Lattice QCD calculations in Muon $g-2$

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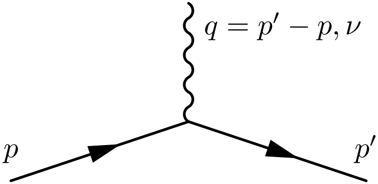
(MIP2025)
湖南大学

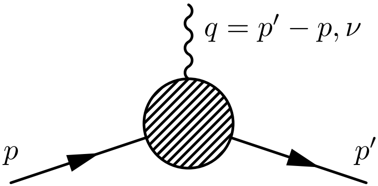
Anomalous magnetic moments



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$

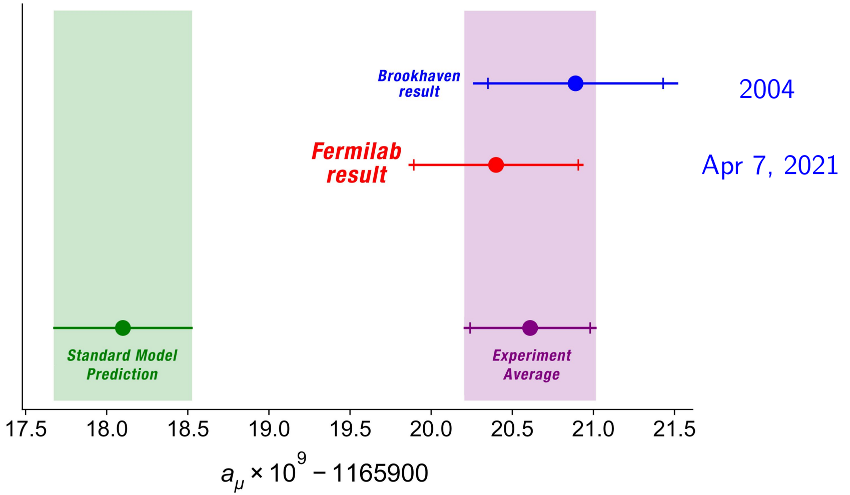


$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

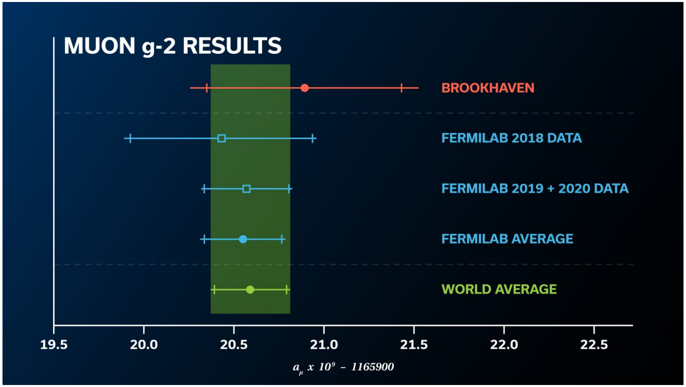
(Euclidean space time)

$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

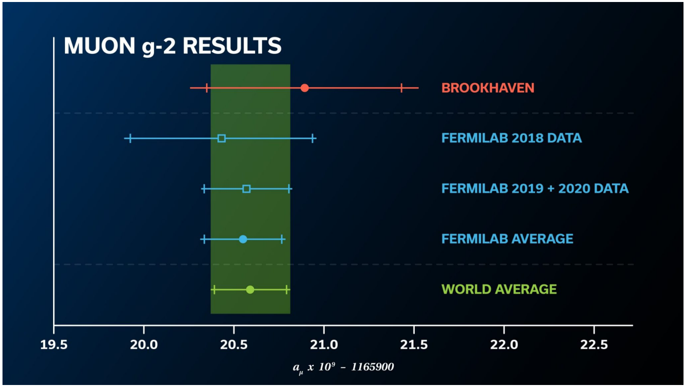
- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.



- Muon $g-2$ [arXiv:2104.03281 [hep-ex]]
- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g-2$ experiment.



- Muon $g-2$ [arXiv:2308.06230 [hep-ex]]
- New results (August 10, 2023) reduced the uncertainty by a factor of two.
- The final result from Fermilab will be released soon.
- All results so far are consistent.
- J-PARC is working on a separate muon $g-2$ experiment using a different setup (very cool muon).



- Muon $g-2$ [arXiv:2308.06230 [hep-ex]]
- The standard model prediction is missing in the above plot.
- Compared with previous WP20 results, EXP - WP20 $= 24.9(2.2)_{\text{exp}}(4.3)_{\text{WP20}} \times 10^{-10}$
- Due to the new lattice calculations and new $e^+e^- \to$ hadrons experimental results, the standard model prediction, which will be summarized in the new "Muon $g-2$ Theory Initiative White paper" (will be released soon), will likely be quite different from the previous white paper.

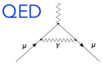
Muon $g-2$ from the Standard Model

Muon $g-2$ Theory Initiative White paper posted 10 June 2020.

