第四届强子与重味物理理论实验联合研讨会

Detecting Light Dark Matter with Plasmons

武雷

南京师范大学

in collaboration with 梁正良, 苏亮亮, 祝斌

2501.07591

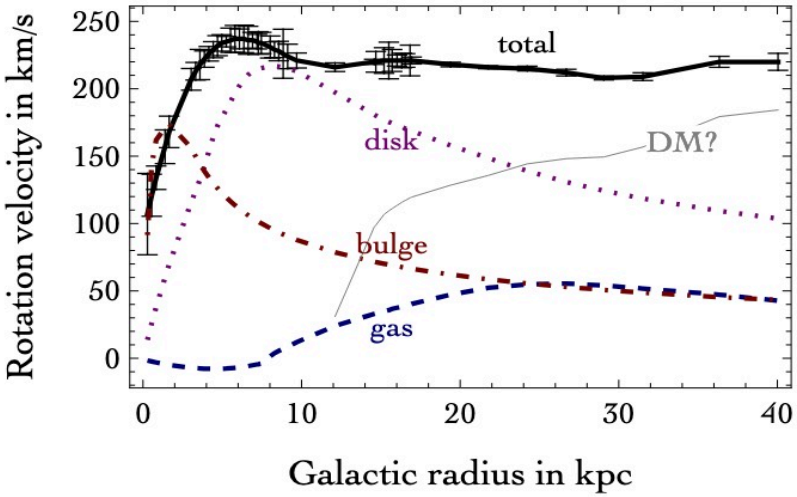
PRL 134 (2025) 7, 071001

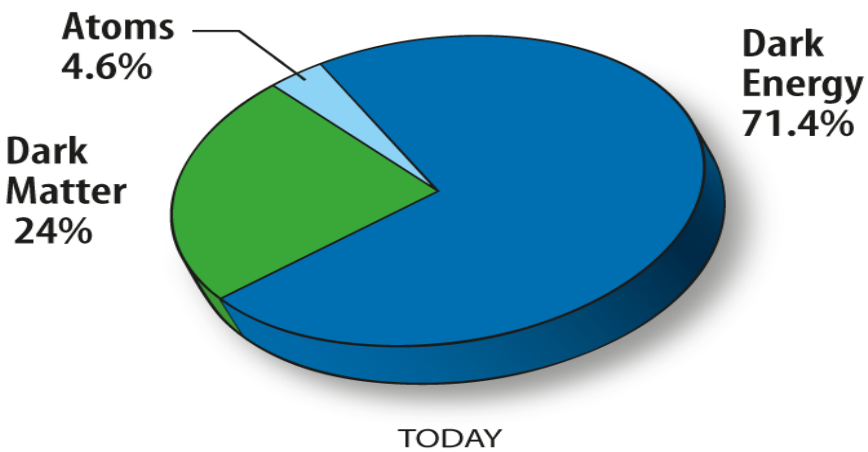
Outline

- From WIMPs to Light Dark Matter (LDM)
- Electronic Collective Excitations: Plasmons
- Detecting LDM with Plasmons
- Conclusions

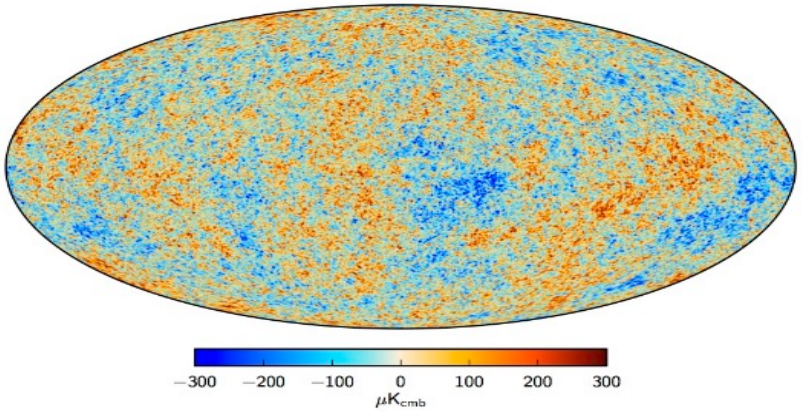
1. From WIMPs to Light DM

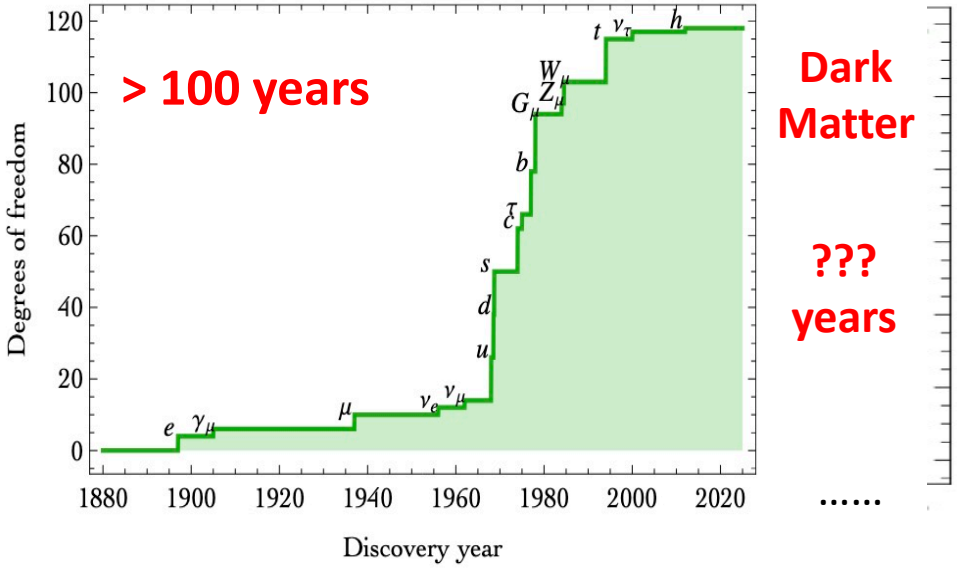
天文学观测





宇宙学观测





Dark Matter

??? years

……

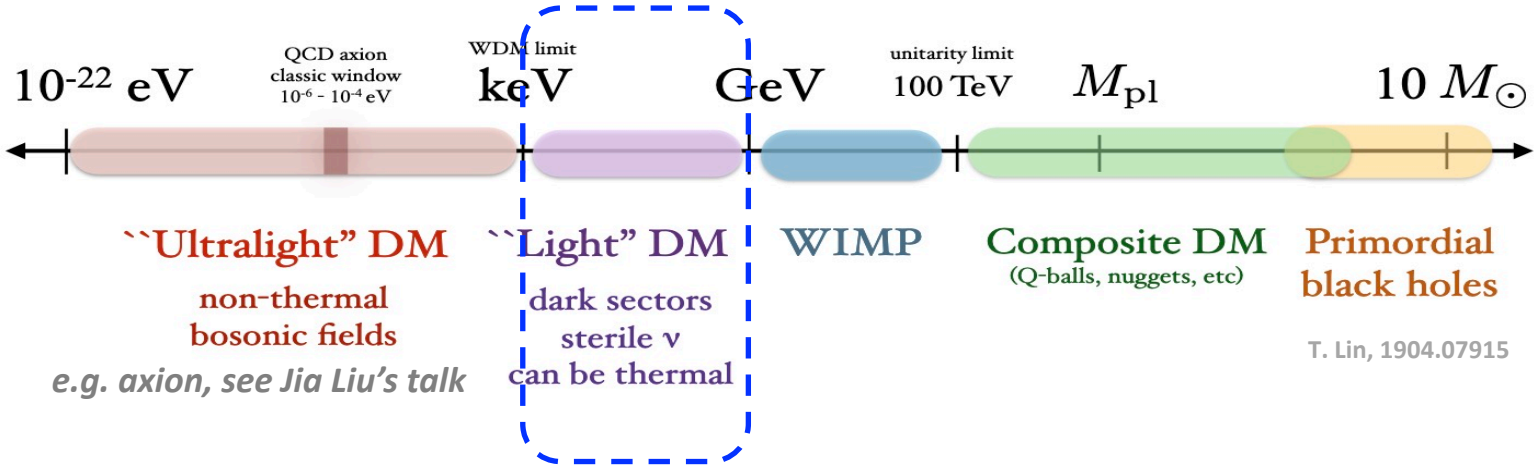
1. From WIMPs to Light DM

天体物理	Primordial BH	1966, 1971	Zeldovich & Novikov, Hawking
	MACHOs	1981, 1986	Petrou; Paczynski
粒子物理	Gravitinos	1981, 1982	Fayet; Witten; Pagel & Primack
	Axions	1983	Preskill, Wise & Wilczek
	Neutralinos	1984	Ellis et al.
	Strangelets	1984	Witten; Fahri & Jaffe
	Q-balls	1984	Witten
	Extra-dimensional DM	1984	Kolb & Slansky; Servant & Tait
	WIMPs	1985	Steigman & Turner
	Sterile neutrinos	1993	Dodelson & Widrow
	Fuzzy DM	2000	Hu, Barkana & Gruzinov
	Sub-GeV DM	2003	Boehm, Fayet et al.

