|V_{ub}|$ can be measured with exclusive and inclusive processes

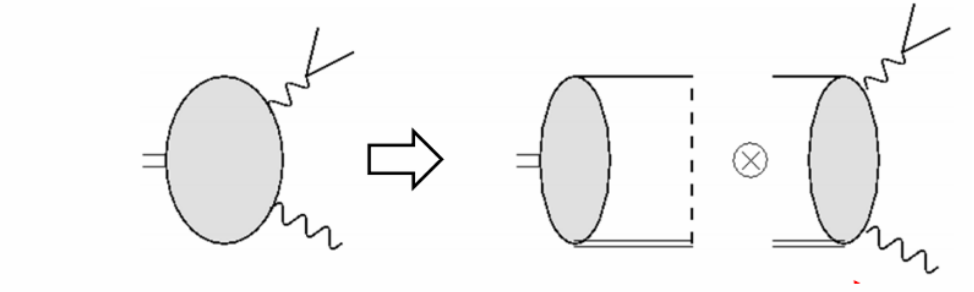
$$|V_{ub}| = (4.13 \pm 0.12 \,{}^{+\,0.13}_{-\,0.14} \pm 0.18) \times 10^{-3} \quad \text{(inclusive)},$$ via $B \to X_u \ell \nu$ PDG (2024)

$$|V_{ub}| = (3.70 \pm 0.10 \pm 0.12) \times 10^{-3} \quad \text{(exclusive)},$$ via $B \to \pi \ell \nu$ PDG (2024)

The V_{ub} puzzle

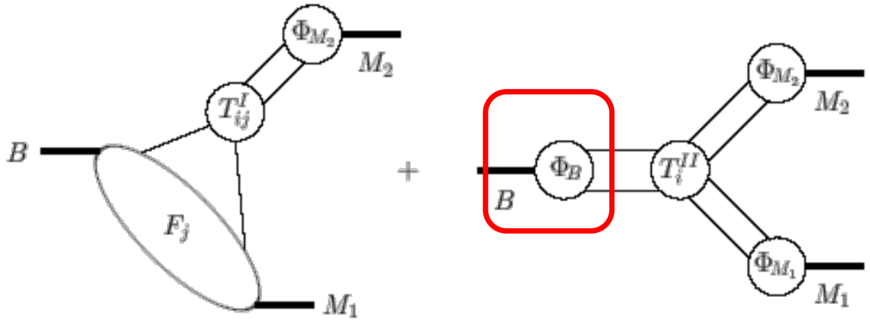
QCD factorization for exclusive B decays

- QCD factorization for $B \to \gamma \ell \nu$



$$Amp \propto \int_0^\infty \frac{d\omega}{\omega} T(\omega, m_b, \mu; \alpha_s) \phi_+^B(\omega, \mu)$$

- QCD factorization for B->MM



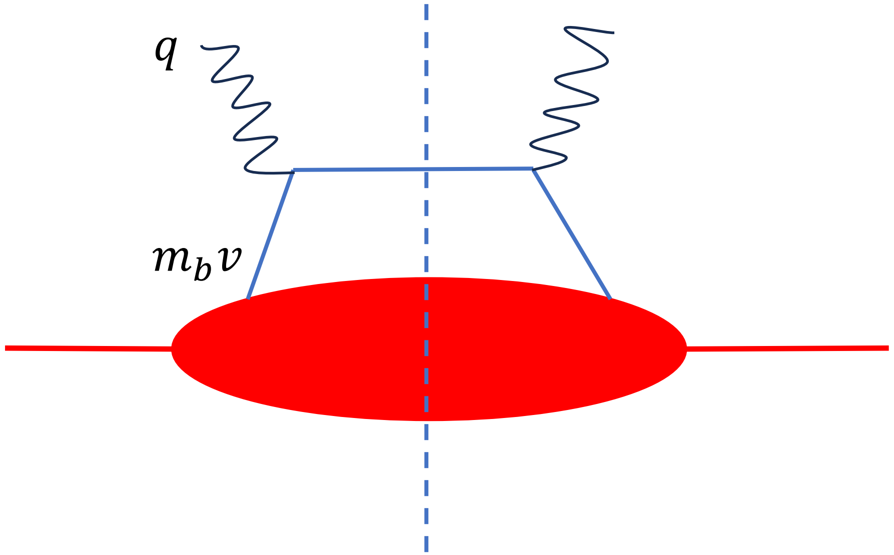
$$A(B \to \pi\pi) \propto \phi_\pi \otimes T_I \otimes F^{B\pi} + \phi_\pi \otimes T_{II} \otimes \phi_B \otimes \phi_\pi$$

