Intertwined Carrollian diffeomorphisms and Carrollian amplitudes

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Carrollian diffeomorphism

- ▷ Minkowski spacetime has a null boundary $\mathscr{I}^- \cup \mathscr{I}^+$ each of which is a Carrollian manifold. For massless scattering, in/out states are located at $\mathscr{I}^-/\mathscr{I}^+$.

- ▷ $\mathscr{I}^+ = \mathbb{R} \times S^2$ has a degenerate metric

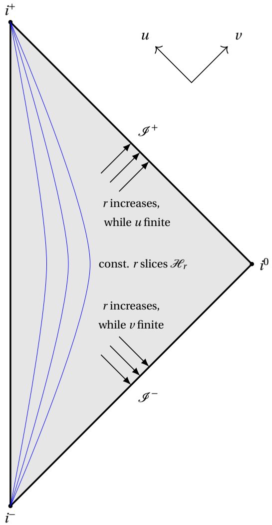
$$ds^2_{\mathscr{I}^+} = \gamma = d\theta^2 + \sin^2\theta\, d\phi^2,$$

and $\chi = \partial_u$ to generate time direction.

- ▷ Carrollian diffeomorphism is generated by

$$\mathscr{L}_\xi \chi = \mu\chi \quad \Rightarrow \quad \xi = f(u,\Omega)\partial_u + Y^A(\Omega)\partial_A,$$

and consists of $\mathrm{Diff}(S^2) \ltimes C^\infty(\mathscr{I}^+)$.



Asymptotic symmetries

- ▷ At large r, a diffeomorphism can be parameterized as

$$
\begin{aligned}
\xi_{f,Y} = & f\partial_u + \frac{1}{2}\nabla_A\nabla^A f\partial_r - \frac{1}{r}\nabla^A f\partial_A + \cdots \\
& + \frac{1}{2}u\nabla_A Y^A\partial_u - \frac{1}{2}r\nabla_A Y^A\partial_r + \frac{u}{4}\nabla_C\nabla^C\nabla\cdot Y\partial_r + (Y^A - \frac{u}{2r}\nabla^A\nabla\cdot Y)\partial_A + \cdots .
\end{aligned} \tag{1}
$$

- ▷ Asymptotic extensions of bulk symmetry

$$f = a^\mu n_\mu,\ Y^A = \omega^{\mu\nu} Y^A_{\mu\nu} \quad \Rightarrow \quad \dot f = 0,\ Y^A = \omega^{\mu\nu} Y^A_{\mu\nu}$$

Poincaré (1900s) ⇒ original BMS (1960s)

$$\Rightarrow \quad \dot f = 0,\ Y^z \text{ meromorphic} \quad \text{or} \quad \dot f = 0,\ \dot Y = 0$$

extended BMS (2010s) or generalized BMS (2010s)

- ▷ BMS is justified by asymptotic Lie derivatives, while eBMS or gBMS is justified by soft graviton theorems. Once allowing $\dot f \neq 0$, we get Carrollian diffeomorphism.

Fundamental fields and symplectic form

- For spin $s \in \mathbb{N}$, the fundamental field $F_{A(s)}$ is defined (2311.11361)

$$f_{A(s)}(t,\boldsymbol{x}) = r^{s-1} F_{A(s)}(u,\Omega) + \mathcal{O}(r^{s-2}), \tag{2}$$

for instance,

$$s=1: \quad a_A(t,\boldsymbol{x}) = A_A(u,\Omega) + \mathcal{O}(r^{-1}), \tag{3a}$$

$$s=2: \quad g_{AB}(t,\boldsymbol{x}) = r^2\gamma_{AB} + rC_{AB}(u,\Omega) + \mathcal{O}(r^0). \tag{3b}$$

- It is easy to find the boundary symplectic form

$$\Omega_s(\delta F;\delta F) = \int du d\Omega\, \delta F^{A(s)} \wedge \delta \dot{F}_{A(s)}. \tag{4}$$

There is no constraint on $F_{A(s)}$ which can determine subleading components through equations of motion (and initial conditions).