Relic Abundance

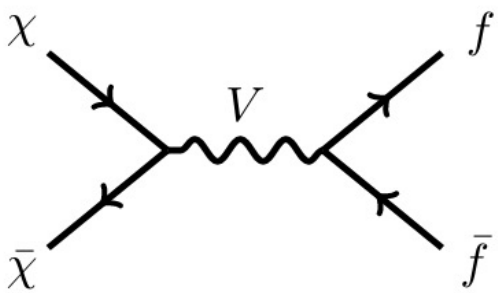
Solution to Big problems

1. From WIMPs to Light DM



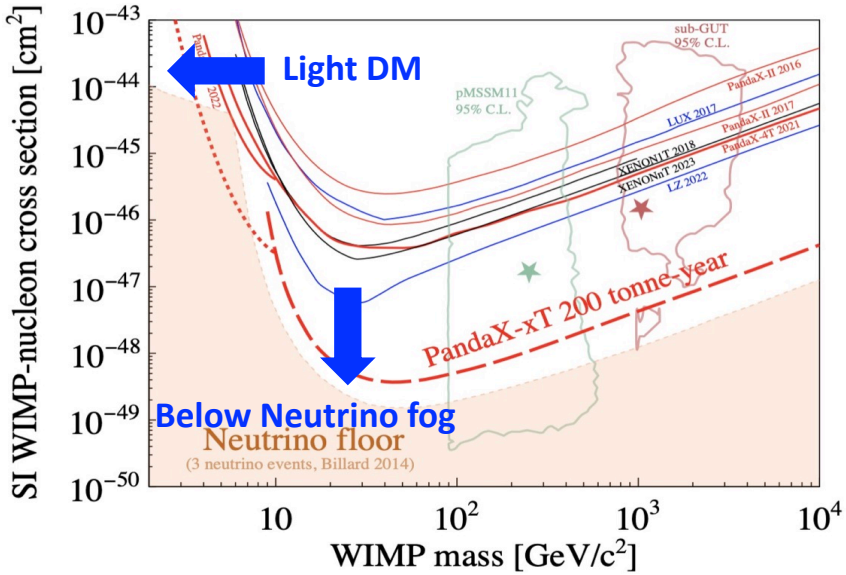
T. Lin, 1904.07915

WIMPs



$m_\chi \sim 100$ GeV, $g_\chi \sim 0.6 \rightarrow \Omega_\chi \sim 0.1$

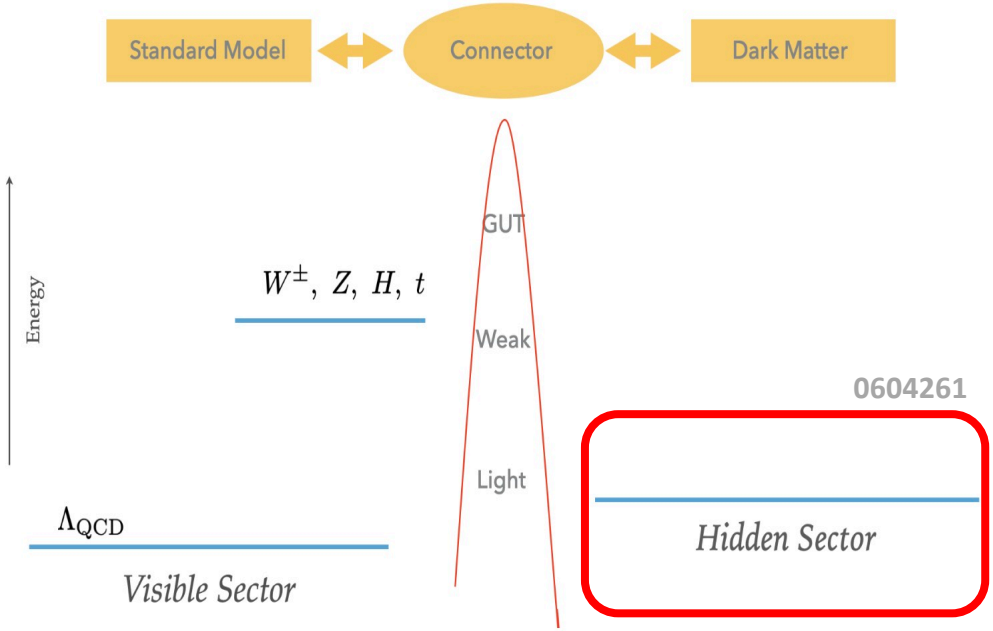
SUSY



PandaX collaboration, 2402.03596

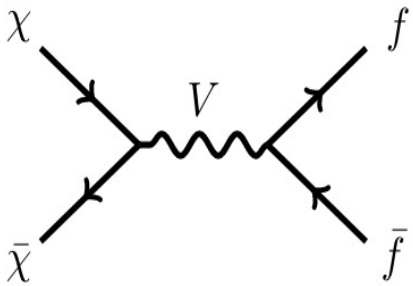
1. From WIMPs to Light DM

Hidden Valley DM



+

- SIMPs [YH, Kuflik, Volansky, Wacker, 2014; YH, Kuflik, Murayama, Volansky, Wacker, 2015]
- ELDERs [Kuflik, Perelstein, Rey-Le Lorier, Tsai, 2016 & 2017]



- $m_V > m_\chi \quad \langle \sigma v \rangle \simeq \dfrac{16\pi\alpha_\chi\alpha_f m_\chi^2}{m_V^4}$

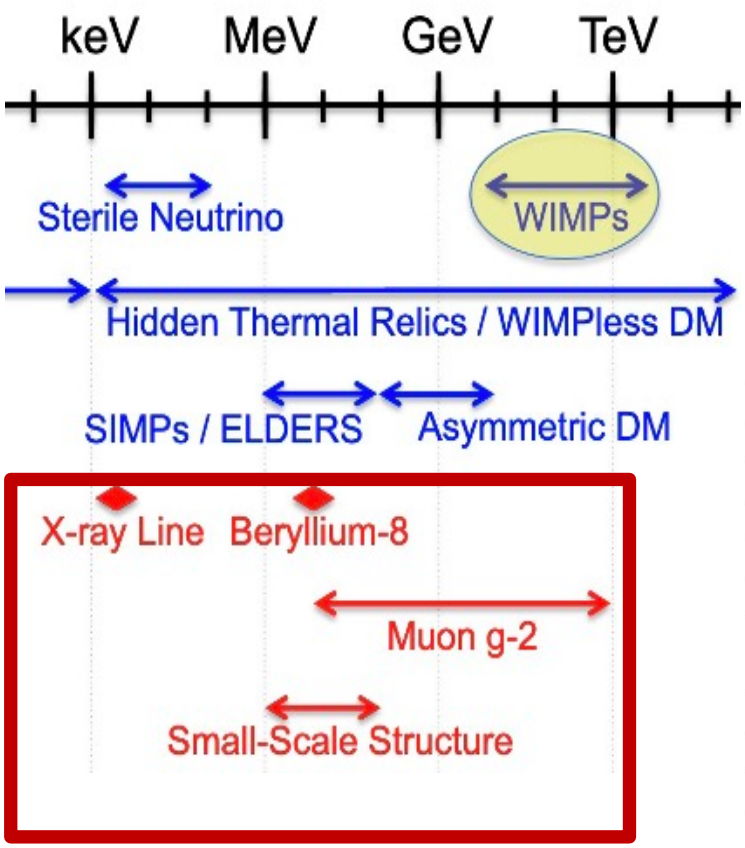
0305261
0711.4866
0803.4196

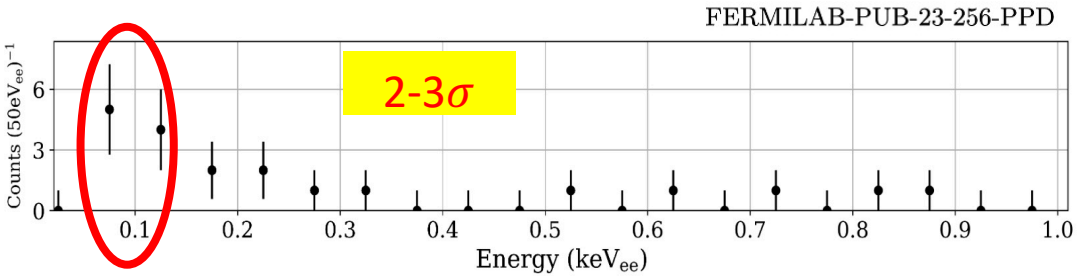
Light non-SM mediator

- $m_V < m_\chi \quad \langle \sigma v \rangle \simeq \dfrac{\pi\alpha_\chi\alpha_f}{m_\chi^2}$

1. From WIMPs to Light DM

Data

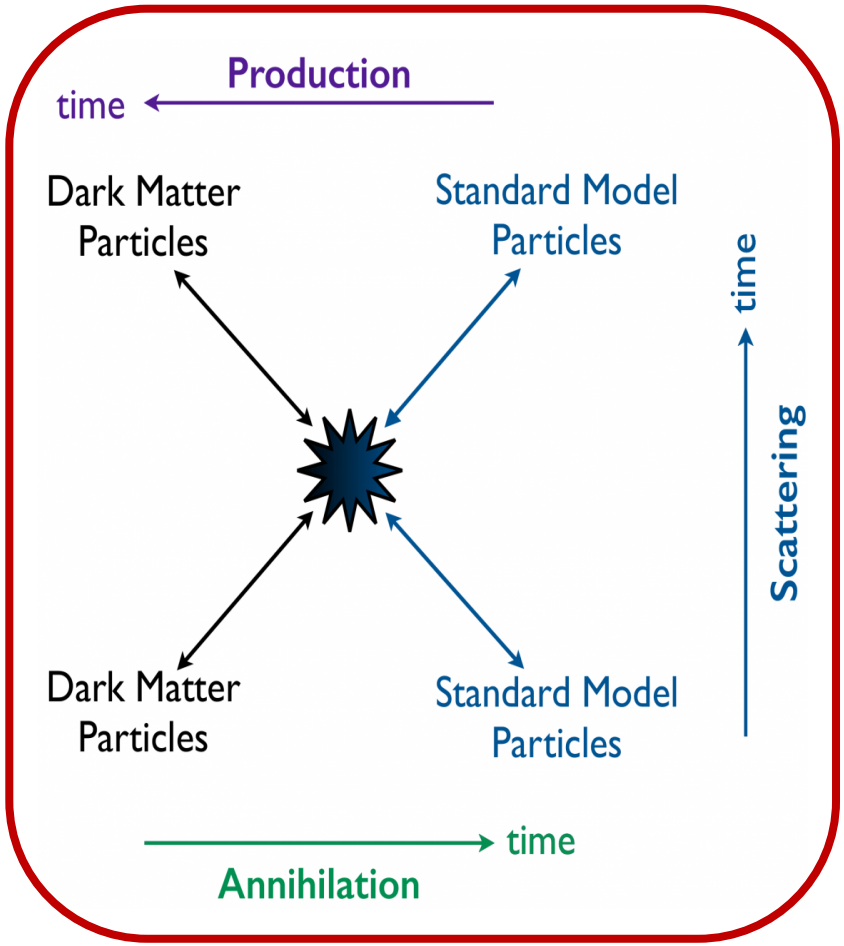




2002.06937

Readout Type	Target	Resolution	Exposure	Threshold	Excess Rate (Hz/kg)	Depth	Reference
Charge (E_e)	Ge	1.6 e^-	80 g·d	0.5 eVee (~1e^-)[a]	[20, 100]	.7 km	EDELWEISS [6]
	Si	~0.2 e^-	0.18 g·d	1.2 eVee (<1 e^-)	[6, 400]	100 m	SENSEI [4]
	Si	0.1 e^-	0.5 g·d	1.2 eVee (<1 e^-)	[10, 2000]	~1 m	CDMS HVeV [3]
	Si	1.6 e^-	200 g·d	1.2 eVee (~1e^-)	[1 ×10^{-3}, 7]	2 km	DAMIC [7]
Energy (E_{det})	Ge	18 eV	200 g·d	60 eV	> 2	~1 m	EDELWEISS [1]
	$CaWO_4$	4.6 eV	3600 g·d	30 eV	> 3 ×10^{-3}	.4 km	CRESST-III [2]
	Al_2O_3	3.8 eV	0.046 g·d	20 eV	> 30	~1 m	νCLEUS [8]
Photo e^-	Xe	6.7 PE (~ 0.25 e^-)	15 kg·d	12.1 eVee (~14 PE)	[0.5, 3] ×10^{-4}	.4 km	XENON10 [5, 9]
	Xe	6.2 PE (~ 0.31 e^-)	30 kg·yr	~70 eVee (~80 PE)	> 2.2 ×10^{-5}	.4 km	XENON100 [5]
	Xe	< 10 PE	60 kg·yr	~140 eVee (~90 PE)	> 1.7 ×10^{-6}	.4 km	XENON1T [10]
	Ar	~15 PE (~ 0.5 e^-)	6780 kg·d	50 eVee	> 6 ×10^{-4}	.4 km	Darkside50 [11]

1. From WIMPs to Light DM



- **BBN, CMB**
- **Large Scale Structure**
- **Stellar Evolution and Cooling**
- **Meson decays**

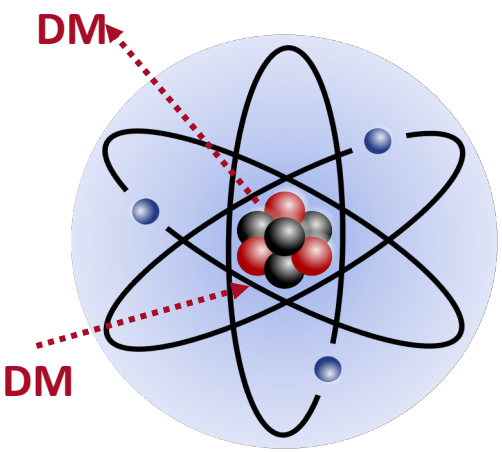
2. Electronic Collective Excitations : Plasmons