132 authors from worldwide theory + experiment community. [Phys. Rept. 887 (2020) 1-166]

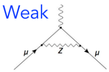
$$\boxed{a_\mu = a_\mu(\mathrm{QED}) + a_\mu(\mathrm{Weak}) + a_\mu(\mathrm{Hadronic})}$$

QED



$+\ldots$ $116\,584\,718.9\,(1) \times 10^{-11}$ 0.001 ppm

Weak

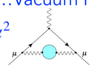


$+\ldots$ $153.6\,(1.0) \times 10^{-11}$ 0.01 ppm

Hadronic…

…Vacuum Polarization (HVP)

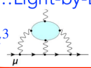
α^2



$+\ldots$ $6845\,(40) \times 10^{-11}$ [0.6%] 0.34 ppm

…Light-by-Light (HLbL)

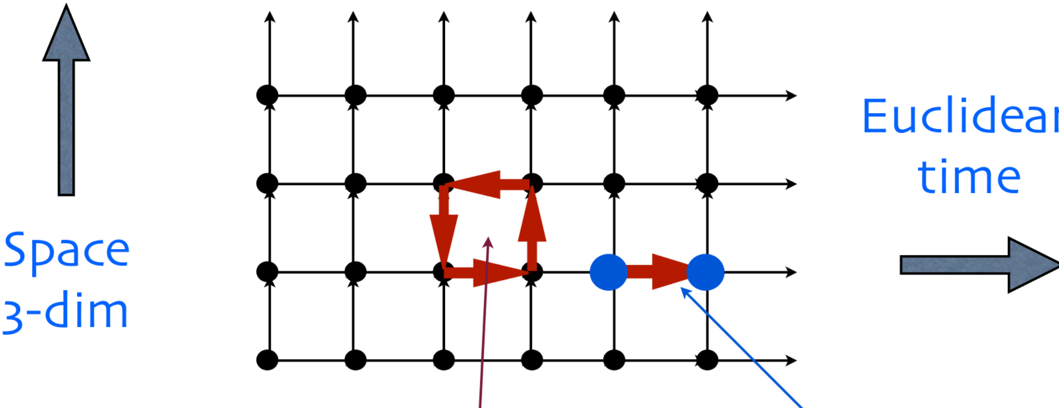
α^3



$+\ldots$ $92\,(18) \times 10^{-11}$ [20%] 0.15 ppm

From Aida El-Khadra's theory talk during the Fermilab $g-2$ result announcement.

- Two methods: dispersive + data $\leftrightarrow$ lattice QCD
- New white paper will be released soon. The contents are confidential before its official release. So, I will only show published results from each lattice groups, but not the averages.



$$S_E^{\mathrm{latt}} = -\sum_{\Box} \frac{6}{g^2} \mathrm{Re}\, \mathrm{tr}_N (U_{\Box,\mu\nu}) - \sum_q \overline{q}(D_\mu^{\mathrm{lat}} \gamma_\mu + a m_q) q$$

Wilson gauge action Lattice fermion action

Figure credit: Stephen R. Sharpe.

$$\langle \mathcal{O}(U, q, \bar{q}) \rangle = \frac{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}} \mathcal{O}(U, q, \bar{q})}{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}}}$$
$$= \frac{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}} \gamma_\mu + am_q\right) \tilde{\mathcal{O}}(U)}{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}} \gamma_\mu + am_q\right)}$$

Monte Carlo:

- The integration is performed for all the link variables: U. Dimension is $L^3 \times T \times 4 \times 8$.
- Sample points the following distribution:

$$e^{-S_{\text{gauge}}^{\text{latt}}(U)} \prod_q \det\left(D_\mu^{\text{latt}}(U) \gamma_\mu + am_q\right)$$

- Therefore:

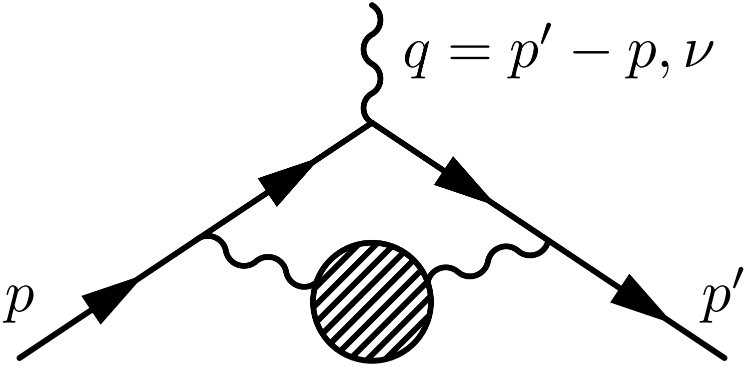
$$\langle \mathcal{O}(U, q, \bar{q}) \rangle = \frac{1}{N_{\text{conf}}} \sum_{k=1}^{N_{\text{conf}}} \tilde{\mathcal{O}}(U^{(k)})$$

- Parameters in lattice QCD calculations (e.g. isospin symmetric ($m_u = m_d = m_l$) and four flavor u, d, s, c theory):

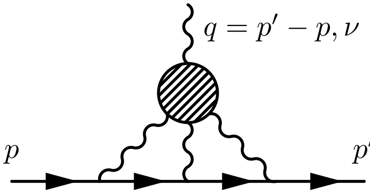
$$g \qquad am_l \qquad am_s \qquad am_c$$

  Note that lattice spacing a is determined by g via the renormalization group equation.

- The experimental inputs needed to determine these parameters can be: m_π / m_Ω, m_K / m_Ω. m_{D_s} / m_Ω.