Quantum flux operators

- One can use Hamilton's equation or the integral of current

$$i_\xi \Omega(\delta F, \delta F) = \delta H_\xi \qquad \text{or} \qquad \mathscr{F}_\xi = -\int_{\mathscr{I}^+} (d^3x)^\mu T_{\mu\nu}\xi^\nu, \tag{5}$$

  to derive Hamiltonians (as generator) or fluxes (as integral over null hypersurface).

- We construct $\mathscr{T}_f^s$ (supertranslation), $\mathscr{M}_Y^s$ (superrotation) and $\mathscr{O}_h^s$ (superduality)

$$\mathscr{T}_f^s = \int du d\Omega\, f(u,\Omega) : \dot{F}_{A(s)} \dot{F}^{A(s)} : , \tag{6a}$$

$$\mathscr{M}_Y^s = \frac{1}{2}\int du d\Omega\, Y_A(\Omega) (: \dot{F}_{B(s)} \nabla_C F_{D(s)} - F_{B(s)} \nabla_C \dot{F}_{D(s)} :) P^{AB(s)CD(s)}, \tag{6b}$$

$$\mathscr{O}_h^s = \int du d\Omega\, h(\Omega) : \dot{F}^{A(s)} \widetilde{F}_{A(s)} : . \tag{6c}$$

Fundamental commutators and correlators

- Plane wave can be expanded as spherical waves

$$e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = 4\pi \sum_{\ell,m} i^{\ell} j_{\ell}(\omega r)\, Y^{*}_{\ell,m}(\Omega_p)\, Y_{\ell,m}(\Omega), \tag{7}$$

and at large r, we find ($\omega \neq 0$)

$$e^{ip\cdot x} \sim \frac{2\pi}{i\omega r} e^{-i\omega u} \delta(\Omega - \Omega_p) + \frac{\pi}{\omega^2 r^2} e^{-i\omega u} \sum_{\ell m} \ell(\ell+1)\, Y^{*}_{\ell,m}(\Omega_p)\, Y_{\ell,m}(\Omega). \tag{8}$$

- From the boundary symplectic form or mode expansion, we can work out the fundamental commutator

$$[F_{A(s)}(u,\Omega), \dot{F}_{B(s)}(u',\Omega')] = \frac{i}{2} X_{A(s)B(s)} \delta(u-u')\delta(\Omega - \Omega'). \tag{9}$$

- We also have fundamental correlator

$$\langle 0|\, F_{A(s)}(u,\Omega) \dot{F}_{B(s)}(u',\Omega')\, |0\rangle = X_{A(s)B(s)} \frac{\delta(\Omega-\Omega')}{4\pi(u-u'-i\epsilon)}. \tag{10}$$

Fluxes as generators

- All the physical operators have the same form

$$\int du d\Omega : \dot{F}^{A(s)} \delta F_{A(s)} : \tag{11}$$

which makes sure they can generate the corresponding variations through commutators.

- For supertranslations, we obtain

$$\delta_f F_{A(s)} \equiv \mathscr{L}_{\xi_f} F_{A(s)} = i[\mathcal{T}_f^s, F_{A(s)}] = f \dot{F}_{A(s)}. \tag{12}$$

- For superduality, we have

$$\delta_h F_{A(s)} = h Q_{A(s)B(s)} F^{B(s)} \equiv i[\mathcal{O}_h^s, F_{A(s)}]. \tag{13}$$

- For superrotation, there is some subtlety.