PHYSICAL REVIEW D VOLUME 31, NUMBER 12

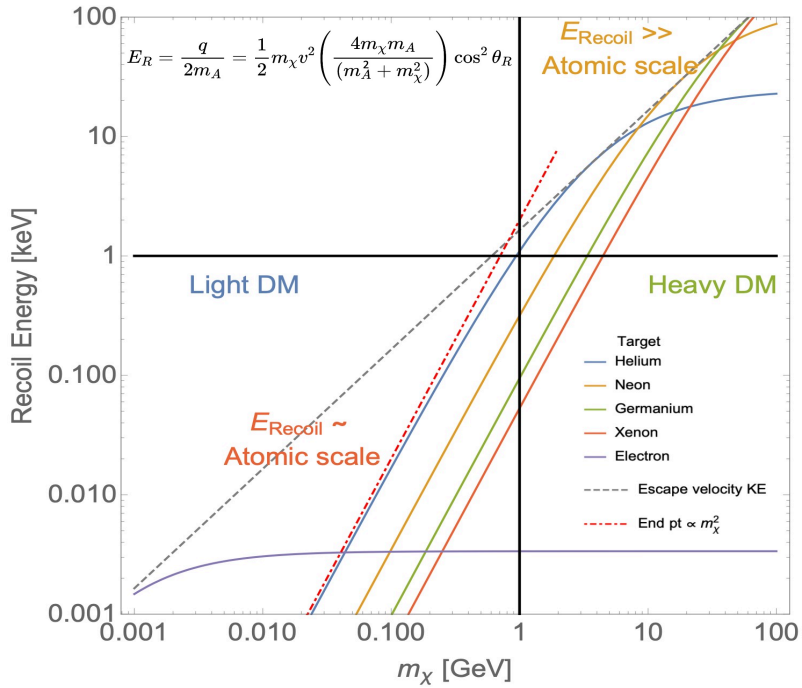
Detectability of certain dark-matter candidates

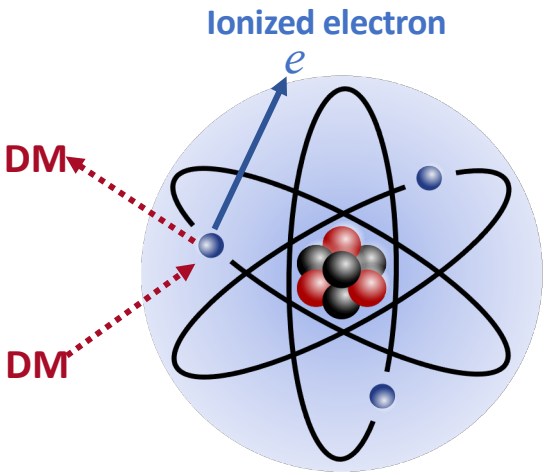
Mark W. Goodman and Edward Witten
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 7 January 1985)

We consider the possibility that the neutral-current neutrino detector recently proposed by Drukier and Stodolsky could be used to detect some possible candidates for the dark matter in galactic halos. This may be feasible if the galactic halos are made of particles with coherent weak interactions and masses $1-10^6$ GeV; particles with spin-dependent interactions of typical weak strength and masses $1-10^2$ GeV; or strongly interacting particles of masses $1-10^{13}$ GeV.



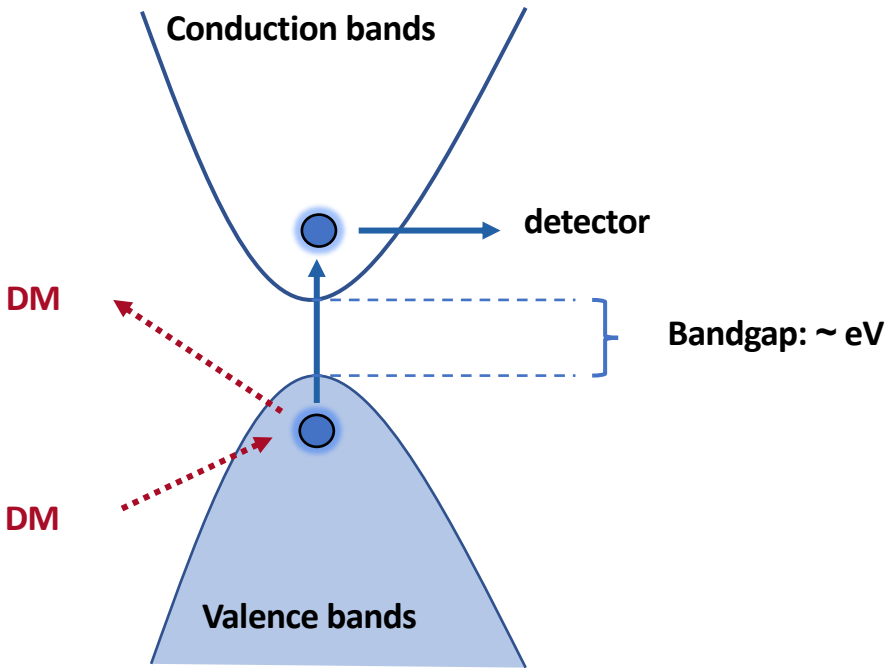
~ keV

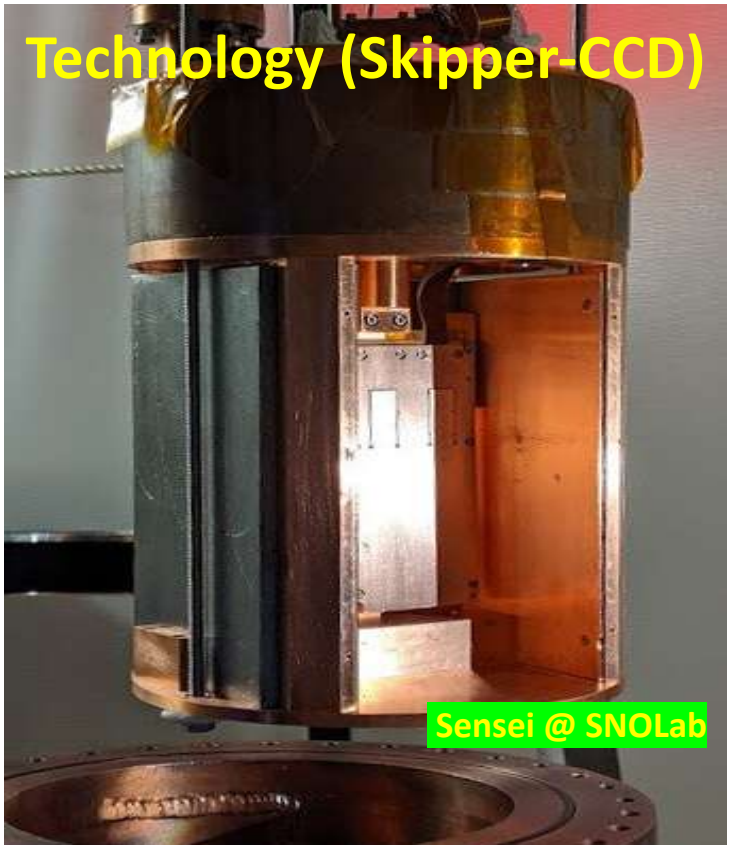




~ 0. KeV

2. Electronic Collective Excitations : Plasmons



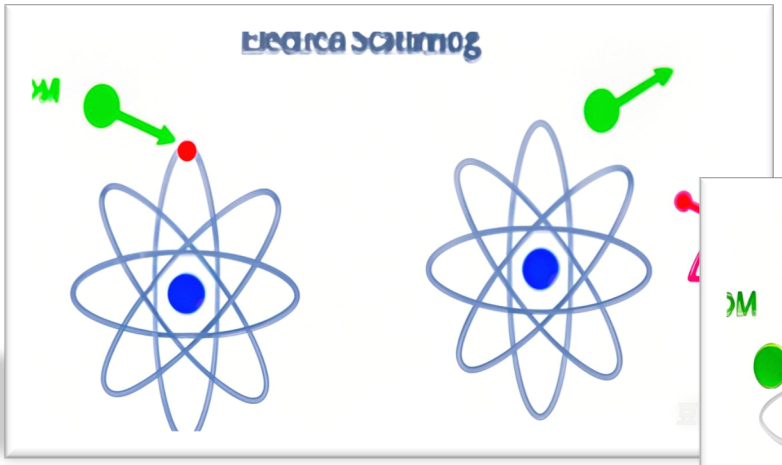


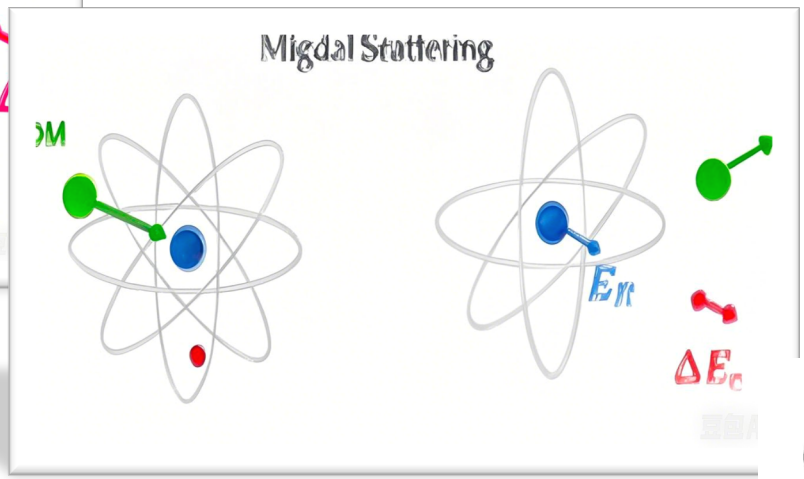
Electron recoil ionizing 1–6 e^- in the detector, which corresponds to (~1.2–20 eV) energy depositions

2. Electronic Collective Excitations: Plasmons

Ionization signals

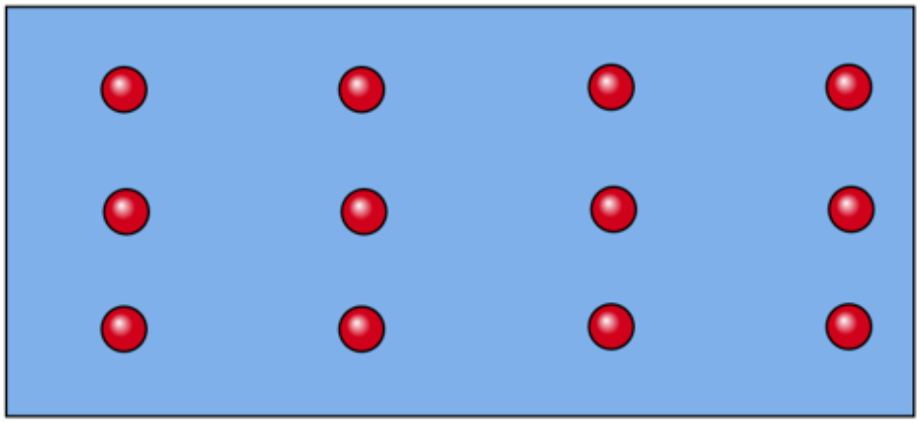
(DM-e, Migdal effect)