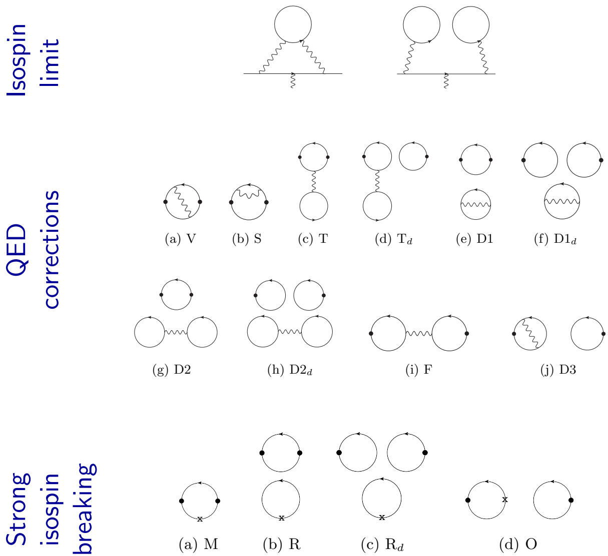


HVP



HLbL

Muon $g - 2$ HVP: diagrams



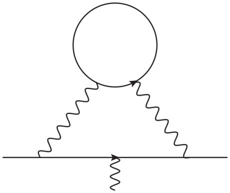
- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- Need to calculate and cross check all the contributions.

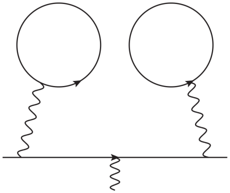
Muon $g-2$ HVP: lattice approach

T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\mathrm{QCD}}$$

$$a_\mu^{\mathrm{HVP\ LO}} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2) = \sum_{t=0}^{+\infty} w(t)C(t)$$





QED

and

strong isospin

breaking

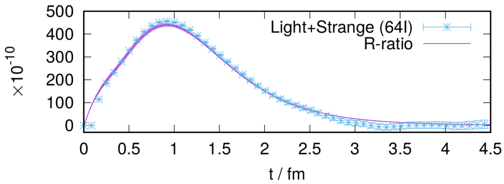
$a_\mu^{\mathrm{HVP,\,LO}}(ud)$	$a_\mu^{\mathrm{HVP,\,LO}}(s)$	$a_\mu^{\mathrm{HVP,\,LO}}(c)$	$a_{\mu,\mathrm{disc}}^{\mathrm{HVP,\,LO}}$	$\delta a_\mu^{\mathrm{HVP}}$
650.2(11.6)	53.2(0.3)	14.6(0.1)	−13.7(2.9)	7.2(3.4)

- From muon $g-2$ theory initiative white paper (2020). Value in unit of 10^{-10}
- Light quark connected diagram has the largest contribution and largest uncertainty.

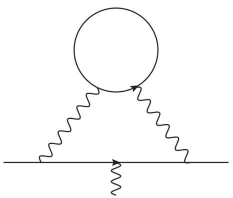
T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\text{QCD}}$$

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$



- Statistical error is mostly from:
  Light quark connected diagram at $t \gtrsim 1.5$ fm
  - More configurations (BMW 20 used $\sim 20,000$).
  - Use low modes averaging to gain full volume average.
  - Bounding method on the long distance tail.
  - Study the $\pi\pi$ system spectrum to calculate $C(t)$ with large t.
- Finite volume effects can be controlled.
- Systematic error is mostly from the **continuum extrapolation**.

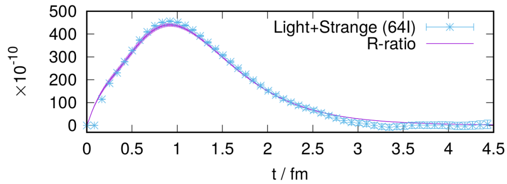


Muon $g-2$ HVP: uncertainties

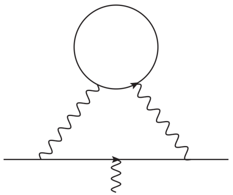
T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\mathrm{QCD}}$$

$$a_\mu^{\mathrm{HVP\ LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$



- Statistical error is mostly from:
  Light quark connected diagram at $t \gtrsim 1.5$ fm
- Finite volume effects can be controlled.
  - Compare with larger volume lattices
    $6.3 \sim 10.8$ fm (BMW), $5.5 \sim 7.3$ fm (RBC-UKQCD), etc.
  - Analytical calculation based on $F_\pi(Q^2)$.
    [Hansen & Patella, arXiv:1904.10010], ChiPT,
    Meyer-Lellouch-Luscher-Gounaris-Sakurai model etc.
- Systematic error is mostly from the **continuum extrapolation**.



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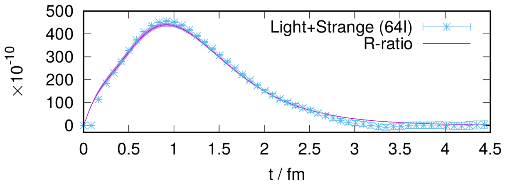
$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$



- Statistical error is mostly from:

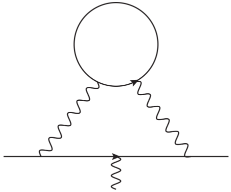
  Light quark connected diagram at $t \gtrsim 1.5$ fm

- Finite volume effects can be controlled.

- Systematic error is mostly from the **continuum extrapolation**.

  - More and finer lattice spacings.

  - Cross checks with different lattice gauge and fermion actions.