Intertwined Carrollian diffeomorphism

Quantum flux operators realize intertwined Carrollian diffeomorphism

$$[\mathscr{T}^s_{f_1},\mathscr{T}^s_{f_2}] = \mathrm{C}^s_T(f_1,f_2) + i\mathscr{T}^s_{f_1\dot{f}_2-f_2\dot{f}_1}, \tag{14a}$$

$$[\mathscr{T}^s_{f},\mathscr{M}^s_{Y}] = -i\mathscr{T}^s_{Y^A\nabla_A f}, \tag{14b}$$

$$[\mathscr{M}^s_{Y},\mathscr{M}^s_{Z}] = i\mathscr{M}^s_{[Y,Z]} + is\mathscr{O}^s_{o(Y,Z)}, \tag{14c}$$

$$[\mathscr{T}^s_{f},\mathscr{O}^s_{h}] = 0, \tag{14d}$$

$$[\mathscr{M}^s_{Y},\mathscr{O}^s_{h}] = i\mathscr{O}^s_{Y^A\nabla_A h}, \tag{14e}$$

$$[\mathscr{O}^s_{h_1},\mathscr{O}^s_{h_2}] = 0. \tag{14f}$$

We get a new central charge

$$\mathrm{C}^{(s=0)}_T = -\frac{i}{48\pi}\delta^{(2)}(0)\int du d\Omega(f_1\partial_u^3 f_2 - f_2\partial_u^3 f_1), \tag{15}$$

and need a helicity flux $\mathscr{O}^s_h$ to close the algebra which concerns electromagnetic duality.

Helicity flux and superduality

- Infinitesimal duality transformations at $\mathscr{I}^+$ are

$$s=1: \quad \delta_\epsilon A_A = \epsilon \widetilde{A}_A, \qquad \delta_\epsilon \widetilde{A}_A = -\epsilon A_A, \tag{16a}$$

$$s=2: \quad \delta_\epsilon C_{AB} = \epsilon \widetilde{C}_{AB}, \qquad \delta_\epsilon \widetilde{C}_{AB} = -\epsilon C_{AB}. \tag{16b}$$

- $\widetilde{A}_A$ and $\widetilde{C}_{AB}$ are dual vector field and shear tensor

$$\widetilde{A}_A = \epsilon_{BA} A^B, \qquad \widetilde{C}_{AB} = \epsilon_{CA} C^C_B. \tag{17}$$

- Original EM duality corresponds to $h=$ const., an $SO(2)$ transformation for which $\mathscr{O}_h$ measures the particle number difference between left and right hand helicities.
- Here $h \in C^\infty(S^2)$ lifts $so(2)$ to an infinite-dimensional algebra, and generalizes global transformation to be local.
- Compute helicity flux $\mathscr{O}_h^{(s=2)}$ in multipole expansion for two-body systems in 2403.18627. It is an observable about gravitational radiation.

Carrollian amplitudes (CA)

- To define CA, we should first construct states in Carrollian space (2402.04120)

$$\Sigma(u,\Omega)=\int\frac{d\omega}{8\pi^2 i}[a_{\boldsymbol{p}}e^{-i\omega u}-a^\dagger_{\boldsymbol{p}}e^{i\omega u}] \tag{18}$$

$$\Rightarrow\qquad |\Sigma(u,\Omega)\rangle=\Sigma(u,\Omega)|0\rangle=\frac{i}{8\pi^2}\int_0^\infty d\omega e^{i\omega u}|\boldsymbol{p}\rangle,\quad |\boldsymbol{p}\rangle\equiv a^\dagger_{\boldsymbol{p}}|0\rangle\,. \tag{19}$$

- Then the S matrix reads

$${}_{\text{out}}\langle\prod_{k=m+1}^{m+n}\Sigma(u_k,\Omega_k)|\prod_{k=1}^{m}\Sigma(u_k,\Omega_k)\rangle_{\text{in}}=(\frac{1}{8\pi^2 i})^{m+n}\prod_{j=1}^{m+n}\int d\omega_j e^{-i\sigma_j\omega_j u_j}\langle\boldsymbol{p}_{m+1}\cdots\boldsymbol{p}_{m+n}|S|\boldsymbol{p}_1\cdots\boldsymbol{p}_m\rangle.$$