Beneke,Buchalla,Neubert,Sachrajda, 1999; Bauer, Pirjol, Stewart, 2001

- Needs the input of LCDA

QCD factorization for inclusive B decays

- Inclusive decay rate is related to the hadronic tensor



$$W^{\mu\nu} = \frac{1}{\pi}\,\mathrm{Im}\,\frac{\langle \bar{B}(v)|\, T^{\mu\nu}\, |\bar{B}(v)\rangle}{2M_B}$$

$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\;\mathrm{T}\left\{J^{\dagger\mu}(0), J^{\nu}(x)\right\}$$

$$J^{\mu} = \bar{u}\gamma^{\mu}(1-\gamma_5)b$$

$p = m_b v - q$: momentum of light partons

- When $p^{\mu} \sim m_b \gg \Lambda_{QCD}$: heavy quark expansion (local OPE)

- When $p^{\mu} = (p^{+}, p^{-}, p_{\perp}) \sim E\left(\lambda, 1, \sqrt{\lambda}\right)$: twist expansion (light-cone OPE)

$$\lambda = \frac{\Lambda_{QCD}}{E}$$

B meson inclusive decays and shape function

- The differential decay rates are expressed in terms of shape function

$B \to X_u \ell \nu$

$$\frac{d\Gamma}{dE_\ell} = \frac{G_F^2 |V_{ub}|^2 m_b^4}{96\pi^3} \int d\omega \theta(m_b - 2E_\ell - \omega) S(\omega)$$

$B \to X_s \gamma$

$$\frac{d\Gamma}{dE_\gamma} = \frac{G_F^2 |V_{tb} V_{ts}^*|^2 \alpha |C_7^{\text{eff}}|^2 m_b^5}{32\pi^4} S(E_\gamma)$$

Bauer, Manohar, 2003; Bosch, Lange, Neubert, Paz, 2004,…

- Shape function reflects the fermi motion of b-quark inside B-meson

- Shape function and LCDA are crucial for understanding heavy meson structure

- Shape function and LCDA are defined with matrix elements of nonlocal HQET operators

How to access the nonperturbative functions?

- Extract from experiment
- Extract from lattice simulations

? Light-cone separation

Can be solved with the Large Momentum Effective Theory (or pseudo distributions,etc.)

Kawamura, Tanaka, 2018; Wang,Wang,Xu,**SZ**,2019; **SZ**,Radyushkin,2020

Xu, Zhang, **SZ**, 2022; Xu, Zhang, 2022; Hu, Wang, Xu, **SZ**, 2023; Hu, Xu, **SZ**,2024

? Heavy quark on the lattice

Back to active heavy quarks, i.e., heavy quarks in QCD

Large momentum effective theory

Parton Physics on a Euclidean Lattice #1

Xiangdong Ji (Maryland U. and Shanghai Jiaotong U.) (May 7, 2013)

Published in: *Phys.Rev.Lett.* 110 (2013) 262002 • e-Print: 1305.1539 [hep-ph]