- BMW: 4stout & 4HEX gauge and Staggerred fermion.
- Fermilab/HPQCD/MILC: one-loop Symanzik improved gauge and highly improved staggered quarks (HISQ)
- ABGP: same as FNAL/HPQCD/MILC.
- LM: same as FNAL/HPQCD/MILC.

- Mainz: Lüscher-Weisz gauge and $O(a)$ improved Wilson-clover fermion.
- PACS: Iwasaki gauge and $O(a)$ improved Wilson-clover fermion.
- ETMC: Iwasaki gauge and Wilson-clover twisted-mass quarks.

- RBC-UKQCD: Iwasaki gauge and Domain wall fermion.
- χQCD: same as RBC-UKQCD except overlap valance fermion.

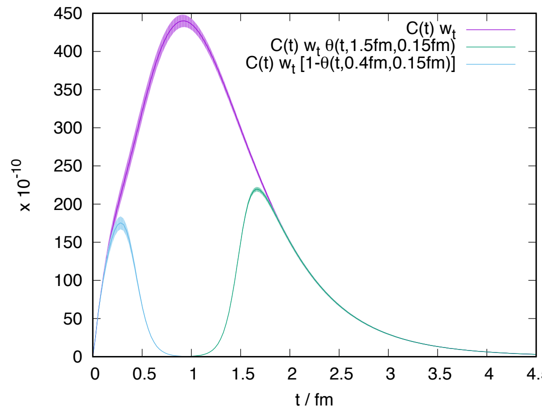
RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t) C(t)$$

$$w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
  - short distance $\Leftarrow$ discretization effects
  - long distance $\Leftarrow$ noisy $\pi\pi$ tail
  - intermediate (Window): sweet spot
- Can form windows from $R(e^+e^-)$ dispersive analysis too.
- Compare intermediate "window" among lattice-QCD calculations
- Compare "short distance" window among lattice-QCD calculations



Muon $g-2$ HVP: controlling systematics

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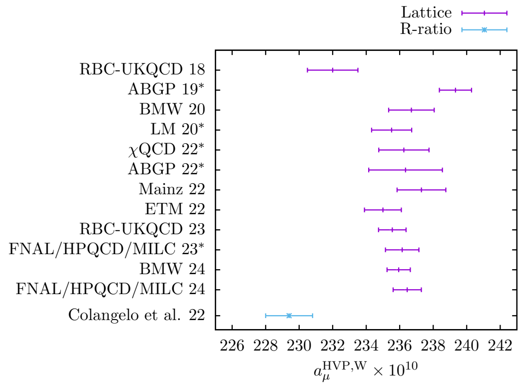

- Lattice QCD community has already reached consensus on the window contribution!
- *: light quark connected contribution plus remaining contributions from FNAL/HPQCD/MILC 24.)

Muon $g-2$ HVP: controlling systematics

RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t) C(t)$$

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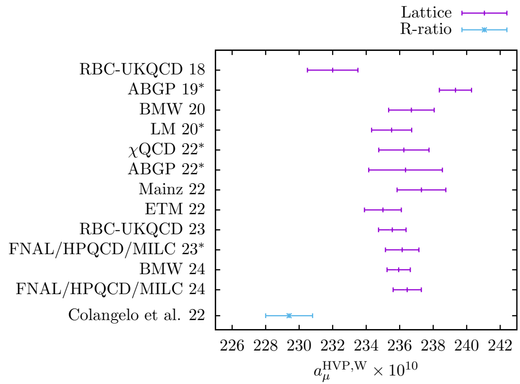

- The consensus has a noticeable tension with the dispersive results. (New CMD-3 results add some additional complications to the dispersive results.)

RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t) C(t) \qquad w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
  - short distance $\Leftarrow$ discretization effects
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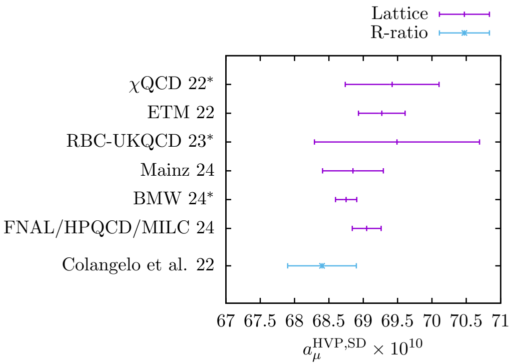

- *: light quark connected contribution plus remaining contributions from FNAL/HPQCD/MILC 24.)

HVP long-distance integrand recontruction from $IJ = 11$ $\pi\pi$ states.

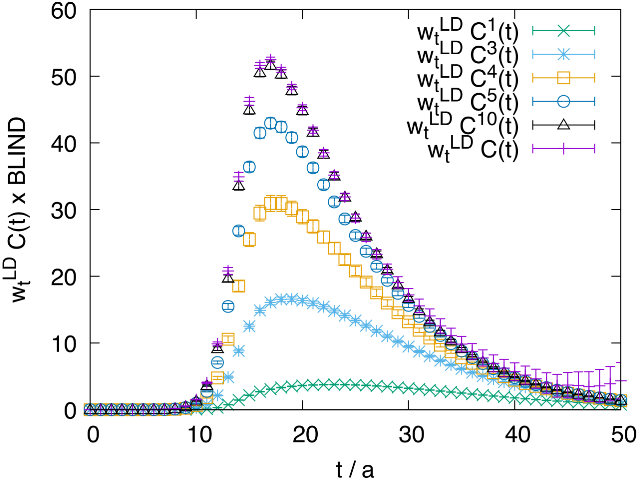

FIG. 1. Reconstruction of the lowest N states $C^N(t)$ compared to the inclusive $C(t)$ for the 96I ensemble with $w_t^{\rm LD} = w_t\Theta(t, t_1, \Delta)$. The exponential growth of statistical noise in $C(t)$ is absent in the reconstruction.