- Taking connected and amputated parts, we get CA

$$\langle\prod_{j=1}^{m+n}\Sigma_j(u_j,\Omega_j;\sigma_j)\rangle=(\frac{1}{8\pi^2 i})^{m+n}\prod_{j=1}^{m+n}\int d\omega_j e^{-i\sigma_j\omega_j u_j}(2\pi)^4\delta^{(4)}(\sum_{j=1}^{m+n}p_j)i\mathcal{M}(p_1,p_2,\cdots,p_{m+n}). \tag{20}$$

Poincaré transformation law of CA

- Under translation $u' = u - a\cdot n$, $\Omega' = \Omega$, CA changes as

$$\langle\prod_{j=1}^{n}\Sigma_j(u'_j,\Omega_j)\rangle = \langle\prod_{j=1}^{n}\Sigma_j(u_j,\Omega_j)\rangle, \quad u'_j = u_j - a\cdot n_j. \tag{21}$$

- In stereographic coordinates, Lorentz transformation reads

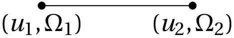
$$u\to u' = \Gamma^{-1}u, \quad z\to z' = \frac{az+b}{cz+d}, \quad \text{where} \quad ad-bc=1, \quad \Gamma = \frac{|az+b|^2+|cz+d|^2}{1+z\bar{z}}, \tag{22}$$

under which CA transforms as

$$\langle\prod_{j=1}^{n}\Sigma_j(u'_j,z'_j,\bar{z}'_j)\rangle = \left(\prod_{j=1}^{n}\Gamma_j\right)\langle\prod_{j=1}^{n}\Sigma_j(u_j,z_j,\bar{z}_j)\rangle. \tag{23}$$

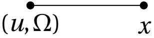
Feynman rules for CA

- ▷ CA can be obtained from Fourier transform of momentum space amplitudes.
- ▷ CA can also be constructed using the following Feynman rules (for Φ^4 theory):
  - – Boundary-to-boundary propagator



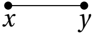
$$= -\beta(u_2 - u_1)\delta(\Omega_1 - \Omega_2); \tag{24}$$

  - – External line



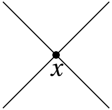
$$= -\frac{\sigma}{8\pi^2(u + n\cdot x - i\sigma\epsilon)}; \tag{25}$$

  - – Feynman propagator



$$= \int \frac{d^4p}{(2\pi)^4} G_F(p) e^{ip\cdot(x-y)} = \frac{1}{4\pi^2\left((x-y)^2 + i\epsilon\right)}; \tag{26}$$

  - – Vertex



$$= -i\lambda \int d^4x. \tag{27}$$

Problems in ongoing works

- The violation of Jacobi identity due to center charge (for $s=0$)

$$[\mathscr{T}_{f_1},[\mathscr{T}_{f_2},\mathscr{M}_Y]]+(\text{perm.})=-\frac{1}{48\pi}\delta^{(2)}(0)\int du d\Omega\, Y^A\nabla_A(f_1\partial_u^3 f_2-f_2\partial_u^3 f_1). \tag{28}$$

- How to understand chiral anomaly in asymptotic QED

$$\partial_\mu j_5^\mu=-\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}. \tag{29}$$

- How to explain the isomorphism among various (local) $U(1)$

Groups	Definitions/actions
Boundary little group	Preserve momentum $p^\mu\sim\omega n^\mu$
Local rotation of vielbeins	$e'^A_a=R_a{}^b e^A_b$, preserve $\mathbb{R}^2$
Superduality	Reduce along r, $\delta_h A_A=h\epsilon_{BA}A^B$
Phase sym. of complex scalar	$\Phi\to e^{-i\alpha}\Phi$
Chiral sym. of fermion	$\Psi_M\to e^{-i\alpha\gamma_5}\Psi_M$, $\delta\chi=i\alpha\chi$
$\mathcal{N}=1$ R sym.	$\Phi\to e^{2i\alpha/3}\Phi$, $\chi\to e^{-i\alpha/3}\chi$

Discussions

- We have a family of spinor Lie derivatives

$$\mathscr{L}_\xi \Psi = \xi^\mu \nabla_\mu \Psi - \frac{1}{4}\nabla_{[\mu}\xi_{\nu]}\gamma^\mu\gamma^\nu\Psi + \alpha\nabla_\mu\xi^\mu\Psi, \tag{30}$$

where $\alpha = 0$ is for Kosmann, $\alpha = -1/4$ is for Penrose, but we need $\alpha = 1/4$.