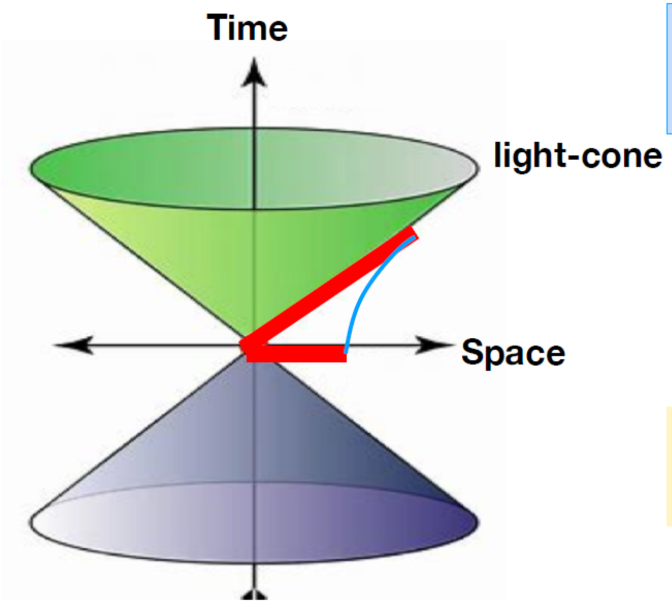
pdf DOI cite claim reference search 559 citations

Large-momentum effective theory #10

Xiangdong Ji (Maryland U. and Tsung-Dao Lee Inst., Shanghai), Yu-Sheng Liu (Tsung-Dao Lee Inst., Shanghai), Yizhuang Liu (Tsung-Dao Lee Inst., Shanghai), Jian-Hui Zhang (Beijing Normal U.), Yong Zhao (Brookhaven) (Apr 7, 2020)

Published in: *Rev.Mod.Phys.* 93 (2021) 3, 035005 • e-Print: 2004.03543 [hep-ph]

pdf DOI cite claim reference search 154 citations



Physical PDF

$$f_{q/H}(x,\Lambda) = \int_{-\infty}^{\infty} \frac{d\xi^-}{4\pi} e^{ix\xi^- P^+} \left\langle P \middle| \bar{\psi}(\xi^-)\gamma^+ \exp\left(-ig\int_0^{\xi^-} dz^- A^+(z^-)\right)\psi(0) \middle| P\right\rangle$$

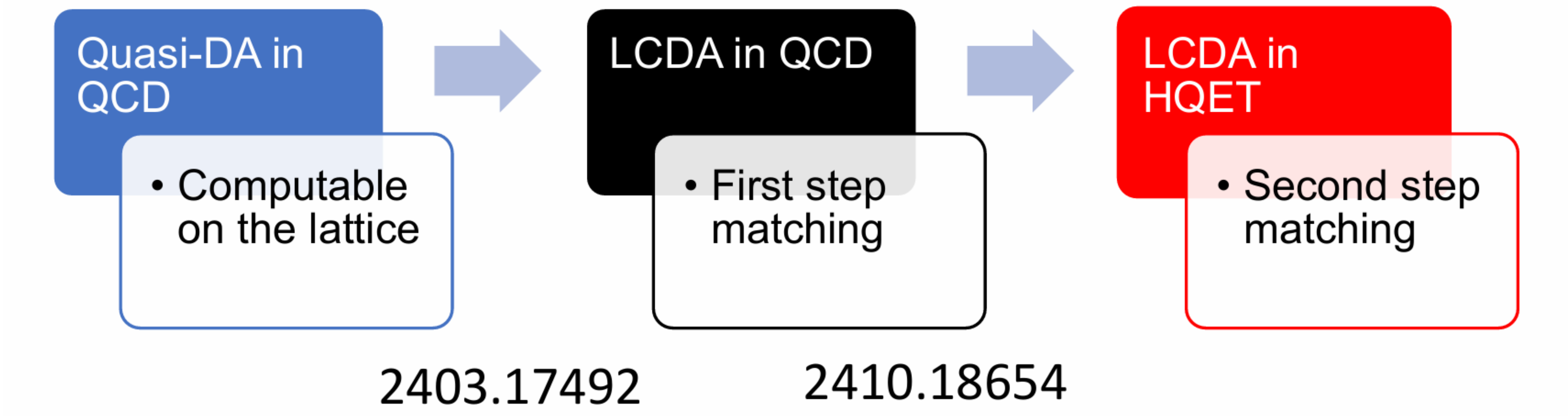
$$\tilde{f}_{q/H}(x,P^z,\Lambda) = \int_{-\infty}^{\infty}\frac{dz}{4\pi} e^{-ixzP^z}\left\langle P\middle|\bar{\psi}(z)\gamma^z \exp\left(ig\int_0^z d\eta A^z(\eta)\right)\psi(0)\middle|P\right\rangle$$