[arXiv:2410.20590]

Controlling the long distance statistical error is very challenging.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$

$$w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
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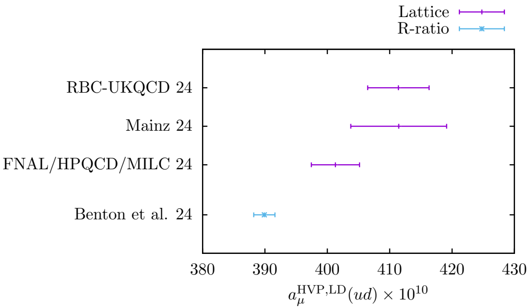


- Compare "long distance" window among lattice-QCD calculations

- QED and related quark mass (IB) corrections for the "long distance" window is not available yet! There are, however, some results on the IB corrections for the entire HVP.

- Using lattice results in the BMW20 scheme: $M_\pi = 0.13497$ GeV, $M_{ss*} = 0.6898$ GeV, $w_0 = 0.17236$ fm.

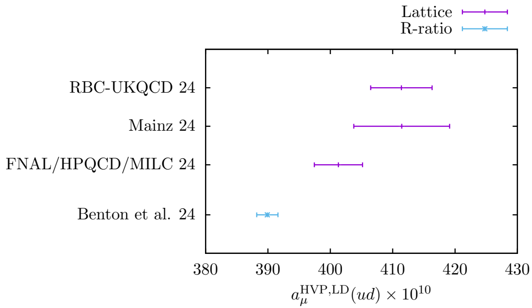
Muon $g-2$ HVP: "long distance" window

Controlling the long distance statistical error is very challenging.

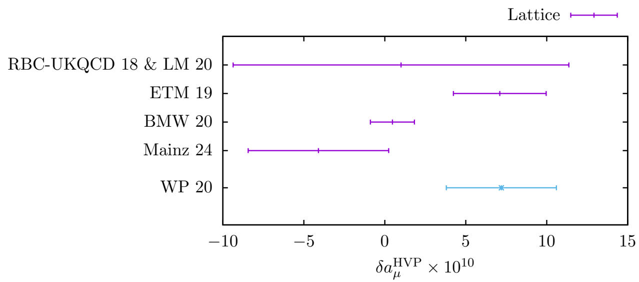
$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$

$$w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
  - short distance $\Leftarrow$ discretization effects
  - long distance $\Leftarrow$ noisy $\pi\pi$ tail
  - intermediate (Window): sweet spot
- Compare "long distance" window among lattice-QCD calculations

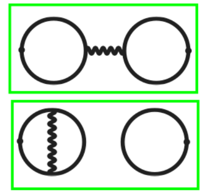


- QED and related quark mass (IB) corrections for the "long distance" window is not available yet! There are, however, some results on the IB corrections for the entire HVP.
- Benton et al. 24: Data-driven method to obtain the light-quark-connected contribution. Based on KNT19 (pre-CMD-3) data.



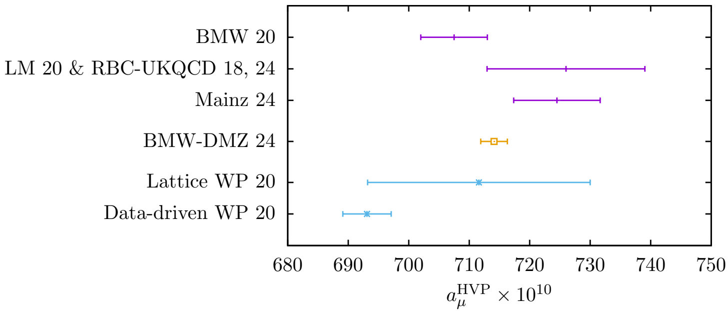
- QED and corrections due to quark mass shift compared with the specified isospin-symmetric QCD world.
- Central values are not directly comparable due to the differences in the definition of the isospin-symmetric QCD world.
- The to be released white paper will have these results converted to use the same WP5 scheme.
- Previous focus of many lattice groups have been the intermediate and long distance window contribution in the isospin-symmetric QCD world. A lot of progress has been made. I expect the situation for the isospin-breaking corrections will improve soon.