Collective Excitations

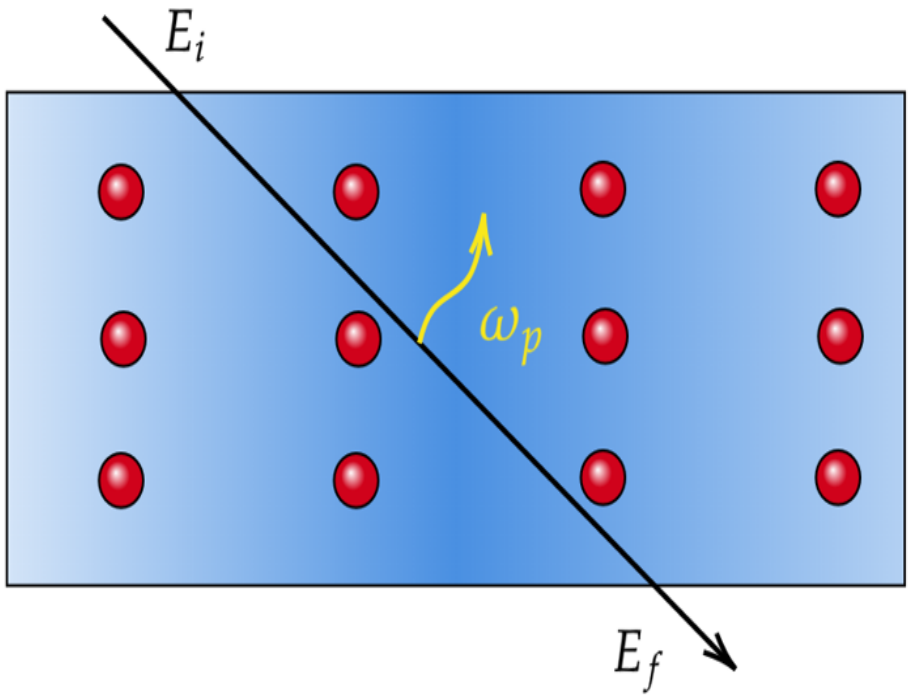
2. Electronic Collective Excitations: Plasmons

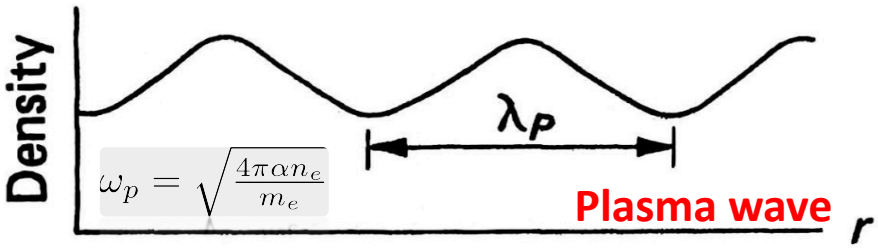


electron gas

+

ionic lattice



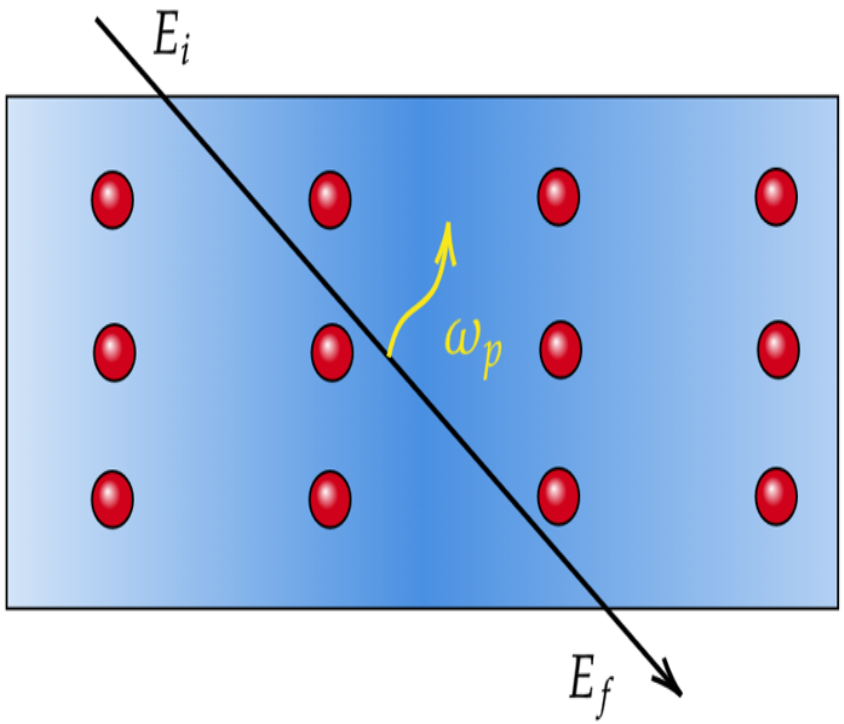


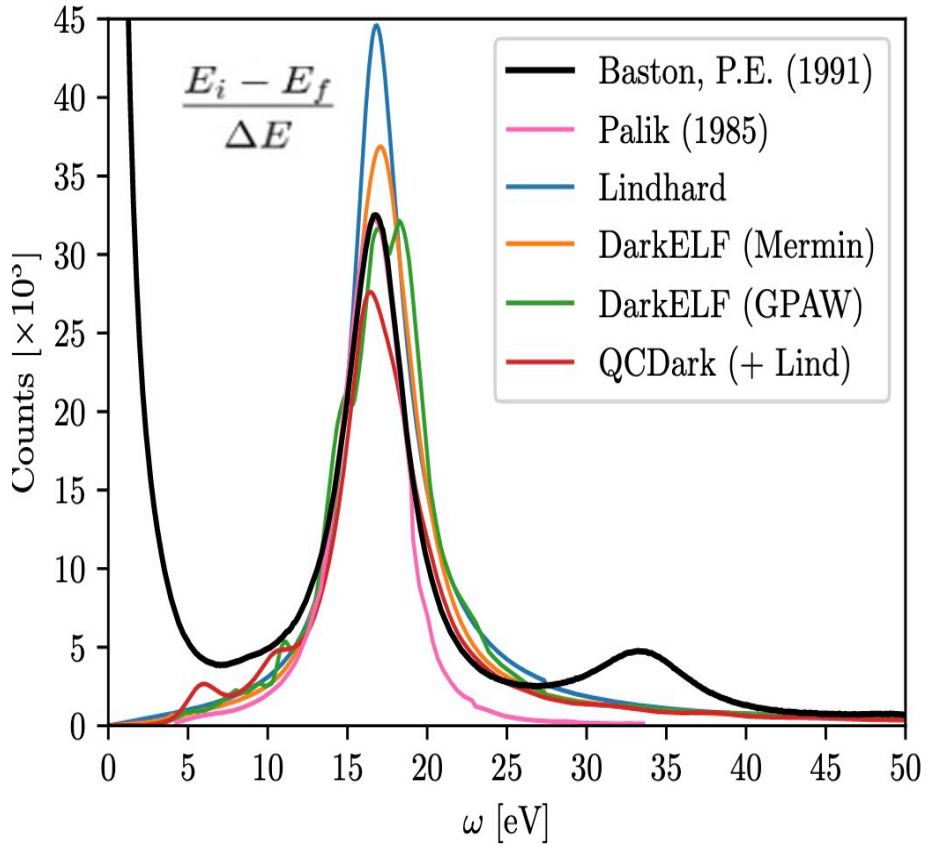
Plasmon

quanta of electron density fluctuation

2. Electronic Collective Excitations: Plasmons

How to identify plasmons: Electron Energy Loss Spectroscopy (EELS)





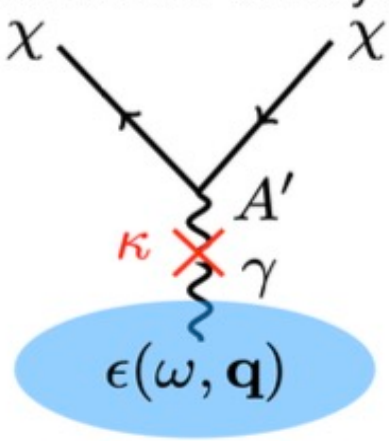
3. Detecting LDM with Plasmons

Liang, Su, LW, Zhu, 2401.11971

◆ DM-induced Plasmon Signals in Solid Detector

$$R \sim \int d^3\mathbf{v} f(\mathbf{v}) \int d^3\mathbf{q} F_{\rm DM}^2(\mathbf{q}) S(\mathbf{q}, \omega_{\mathbf{q}})$$

Interactions in CM systems



Dielectric Function

Electronic Structure (polarization)

$$\varepsilon(Q,\omega) = \varepsilon_1(Q,\omega) + i\varepsilon_2(Q,\omega)$$

Dissipation Processes (absorption and scattering)

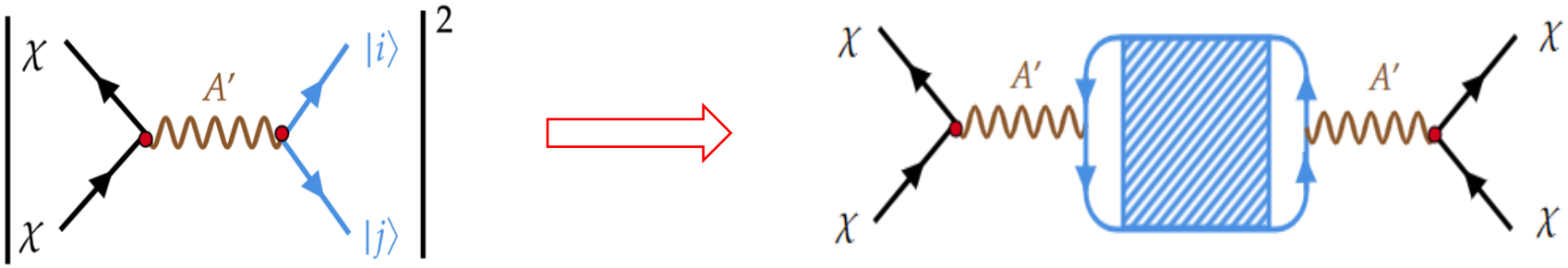
3. Detecting LDM with Plasmons

Liang, Su, LW, Zhu, 2401.11971

Benchmark Model :

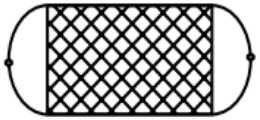
$$\mathcal{L}_{\rm int} \supset g_\chi \bar{\chi}\gamma^\mu \chi A'_\mu + g_e \bar{e}\gamma^\mu e A'_\mu$$

$$\mathcal{L}_e^{\rm eff} \supset g_e A'_0 \psi_e^* \psi_e + \frac{i g_e}{2m_e}\mathbf{A}' \cdot \left(\psi_e^* \overrightarrow{\nabla} \psi_e - \psi_e^* \overleftarrow{\nabla} \psi_e\right) + \cdots$$

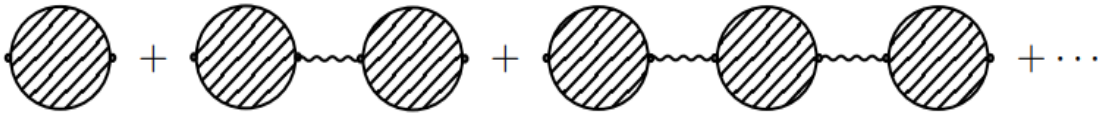


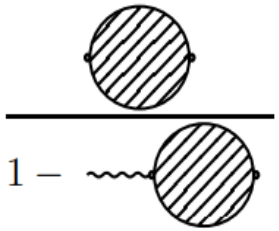
3. Detecting LDM with Plasmons

Liang, Su, LW, Zhu, 2401.11971

$\chi^{\mathrm{r}}_{\hat{\rho}\hat{\rho}} =$ 

Density-Density correlation function (RPA)

$=$ 

$=$ 

$\Pi_e(Q,\omega)$

Lindhard function

$$= \frac{\Pi_e}{1 - V_e \Pi_e},$$

$$\frac{1}{\epsilon(\mathbf{Q},\omega)} = 1 + V_e(Q)\,\chi^{\mathrm{r}}_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\omega)$$

Dielectric Function

Energy Loss Function (ELF)

$$\Gamma(\mathbf{p}_\chi) = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} |V(\mathbf{Q},\omega)|^2 \left[2\frac{Q^2}{e^2}\mathrm{Im}\left(-\frac{1}{\epsilon(\mathbf{Q},\omega)}\right)\right],$$