Quasi-PDF

See the talks by Qi-An Zhang, Jun Hua

The approach of two step matching

$$P_z \gg m_Q \gg \Lambda_{QCD}$$



See the talk by Qi-An Zhang

- 1st step matching: same with pion DA (independent of m_Q)
- 2nd step matching

Ishaq, Jia, Xiong, Yang, PRL2020;
SZ, PRD 2020
Beneke, Finauri, Keri Vos, Wei, JHEP 2023

Two step matching for shape functions

- Shape function is still untouched by lattice QCD

- Defined with hadron-to-hadron matrix element of nonlocal HQET operator

$$\frac{\langle \bar{B}(v)|\, \bar{h}\, \Gamma\, \delta(\omega - in \cdot D)\, h\, |\bar{B}(v)\rangle}{2M_B} = S(\omega)\frac{1}{2}\,\mathrm{tr}\left(\Gamma\, \frac{1+\not{v}}{2}\right) + \ldots$$

- Matching between equal-time (quasi) QCD operator and light-cone operator is known.
  Same with quark PDF

- The relation between shape function and its QCD counter part is unknown.

Shape function in HQET→QCD

- Shape function in HQET

$$S^{\text{HQET}}(\omega,\mu) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega v^+ t} \frac{\langle B(v)|\bar{h}_v(0)\, W(0,tn_+)\, h_v(tn_+)|B(v)\rangle}{\langle B(v)|\bar{h}_v(0)\, h_v(0)|B(v)\rangle}\,.$$

h_v: heavy quark field in HQET
Support: $-\infty < \omega < \bar{\Lambda}$, with $\bar{\Lambda} = m_B - m_b$.

- Activate the heavy quark: shape function in QCD

$$S^{\text{QCD}}(x,\mu) = \int_{-\infty}^{+\infty} \frac{dz^-}{2\pi} e^{-ixp_B^+ z^-} \frac{\langle B(p_B)|\bar{b}(0)\,\Gamma\, W(0,z)\, b(z)|B(p_B)\rangle}{\langle B(p_B)|\bar{b}(0)\,\Gamma b(0)|B(p_B)\rangle}\,.$$

b: heavy quark field in QCD
Support: $0 < x < 1$.

Shape function in QCD

- Two typical scales in heavy meson: Λ_{QCD} and m_Q

- Most of the momentum is carried by the heavy quark.
  Shape function is peaked near the end point, i.e., $x \sim 1$

- The peak is smeared by the gluon radiation.

- Near the peak: soft gluon radiation, nonperturbative. — Peak region
- Far from the peak: hard gluon radiation, perturbative. — Tail region

- Integrate out m_Q: QCD shape function → HQET shape function

Matching at tree-level

- Assuming $p_B^+ = m_B v^+$, $p_b^+ = m_b v^+ + k^+$

  p_B: B meson momentum; p_b: b quark momentum;
  k: residue momentum of b quark

- Tree-level:

  QCD shape function:

  $$S^{QCD}(x) = \delta\left(x - \frac{m_b}{m_B} - \frac{k^+}{m_B v^+}\right) = \delta\left(x - 1 + \frac{\Lambda}{m_B} - \frac{k^+}{m_B v^+}\right)$$

  $\Lambda \equiv m_B - m_b$

  HQET shape function: $S^{HQET}(\omega) = \delta\left(\omega - \frac{k^+}{v^+}\right)$

- Matching relation at tree level:

$$S^{QCD}(x) = Z(x,\omega) \otimes S^{HQET}(\omega)$$

$$Z^{(0)}(x,\omega) = \delta\left(1 - x - \frac{\Lambda}{m_B} - \frac{\omega}{m_B}\right)$$