$a_\mu^{\text{ud, conn, isospin}}$	$649.7(14.2)_{\mathrm{S}}(2.8)_{\mathrm{C}}(3.7)_{\mathrm{V}}(1.5)_{\mathrm{A}}(0.4)_{\mathrm{Z}}(0.1)_{\mathrm{E48}}(0.1)_{\mathrm{E64}}$
$a_\mu^{\text{s, conn, isospin}}$	$53.2(0.4)_{\mathrm{S}}(0.0)_{\mathrm{C}}(0.3)_{\mathrm{A}}(0.0)_{\mathrm{Z}}$
$a_\mu^{\text{c, conn, isospin}}$	$14.3(0.0)_{\mathrm{S}}(0.7)_{\mathrm{C}}(0.1)_{\mathrm{Z}}(0.0)_{\mathrm{M}}$
$a_\mu^{\text{uds, disc, isospin}}$	$-11.2(3.3)_{\mathrm{S}}(0.4)_{\mathrm{V}}(2.3)_{\mathrm{L}}$
$a_\mu^{\text{QED, conn}}$	$5.9(5.7)_{\mathrm{S}}(0.3)_{\mathrm{C}}(1.2)_{\mathrm{V}}(0.0)_{\mathrm{A}}(0.0)_{\mathrm{Z}}(1.1)_{\mathrm{E}}$
$a_\mu^{\text{QED, disc}}$	$-6.9(2.1)_{\mathrm{S}}(0.4)_{\mathrm{C}}(1.4)_{\mathrm{V}}(0.0)_{\mathrm{A}}(0.0)_{\mathrm{Z}}(1.3)_{\mathrm{E}}$
a_μ^{SIB}	$10.6(4.3)_{\mathrm{S}}(0.6)_{\mathrm{C}}(6.6)_{\mathrm{V}}(0.1)_{\mathrm{A}}(0.0)_{\mathrm{Z}}(1.3)_{\mathrm{E48}}$
$a_\mu^{\text{udsc, isospin}}$	$705.9(14.6)_{\mathrm{S}}(2.9)_{\mathrm{C}}(3.7)_{\mathrm{V}}(1.8)_{\mathrm{A}}(0.4)_{\mathrm{Z}}(2.3)_{\mathrm{L}}(0.1)_{\mathrm{E48}}$ $(0.1)_{\mathrm{E64}}(0.0)_{\mathrm{M}}$
$a_\mu^{\text{QED, SIB}}$	$9.5(7.4)_{\mathrm{S}}(0.7)_{\mathrm{C}}(6.9)_{\mathrm{V}}(0.1)_{\mathrm{A}}(0.0)_{\mathrm{Z}}(1.7)_{\mathrm{E}}(1.3)_{\mathrm{E48}}$
$a_\mu^{\text{R}-\text{ratio}}$	
a_μ	$715.4(16.3)_{\mathrm{S}}(3.0)_{\mathrm{C}}(7.8)_{\mathrm{V}}(1.9)_{\mathrm{A}}(0.4)_{\mathrm{Z}}(1.7)_{\mathrm{E}}(2.3)_{\mathrm{L}}$ $(1.5)_{\mathrm{E48}}(0.1)_{\mathrm{E64}}(0.3)_{\mathrm{b}}(0.2)_{\mathrm{c}}(1.1)_{\overline{\mathrm{S}}}(0.3)_{\overline{\mathrm{Q}}}(0.0)_{\mathrm{M}}$



Disconnected $-0.55(15)_{\text{stat}}(10)_{\text{syst}}$

- The left table shows result from RBC-UKQCD 18. The right figure shows the result from BMW 20.
- This discrepancy needs further study and more cross checks.



- LM 20 & RBC-UKQCD 18, 24: Errors mainly from the quite old IB corrections and will be improved soon.

- BMW-DMZ 24: Lattice QCD results are combined with data-driven results. Data-driven results are used to determine the the long-distance tail contribution and account for about 5% of the final result.

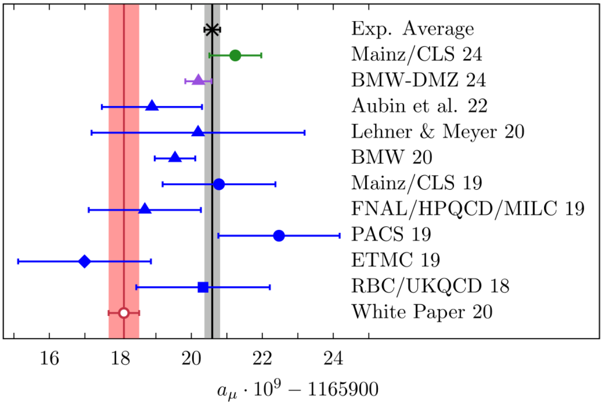
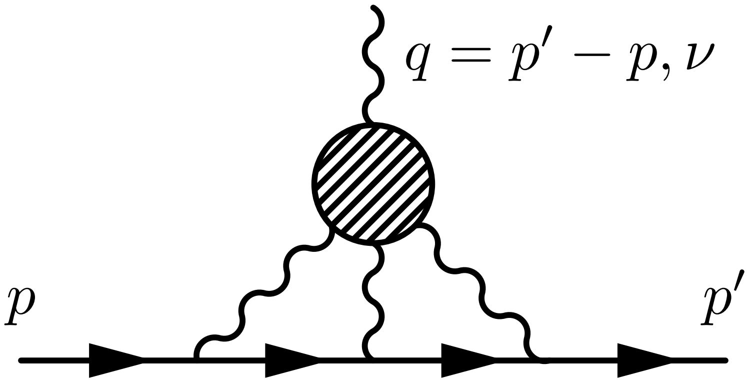
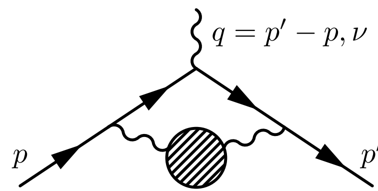


Figure 10. Compilation of results for the leading-order hadronic vacuum polarization contribution, including isospin-breaking effects, combined with the remaining contributions to a_μ as summarized in the 2020 White Paper [1]. Our result quoted in eq. (3.21) is shown in green. Different discretizations of the quark action are denoted by circles (Wilson fermions) [35, 96], triangles (staggered fermions) [14, 15, 17, 97], squares (domain wall fermions) [27] and diamonds (twisted-mass Wilson fermions) [98]. The data-driven estimate from the 2020 White Paper [1] and the current experimental average [12, 13, 110] are represented by the red and grey vertical bands, respectively. The recent estimate by BMW-DMZ [25] is based on a combination of lattice and data-driven evaluations.