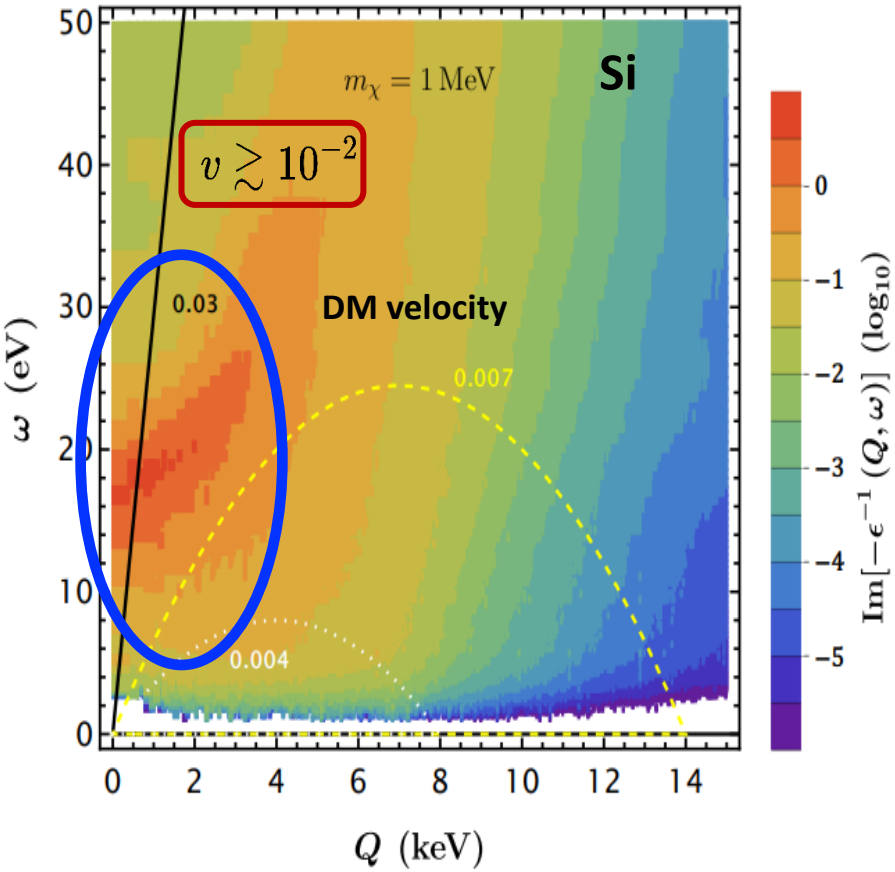
$$\mathrm{Im}\left[\frac{-1}{\epsilon}\right] = \frac{\mathrm{Im}[\epsilon]}{\mathrm{Re}[\epsilon]^2 + \mathrm{Im}[\epsilon]^2}$$

$$\frac{1}{\pi}\frac{\frac{1}{2}\Gamma}{(x-x_0)^2 + \left(\frac{1}{2}\Gamma\right)^2} \sim \boxed{\delta(x - x_0)}$$

DM scattering rate **Scattering potential**

3. Detecting LDM with Plasmons

Liang, Su, LW, Zhu, 2401.11971



- **Energy Loss Function:**

  Density Functional Theory

- **Resonance (plasmon) :**

  |Q| < 5 keV, ω ~ 15 eV

- **To excite plasmon:**

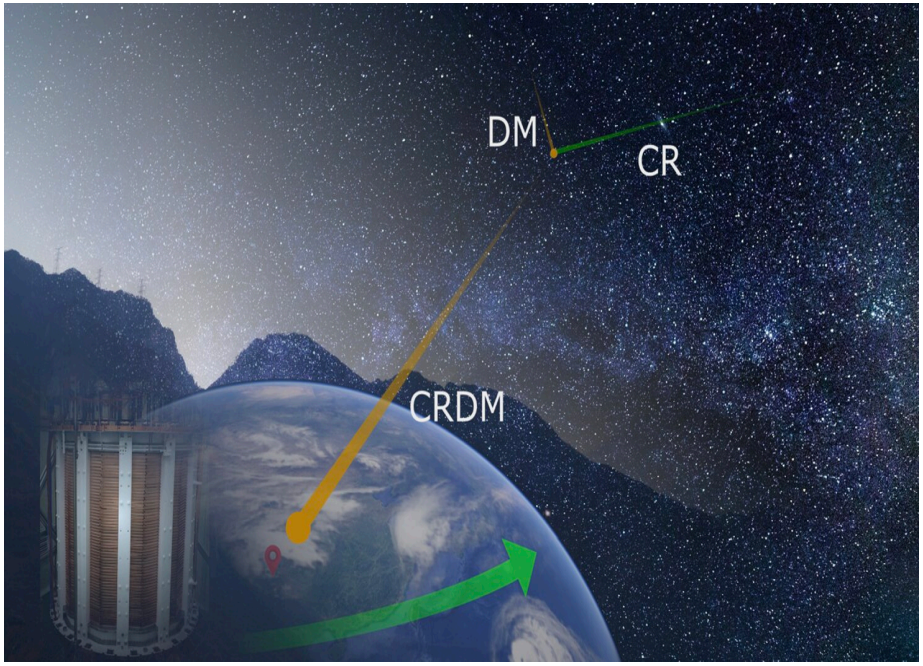
  $V_{min} > q/\omega$

Low deposited energy **Relativistic Dark Matter**

3. Detecting LDM with Plasmons

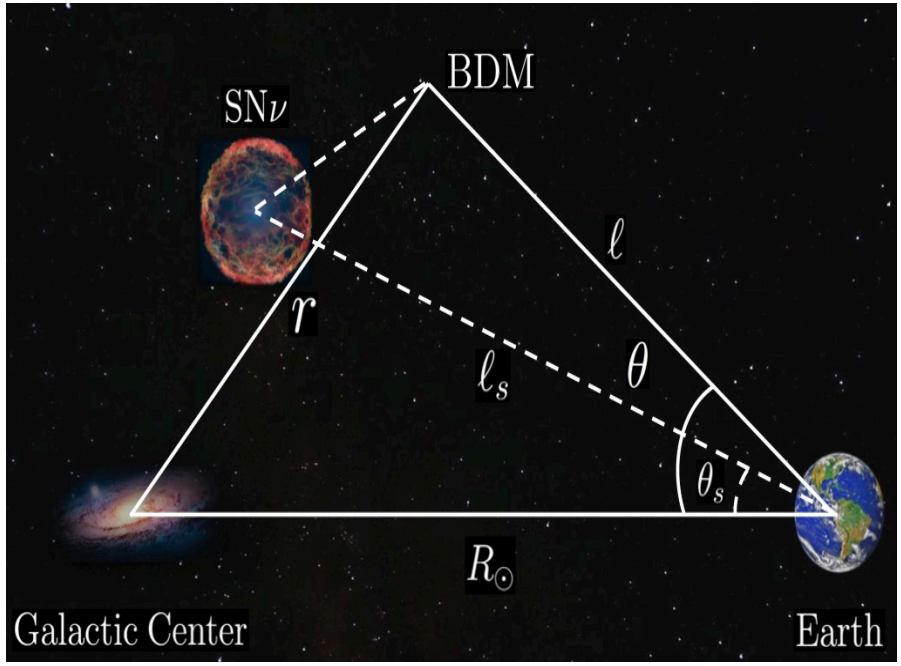
Cosmic Ray

1810.07705
1810.10543



Supernova Neutrino

0908.1790
1506.03825
2501.07591

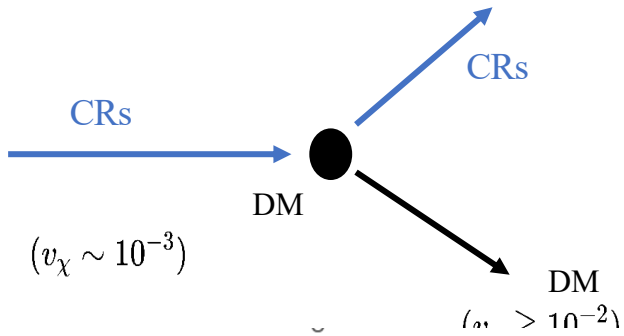


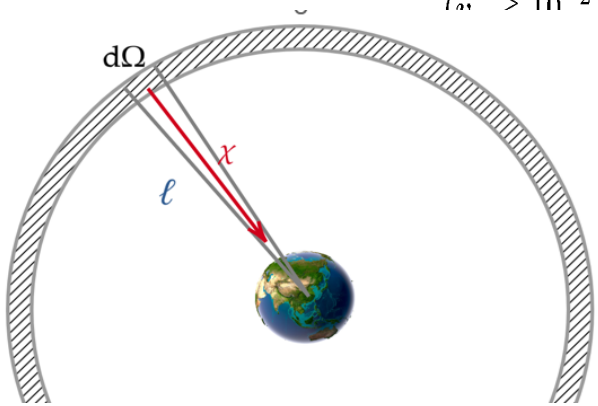
3. Detecting LDM with Plasmons

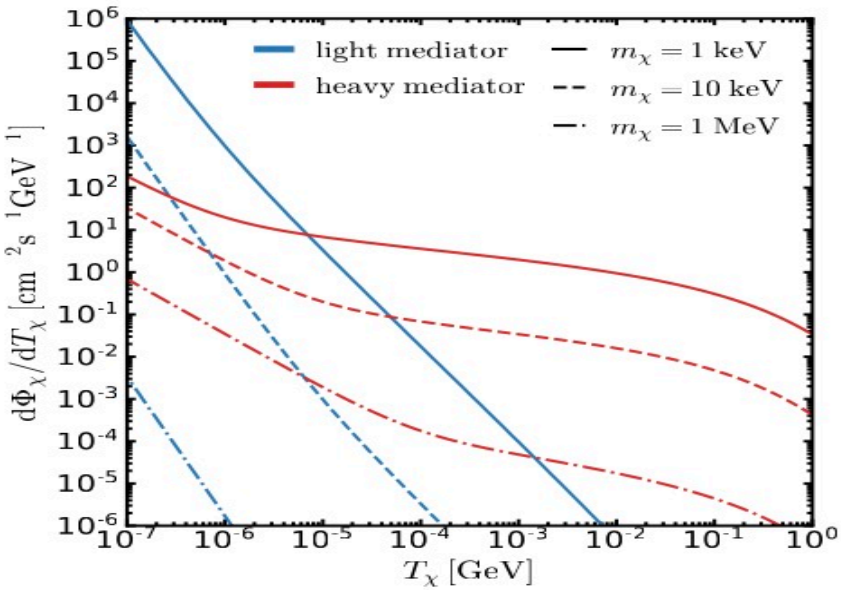
Liang, Su, LW, Zhu, 2401.11971

Cosmic Ray Boosted DM

$$\frac{\mathrm{d}\Phi_\chi}{\mathrm{d}T_\chi} = \int \frac{\mathrm{d}\Omega}{4\pi} \int_{\mathrm{l.o.s.}} \frac{\rho_\chi(\Omega,\ell)}{m_\chi} \mathrm{d}\ell \int_{T_e^{\min}}^{+\infty} \mathrm{d}T_e \frac{\mathrm{d}\sigma_{\chi e}}{\mathrm{d}T_\chi} \frac{\mathrm{d}\Phi_e}{\mathrm{d}T_e}$$