One-loop matching

- The factorization formula between S^{QCD} and S^{HQET}

$$S^{\mathrm{QCD}}(x,\mu) = \begin{cases} Z_{\mathrm{peak}}(x,\omega,\mu) \otimes S^{\mathrm{HQET}}(\omega,\mu)\,, & 1-x \sim O(\Lambda_{QCD}/m_Q) \quad \text{“Peak region”} \\ Z_{\mathrm{tail}}(x,\mu)\,, & 1-x \sim O(1) \quad \text{“Tail region”} \end{cases}$$

- Expand the shape functions and matching coefficient

$$S^{\mathrm{QCD}}(x,\mu) = S^{\mathrm{QCD}(0)}(x,\mu) + \frac{\alpha_s C_F}{2\pi} S^{\mathrm{QCD}(1)}(x,\mu) + \mathcal{O}(\alpha_s^2)\,,$$

$$S^{\mathrm{HQET}}(\omega,\mu) = S^{\mathrm{HQET}(0)}(\omega,\mu) + \frac{\alpha_s C_F}{2\pi} S^{\mathrm{HQET}(1)}(\omega,\mu) + \mathcal{O}(\alpha_s^2)\,.$$

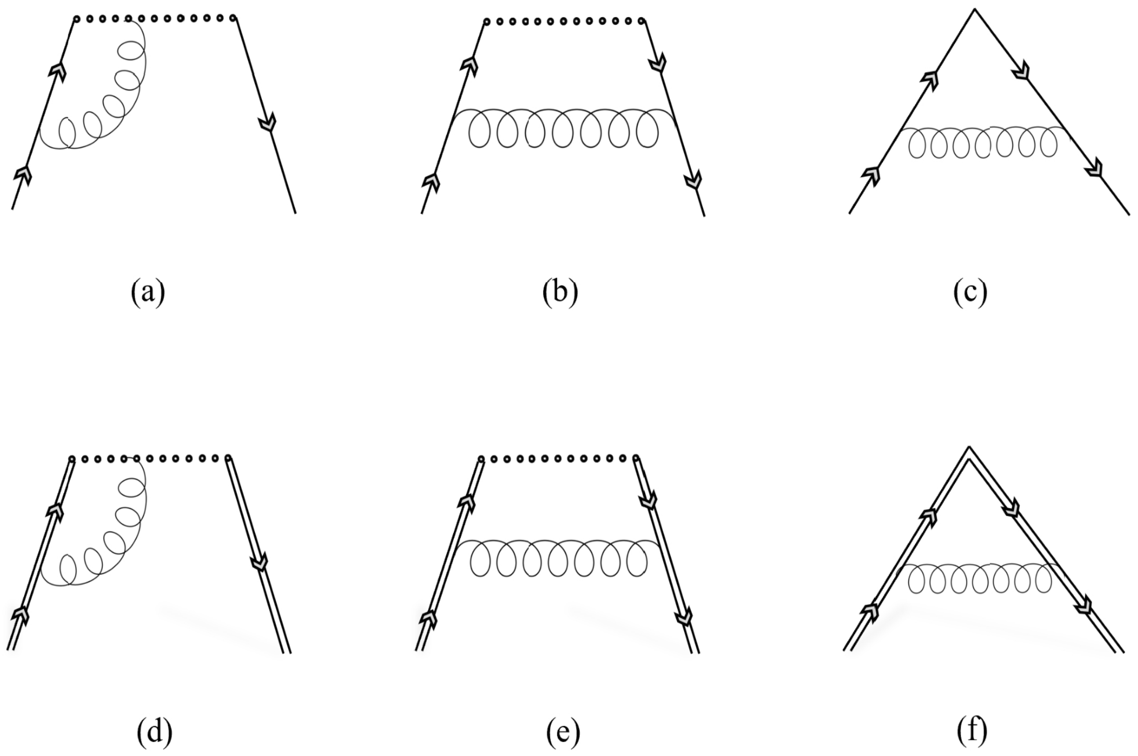
$$Z_{\mathrm{peak}}(x,\omega,\mu) = Z^{(0)}_{\mathrm{peak}}(x,\omega,\mu) + \frac{\alpha_s C_F}{2\pi} Z^{(1)}_{\mathrm{peak}}(x,\omega,\mu) + \mathcal{O}(\alpha_s^2)\,,$$

$$Z_{\mathrm{tail}}(x,\mu) = Z^{(0)}_{\mathrm{tail}}(x,\omega,\mu) + \frac{\alpha_s C_F}{2\pi} Z^{(1)}_{\mathrm{tail}}(x,\mu) + \mathcal{O}(\alpha_s^2)\,.$$