[arXiv:2411.07969]

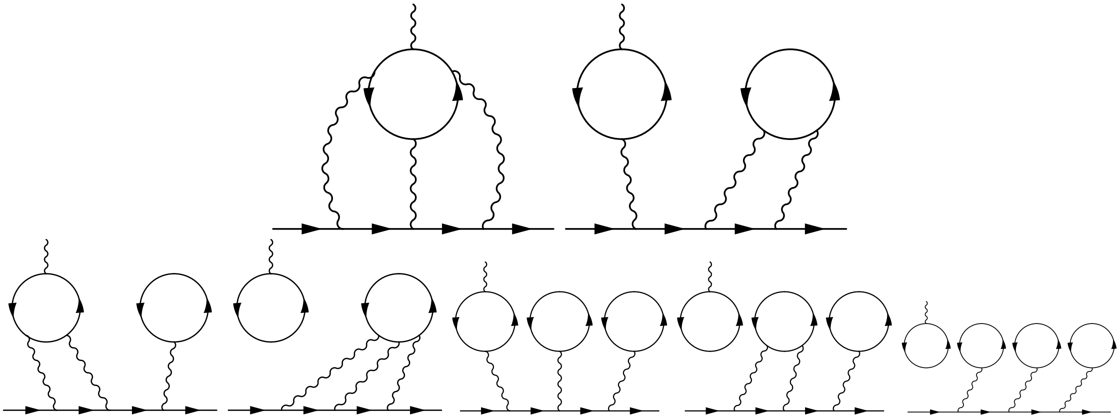


HLbL



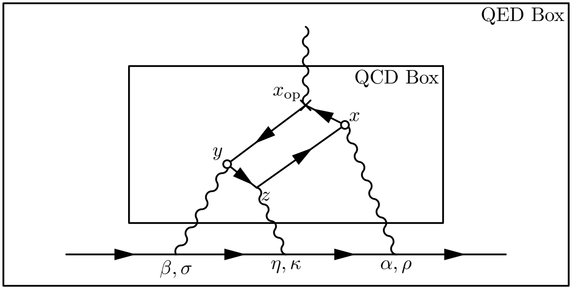
HVP

Muon $g-2$ HLbL: diagrams



- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional different permutations of photons not shown.
- The second row diagrams are suppressed by flavor SU(3) symmetry (and small charge factors, $1/N_c$, etc). The contributions are numerically very small.

HLbL QED$_\infty$: formulation

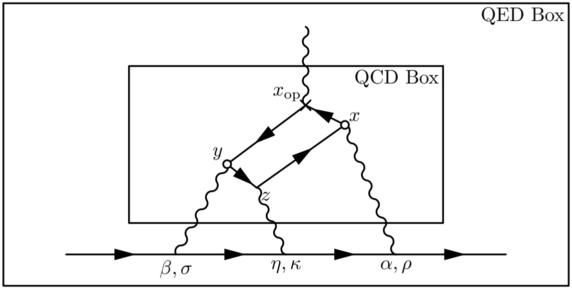


$$a_\mu^{\text{HLbL}} = \frac{2me^2}{3} \frac{1}{VT} \sum_{x_{\text{op}}} \sum_{x,y,z} \frac{1}{2} \epsilon_{i,j,k} \big(x_{\text{op}} - x_{\text{ref}}(x,y,z)\big)_j (6e^4) \mathcal{H}_{k,\rho,\sigma,\kappa}(x_{\text{op}},x,y,z) \mathcal{M}_{i,\rho,\sigma,\kappa}(x,y,z)$$

$$(6e^4)\mathcal{H}_{k,\rho,\sigma,\kappa}(x_{\text{op}},x,y,z) = \langle T J_k(x_{\text{op}}) J_\rho(x) J_\sigma(y) J_\kappa(z) \rangle_{\text{QCD}}$$

$$\mathcal{M}_{i,\rho,\sigma,\kappa}(x,y,z) = \frac{1}{2} \text{Tr}\Big[\frac{1}{6} i^3 \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) \Sigma_i \Big] .$$