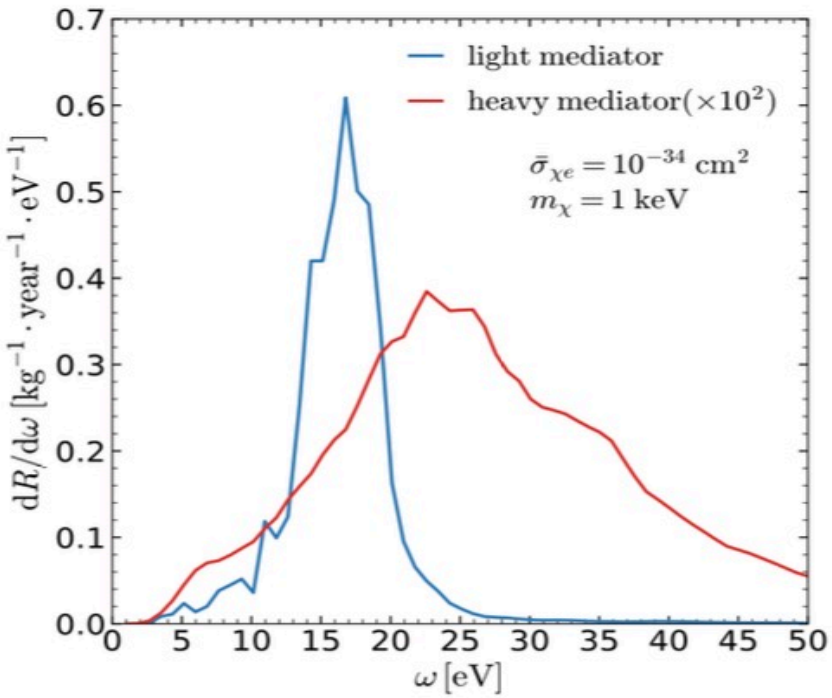


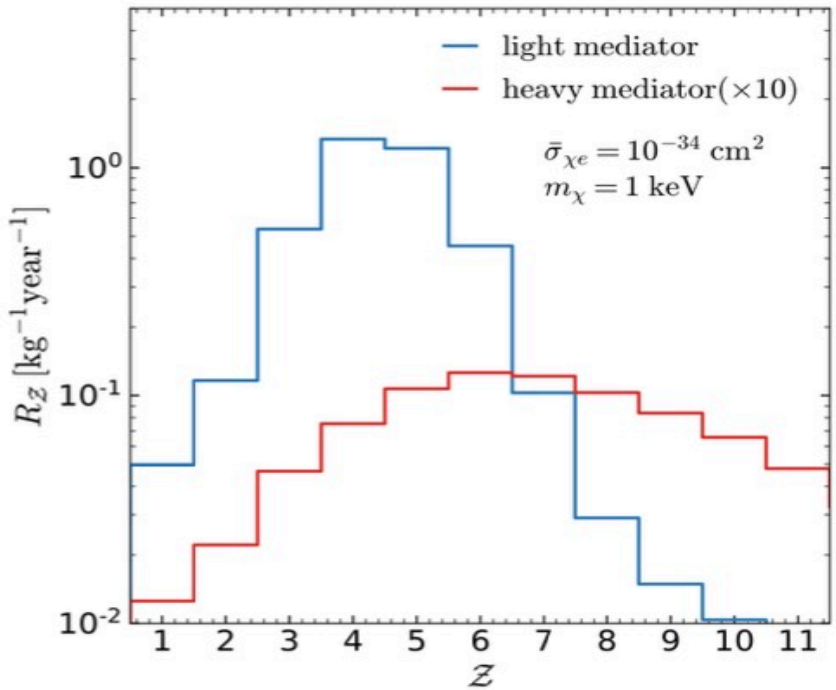
$$R = \frac{1}{\rho_T} \int \mathrm{d}T_\chi \int \frac{\mathrm{d}\Omega}{4\pi} \frac{\mathrm{d}\Phi_\chi}{\mathrm{d}T_\chi} \left(\frac{E_\chi}{p_\chi}\right) \Gamma(p_\chi)$$

3. Detecting LDM with Plasmons

Liang, Su, LW, Zhu, 2401.11971

$$|F_{\rm DM}(q)|^2 = \frac{\left(\alpha^2 m_e^2 + m_{A'}^2\right)^2}{\left(q^2 + m_{A'}^2\right)^2} = \begin{cases} 1 & \text{heavy mediator} \\ \frac{(\alpha m_e)^4}{q^4} & \text{light mediator} \end{cases}$$



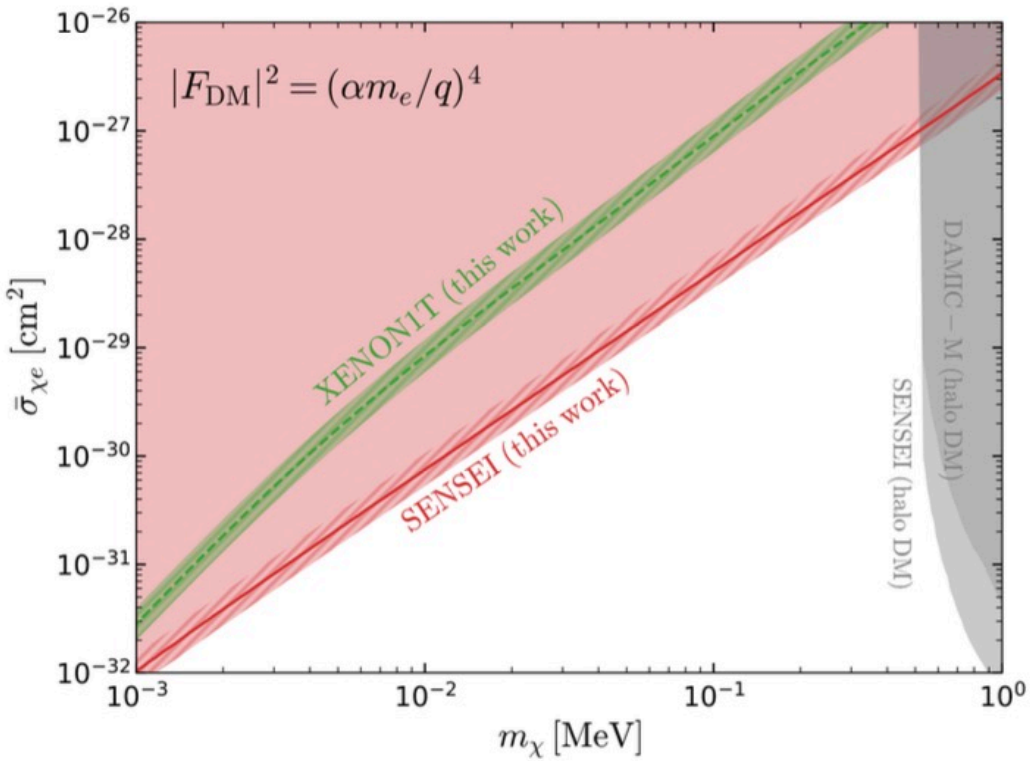


3. Detecting LDM with Plasmons

Liang, Su, WU, Zhu, 2401.11971

2312.13342 **SENSEI @ SNOLAB**

	All data		
Shape	Expo.	Ev.	Bkgd.
2e2p, h	13.58	10	10.66
2e2p, v	16.21	13	12.65
2e2p, d	17.82	32	25.31
2e, all	**46.61**	**55**	**48.62**
3e2p	30.87	3	0.01
3e3p	26.84	1	0.06
3e, all	**57.71**	**4**	**0.07**
4e2p	19.51	0	0.00
4e3p	27.60	0	0.00
4e4p	15.93	0	0.00
4e, all	**63.03**	**0**	**0.00**
5e, all	65.56	0	0.00
6e, all	67.31	0	0.00
7e, all	68.53	0	0.00
8e, all	69.52	0	0.00
9e, all	70.30	0	0.00
10e, all	70.89	0	0.00



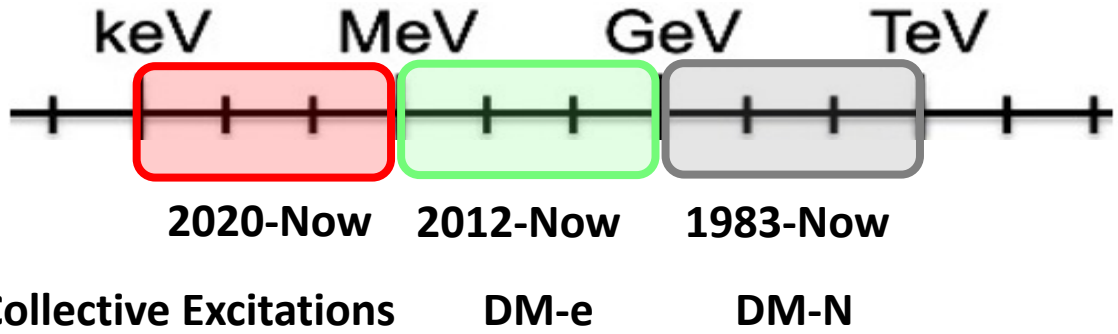
Light Dark Matter	Noble Liquid Detector (large volume, high threshold)	Solid detector (small volume, low threshold)
Non-Relativistic	✓	✓
Relativistic	✓	Collective Excitations