One-loop matching

- This necessitates a region-separated calculation.
- Work in the space-time dimension $d = 4 - 2\epsilon$ and use $\overline{\mathrm{MS}}$ scheme.
- Keep $v \cdot k$ non-zero and use plus function

$$F\left(\omega, k^{+}\right) = \left[F\left(\omega, k^{+}\right)\right]_{\oplus} + \delta\left(\omega - k^{+}\right) \int_{0}^{\Lambda} dt\, F(\omega, t)\,.$$



Matching relation

- The matching function at one-loop

$$Z^{(1)}_{\rm tail}(x,\mu) = \frac{1+x^2}{1-x}\left[\ln\frac{\mu^2}{m_b^2(1-x)^2} - 1\right]$$

$$Z^{(1)}_{\rm peak} = \delta(1 - x - \frac{\Lambda}{m_B} - \frac{\omega}{m_B})\left(\frac{1}{2}\ln^2\frac{\mu^2}{m_b^2} - \frac{3}{2}\ln\frac{\mu^2}{m_b^2} + \frac{\pi^2}{12} - 2\right)$$

- Matching relation verified.

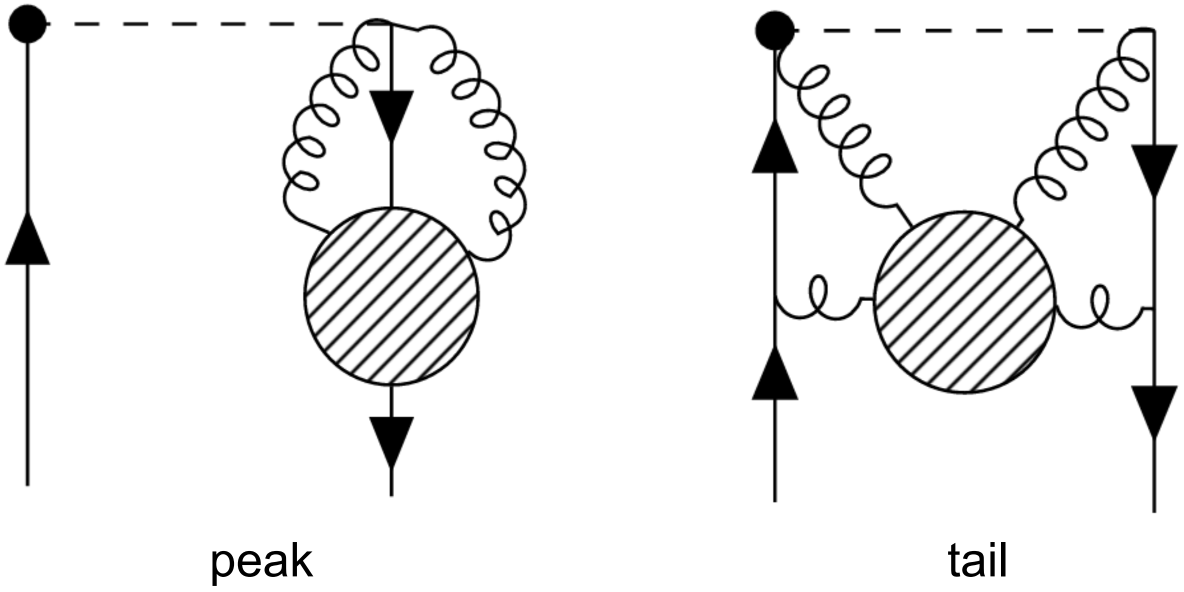
- The plus distributions cancel out in the matching, yielding multiplicative form of factorization formula.

- Similar to the matching calculation for heavy meson LCDA.