arXiv:1705.01067, arXiv:2210.12263

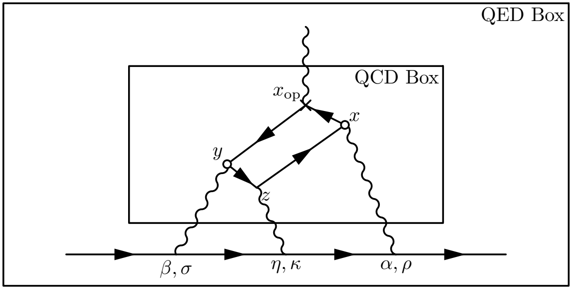


$$
\begin{aligned}
i^3\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) =& \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \mathfrak{G}_{\kappa,\rho,\sigma}(z,x,y) \\
&+ \mathfrak{G}_{\kappa,\sigma,\rho}(z,y,x) + \mathfrak{G}_{\rho,\kappa,\sigma}(x,z,y) + \mathfrak{G}_{\sigma,\rho,\kappa}(y,x,z), \\
\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) =& \lim_{t_{\rm src}\to-\infty, t_{\rm snk}\to\infty} e^{m_\mu(t_{\rm snk}-t_{\rm src})} \int_{\alpha,\beta,\eta} G(x,\alpha)G(y,\beta)G(z,\eta) \\
&\times \int_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} S_\mu\left(x_{\rm snk},\beta\right) i\gamma_\sigma S_\mu(\beta,\eta) i\gamma_\kappa S_\mu(\eta,\alpha) i\gamma_\rho S_\mu\left(\alpha, x_{\rm src}\right),
\end{aligned}
$$

Subtraction to (1) remove infrared divergence; **(2) reduce discretization and finite volume effects.**

$$
\begin{aligned}
\mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) =& \frac{1}{2}\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \frac{1}{2}[\mathfrak{G}_{\rho,\kappa,\sigma}(x,z,y)]^\dagger, \\
\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,x) =& \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(z,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,z).
\end{aligned}
$$

arXiv:1705.01067, arXiv:2210.12263

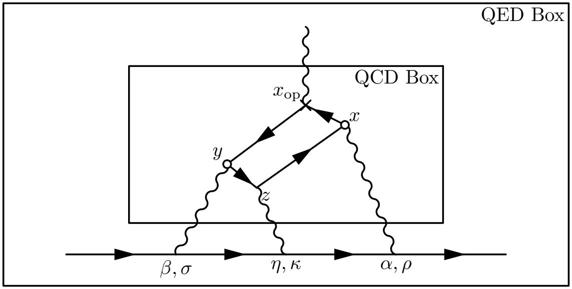


$$\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,x) = \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(z,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,z).$$

- The hadronic four-point function mush satisfy current conservation.
- Change discretization error (when use local current). Change finite volume error.
- **Change the integrand.**
- Does not change the final results (infinite volume & continuum limit).
- The subtraction method is introduced and used by RBC-UKQCD [arXiv:2304.04423]. Mainz [arXiv:2104.02632] and BMW [arXiv:2411.11719] adopt a different subtraction scheme.

arXiv:1705.01067, arXiv:2210.12263

HLbL QED$_\infty$: formulation - $x_{\text{ref}}(x, y, z)$



$$a_\mu^{\text{HLbL}} = \frac{2me^2}{3}\frac{1}{VT}\sum_{x_{\text{op}}}\sum_{x,y,z}\frac{1}{2}\epsilon_{i,j,k}\left(x_{\text{op}} - x_{\text{ref}}(x,y,z)\right)_j (6e^4)\mathcal{H}_{k,\rho,\sigma,\lambda}(x_{\text{op}}, x, y, z)\mathcal{M}_{i,\rho,\sigma,\lambda}(x,y,z)$$

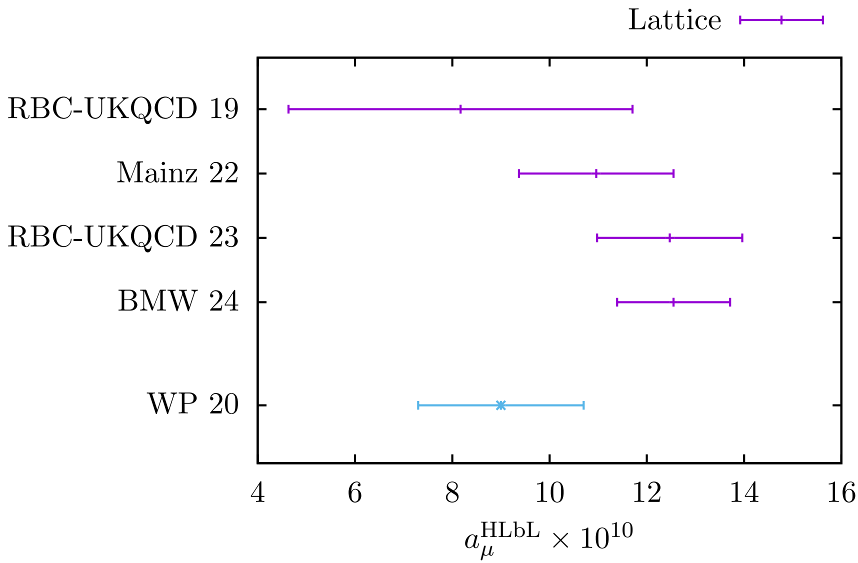
$$x_{\text{ref}}(x,y,z) = x_{\text{ref-far}}(x,y,z) \tag{1}$$

$$= \begin{cases} x & \text{if } |y-z| < \min(|x-y|, |x-z|) \\ y & \text{if } |x-z| < \min(|x-y|, |y-z|) \\ z & \text{if } |x-y| < \min(|x-z|, |y-z|) \\ \frac{1}{3}(x+y+z) & \text{otherwise} \end{cases}$$

$$x_{\text{ref-discon}} = x. \tag{2}$$

arXiv:1705.01067, arXiv:2210.12263

Muon $g-2$ HLbL: results



- A lot of progress made in the lattice front since WP20.
- A very small tension with the WP20 result (mostly determined with data-driven approach).

Thank You!