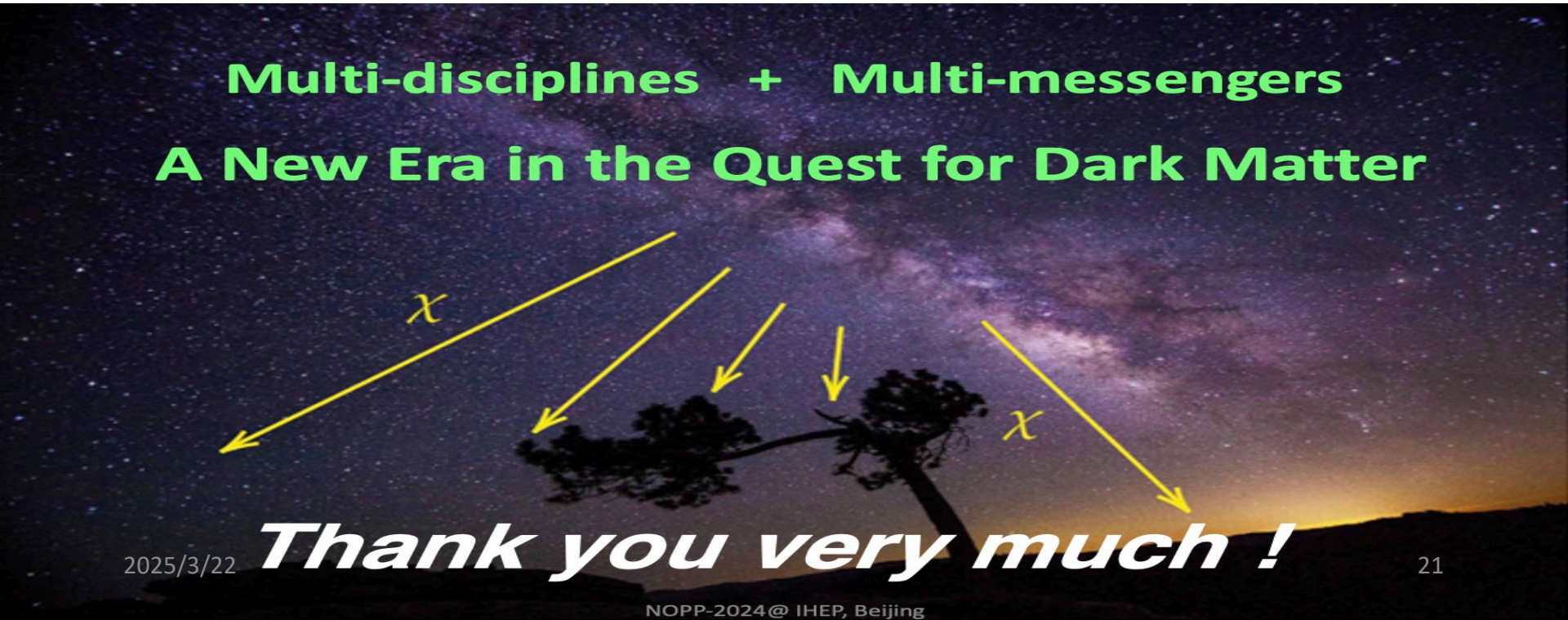
Conclusions

Dark Matter Particle

Direct Detection

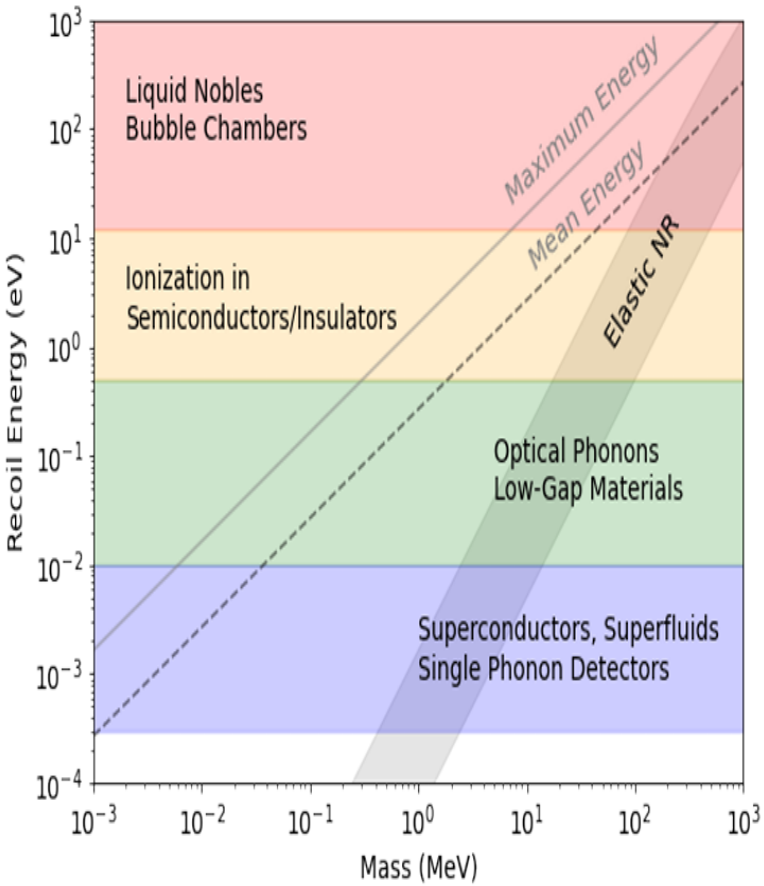
(~ 40 years, 9 orders)



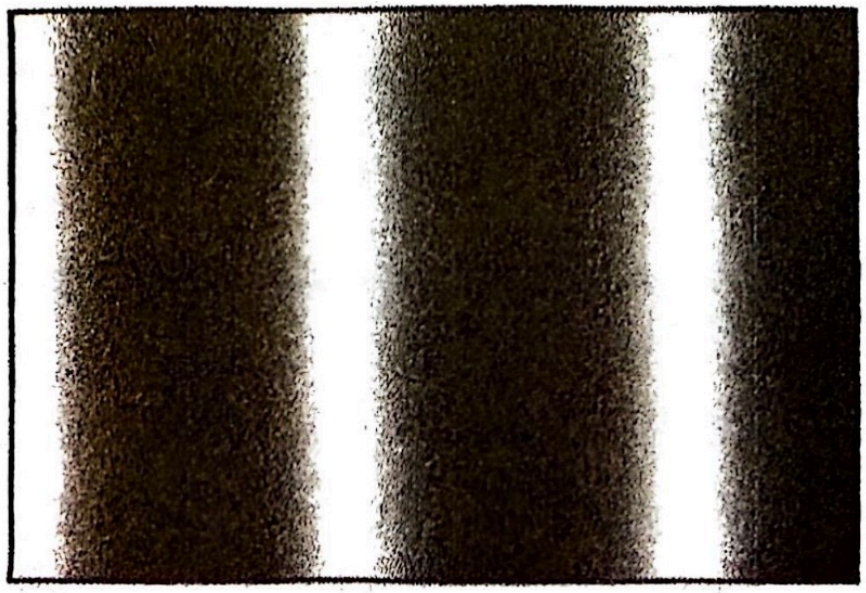


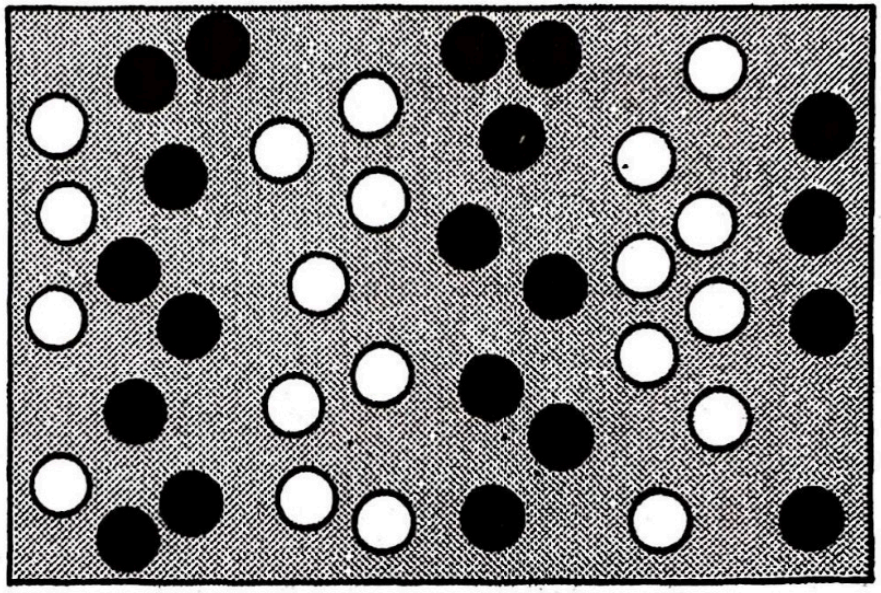
Back Up

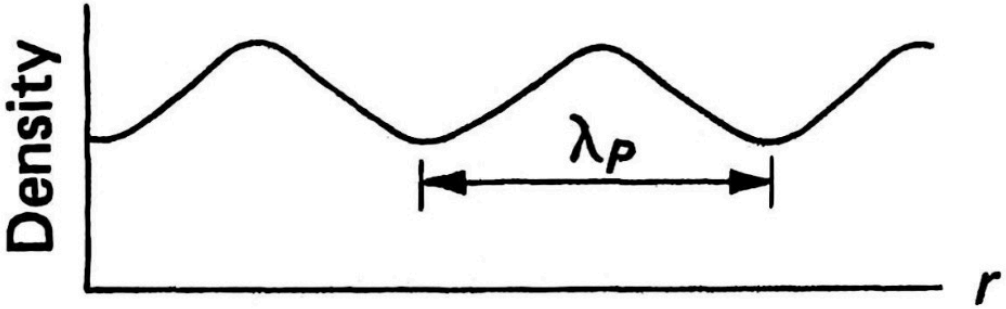
Threshold	Lowest DM Mass Probed	Relevant Techniques, Technologies, and Materials
20 eV	20 MeV (ER) 100 MeV (NR)	Noble Elements (TPCs & SPCs) Solid-State Charge Detectors Phonon Detectors Threshold Detectors
500 meV	1 MeV (ER/NR) 500 meV (Abs.)	Semiconductor Detectors Athermal Phonon Detectors Scintillators NIR Photon Detectors
5 meV	10 keV (CE) 5 meV (Abs.)	Superconductors Low-Gap Materials Athermal Phonon Detectors Polar Materials Superfluids FIR Photon Detectors Magnetic Bubble Chambers Other new ideas



Rouven Essig, *et.al*, arxiv: 2203.08297



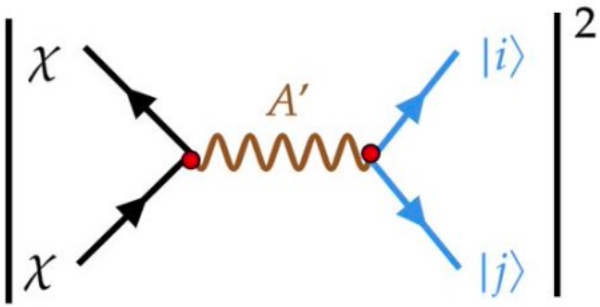




Electron- Particle hole Picture of Plasma Wave

Plasma Wave in Electron Gas

Back Up



$$\sum_{i,j} p_j \left\langle j|e^{-i\mathbf{Q}\cdot\hat{\mathbf{x}}}|i\right\rangle \underline{\left\langle i|e^{i\mathbf{Q}\cdot\hat{\mathbf{x}}}|j\right\rangle} \qquad \int \mathrm{d}^3x e^{i\mathbf{Q}\cdot\mathbf{x}} u_i^*(\mathbf{x})\, u_j(\mathbf{x})$$

$$\hat{\psi}(\mathbf{x}) = \sum_k \hat{a}_k u_k(\mathbf{x})$$

$$\left\langle i|e^{i\mathbf{Q}\cdot\hat{\mathbf{x}}}|j\right\rangle = \int \mathrm{d}^3x e^{i\mathbf{Q}\cdot\mathbf{x}} \left\langle i|\hat{\psi}^\dagger(\mathbf{x})\,\hat{\psi}(\mathbf{x})\,|j\right\rangle = \int \mathrm{d}^3x e^{i\mathbf{Q}\cdot\mathbf{x}} \left\langle i|\hat{\rho}(\mathbf{x})\,|j\right\rangle$$

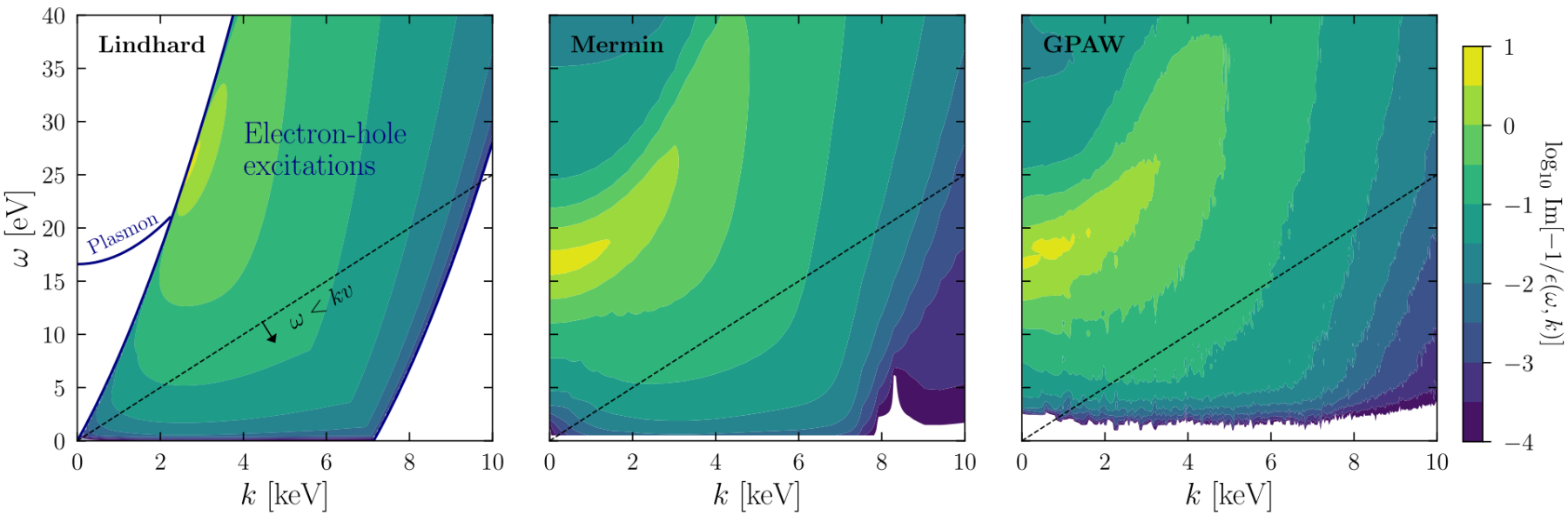
$$\int \mathrm{d}^3x' \mathrm{d}^3x e^{i\mathbf{Q}\cdot(\mathbf{x}-\mathbf{x}')} \sum_j p_j \langle j|\hat{\rho}(\mathbf{x}')\,\hat{\rho}(\mathbf{x})\,|j\rangle \sim \left\langle \hat{\rho}\hat{\rho}\right\rangle$$

Back Up

- The calculation of dielectric function:

  DarkELF: Including the real and imaginary parts of the dielectric function calculated using the DFT software GPAW.