- We find the nonclosure of Lie transport of a spinor around a loop

$$[\mathscr{L}_{\xi_1}, \mathscr{L}_{\xi_2}]\Psi - \mathscr{L}_{[\xi_1,\xi_2]}\Psi = \frac{1}{16}\mathscr{L}_{\xi_1}\eta_{\mu\rho}\mathscr{L}_{\xi_2}\eta_{\nu\sigma}\eta^{\mu\nu}[\gamma^\rho,\gamma^\sigma]\Psi, \tag{31}$$

which implies the appearance of helicity flux in $[\mathscr{M}_Y, \mathscr{M}_Z]$.

- For gauge theories, the Lie derivative of tensors is closed around a loop. We introduce covariant variation by adding connection terms such that $\delta_Y\gamma_{AB} = 0$ and

$$\delta_Y F_{B(s)} = \frac{1}{2}u\nabla_A Y^A \dot{F}_{B(s)} + \frac{1}{2}\nabla_A Y^A F_{B(s)} + Y^A\nabla_A F_{B(s)} + s\nabla_{([B_1}Y_{A]}F^A_{B_2\cdots B_s)}. \tag{32}$$

This makes sure $\delta_Y F_{B(s)} = [i\mathscr{M}_Y + i\mathscr{T}_{f=\frac{1}{2}u\nabla\cdot Y}, F_{B(s)}]$ and implies the appearance of $\mathscr{O}^s$.

Future interests

- ▷ Generalize to $\mathcal{N} = 1$ SYM, supergravity and more supersymmetry.
- ▷ Investigate the relation between superduality and other S duality, e.g., Seiberg duality.
- ▷ Understand spin and orbit angular momentum as well as helicity flux at the null boundary.
- ▷ Construct a reasonable flat holography.

Related papers

[1] W.-B. Liu and J. Long, "Symmetry group at future null infinity: Scalar theory," *Phys. Rev. D* **107** (2023), no. 12, 126002, 2210.00516.

[2] W.-B. Liu and J. Long, "Symmetry group at future null infinity II: Vector theory," *JHEP* **07** (2023) 152, 2304.08347.

[3] W.-B. Liu and J. Long, "Symmetry group at future null infinity III: Gravitational theory," *JHEP* **10** (2023) 117, 2307.01068.

[4] A. Li, W.-B. Liu, J. Long, and R.-Z. Yu, "Quantum flux operators for Carrollian diffeomorphism in general dimensions," *JHEP* **11** (2023) 140, 2309.16572.

[5] W.-B. Liu, J. Long, and X.-H. Zhou, "Quantum flux operators in higher spin theories," *Phys. Rev. D* **109** (2024), no. 08, 086012, 2311.11361.

[6] W.-B. Liu and J. Long, "Holographic dictionary from bulk reduction," *Phys. Rev. D* **109** (2024), no. 06, L061901, 2401.11223.

[7] W.-B. Liu, J. Long, and X.-H. Zhou, "Electromagnetic helicity flux operators in higher dimensions," 2407.20077.

[8] W.-B. Liu, J. Long, and X.-Q. Ye, "Feynman rules and loop structure of Carrollian amplitude," *JHEP* **05** (2024) 213, 2402.04120.

[9] W.-B. Liu, J. Long, H.-Y. Xiao, and J.-L. Yang, "On the definition of Carrollian amplitudes in general dimensions," *JHEP* **11** (2024) 027, 2407.20816.

Thanks for your attention!