Beneke, Finauri, Keri Vos, Wei, JHEP 2023

Matching relation at all orders

- Hard function is local in momentum space to all orders



- The matching at peak region is multiplicative

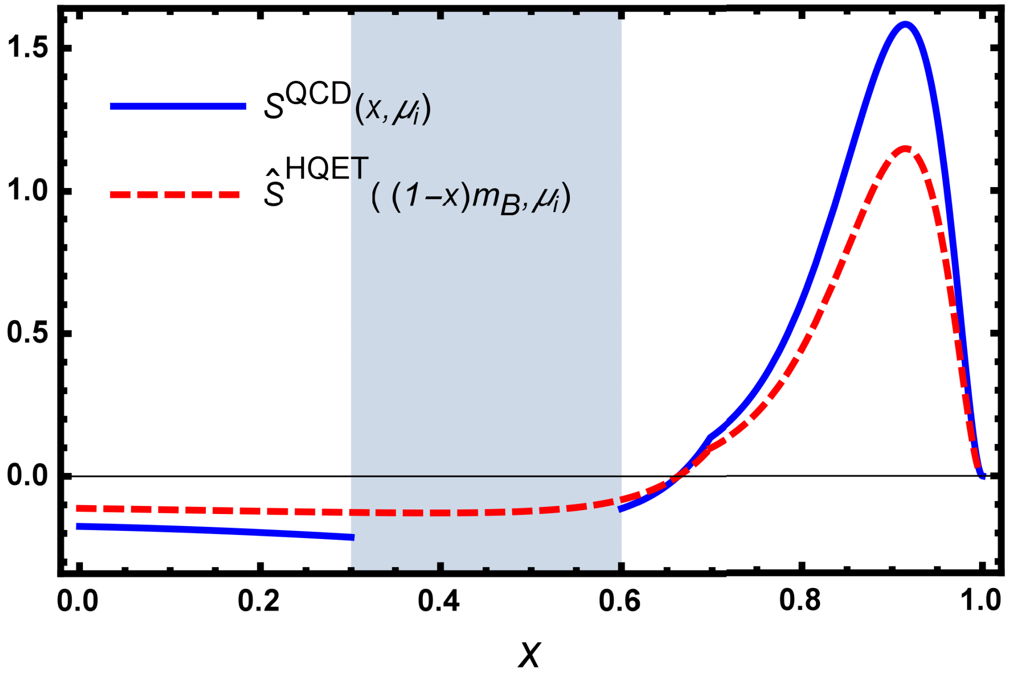
$$Z_{peak}(x, \omega, m_Q, \mu) = Z(m_Q, \mu)\delta(1 - x - \frac{\Lambda}{m_Q} - \frac{\omega}{m_Q})$$

Determining QCD Shape Function from Model

- It is instructive to understand the characteristic feature of QCD shape function and beneficial for future lattice simulations.

- Taking advantage of a widely adopted model of HQET shape function at the soft scale $\mu_i = 1.5\ \mathrm{GeV}$ with $\hat{\omega} = \bar{\Lambda} - \omega \geq 0$.

Neubert et al, NPB, 699 (2004)



$$\hat{S}^{\mathrm{HQET}}(\hat{\omega}, \mu) = \frac{N}{A}\left(\frac{\hat{\omega}}{A}\right)^{b-1} \exp\left(-b\frac{\hat{\omega}}{A}\right) - \frac{\alpha_s C_F}{\pi} \frac{\theta(\hat{\omega} - A - \mu/\sqrt{e})}{\hat{\omega} - A}\left(2\ln\frac{\hat{\omega} - A}{\mu} + 1\right).$$

$$N = \left[1 - \frac{\alpha_s C_F}{\pi}\left(\frac{\pi^2}{24} - \frac{1}{4}\right)\right]\frac{b^b}{\Gamma(b)}.$$

Summary and outlook

- The shape function is a crucial nonperturbative function in heavy meson inclusive decays
- The relation between shape function and its QCD counter part is established
- Can be used in lattice QCD simulation by combining with LaMET
- Could provide insight on the extraction of CKM matrix element and the solution of the puzzles

Thank you!