Simon Knapen, Jonathan Kozaczuk, and Tongyan Lin, Phys. Rev. D **105**, 015014(2021).



- The Lindhard method:

  Assuming homogenous material and neglecting all dissipation effects.

- The Mermin method:

  A generalization of the Lindhard which includes dissipation.

- The GPAW method:

  It relies on a first principles calculation with package GPAW.

Back Up

e.g. Cosmic Ray DM

$$\frac{\mathrm{d}\Phi_\chi}{\mathrm{d}T_\chi} \equiv D_{\mathrm{eff}} \frac{\rho_\chi^{\mathrm{local}}}{m_\chi} \int_{T_{\mathrm{CR}}^{\mathrm{min}}}^{\infty} \mathrm{d}T_{\mathrm{CR}} \frac{\mathrm{d}\sigma}{\mathrm{d}T_\chi} \frac{\mathrm{d}\Phi_{\mathrm{CR}}^{\mathrm{LIS}}}{\mathrm{d}T_{\mathrm{CR}}}$$

differential electronic excitation rate:

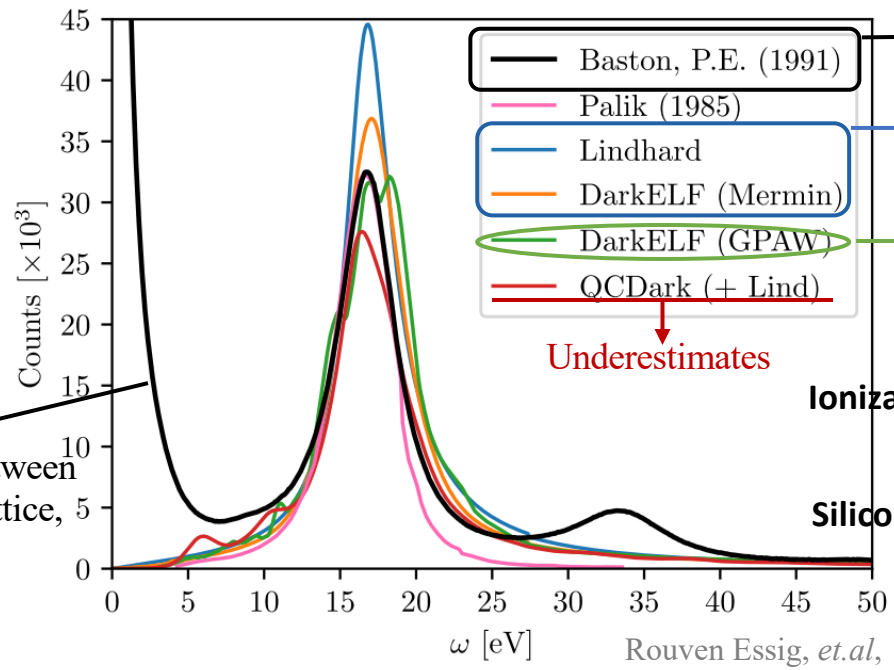
$$\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}\omega} &= \frac{1}{M_{\mathrm{target}}} \int \mathrm{d}E_\chi \int \frac{\mathrm{d}\Omega}{4\pi} \frac{\mathrm{d}\Phi_\chi}{\mathrm{d}E_\chi} \frac{\mathrm{d}\sigma}{\mathrm{d}\omega} \\
&= \frac{1}{\rho_T} \frac{\bar{\sigma}_{\chi e}}{4\alpha\mu_{\chi e}^2} \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \int \mathrm{d}E_\chi \frac{\mathrm{d}\Phi(E_\chi)}{\mathrm{d}E_\chi} \left(\frac{(2E_\chi - \omega)^2 - Q^2}{4E_\chi(E_\chi - \omega)} \right) \left(\frac{\alpha^2 m_e^2 + m_{A'}^2}{Q^2 - \omega^2 + m_{A'}^2} \right)^2 \left(\frac{E_\chi(E_\chi - \omega)}{p_\chi^2} \right) \\
&\times Q\, \mathrm{Im}\left[\frac{-1}{\epsilon(\mathbf{Q}, \omega)} \right] \Theta\left[E_\chi - \sqrt{(p_\chi - Q)^2 + m_\chi^2} - \omega \right],
\end{aligned}$$

Back Up

DM-Electron scattering in semiconductor

- Comparison with EELS data:

The results an incident electron beam kinetic energy of $T = 100$ keV



The data of EELS

Overestimating the response

The position and height of peak is consistent, but GPAW predicts a slightly broader peak.

Underestimates

Ionization charge: $\mathcal{Q}(\omega) = 1 + \lfloor (\omega - E_{\rm gap})/\varepsilon \rfloor$

1509.01598

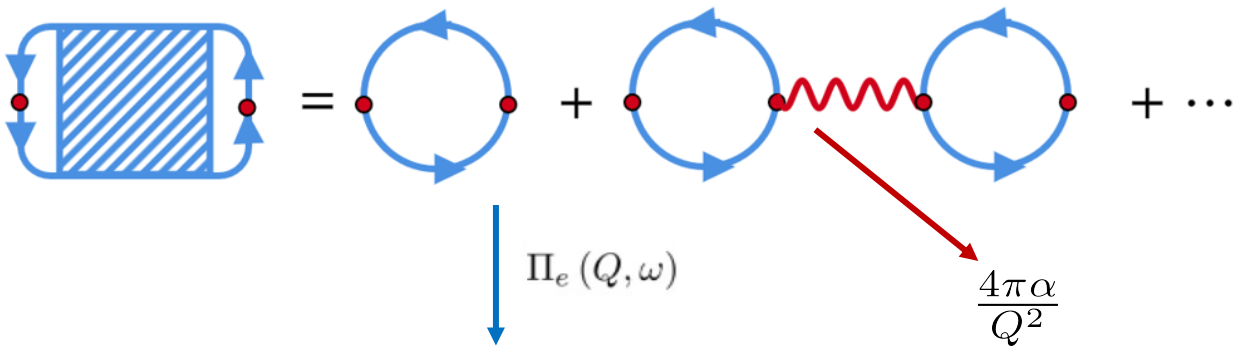
Silicon: E_{gap} = 1.2eV and ε = 3.8eV.

The elastic scattering between incident electrons and lattice, not electron-hole pair excitations.

Rouven Essig, *et.al*, arxiv: 2403.00123

$$\frac{1}{2}m_\chi v_\chi^2 \gtrsim E_{\text{threshold}} \quad \Rightarrow \quad m_\chi \gtrsim 6\ \text{MeV} \cdot \left(\frac{E_{\text{threshold}}}{15\ \text{eV}}\right)$$

where



$$\frac{1}{V}\sum_{i,j}\frac{\left|\langle i|e^{i\mathbf{Q}\cdot\hat{\mathbf{x}}}|j\rangle\right|^2}{\varepsilon_i-\varepsilon_j-\omega-i0^+}\left(f_i-f_j\right)$$

Lindhard function

$$\Gamma(\mathbf{p}_\chi) = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} |V(\mathbf{Q},\omega)|^2 \left[2\frac{Q^2}{e^2}\mathrm{Im}\left(-\frac{1}{\epsilon(\mathbf{Q},\omega)}\right)\right]$$

- Similar Fermi's Golden Rule, but different kinematics

$$Q = |\mathbf{Q}| = \left|\mathbf{p}_\chi - \mathbf{p}'_\chi\right|, \quad \omega = E_\chi - E'_\chi = \sqrt{p_\chi^2 + m_\chi^2} - \sqrt{|\mathbf{p}_\chi - \mathbf{Q}|^2 + m_\chi^2}$$

- Scattering potential

$$|V(\mathbf{Q},\omega)|^2 = \frac{\pi\bar{\sigma}_{\chi e}\left[(2E_\chi - \omega)^2 - Q^2\right]}{4\mu_{\chi e}^2 E_\chi (E_\chi - \omega)} |F_{\mathrm{DM}}(q)|^2,$$

- Dielectric function remains the same
- Event rate

$$R = \frac{1}{\rho_T}\int \mathrm{d}T_\chi \int \frac{\mathrm{d}\Omega}{4\pi}\frac{\mathrm{d}\Phi_\chi}{\mathrm{d}T_\chi}\left(\frac{E_\chi}{p_\chi}\right)\Gamma(p_\chi)$$

$$
\begin{aligned}
\sigma &= \sum_{i,f} \mathcal{P}_i \sigma_{i\to f} \\
&= \int \frac{\mathrm{d}^3 p'_\chi}{(2\pi)^3} \frac{1}{2} \sum_{s,s'} \bar{u}^{s'}_{p'_\chi} \gamma^0 u^s_{p_\chi} \bar{u}^s_{p_\chi} \gamma^0 u^{s'}_{p'_\chi} \frac{2\pi\delta\left(\omega_{pp'} - \omega\right)}{4E_\chi E'_\chi v_\chi} \sum_{i,f} \mathcal{P}_i \left\langle i \middle| e^{-i\mathbf{Q}\cdot\hat{\mathbf{x}}} \middle| f \right\rangle \left\langle f \middle| e^{i\mathbf{Q}\cdot\hat{\mathbf{x}}} \middle| i \right\rangle \\
&= \int \mathrm{d}\omega\, \delta\left(\omega + \varepsilon_i - \varepsilon_f\right) \int \mathrm{d}^3\mathbf{Q}\, \delta^{(3)}\left(\mathbf{Q} + \mathbf{p}'_\chi - \mathbf{p}_\chi\right) \int \frac{\mathrm{d}^3 p'_\chi}{(2\pi)^3} \frac{\pi\bar{\sigma}_{\chi e}[(E_\chi + E'_\chi)^2 - Q^2]}{4\mu^2_{\chi e} E_\chi E'_\chi v_\chi} \left(\frac{\alpha^2 m_e^2 + m^2_{A'}}{Q^2 - \omega^2 + m^2_{A'}}\right)^2 \\
&\times 2\pi\delta\left(\omega_{pp'} - \omega\right) \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}' \sum_{i,f} \mathcal{P}_i \left\langle i \middle| e^{-i\mathbf{Q}\cdot\hat{\mathbf{x}}'} \hat{\rho}_e \middle| f \right\rangle \left\langle f \middle| e^{i\mathbf{Q}\cdot\hat{\mathbf{x}}} \hat{\rho}_e \middle| i \right\rangle \\
&= \int \mathrm{d}\omega \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \frac{\pi\bar{\sigma}_{\chi e}[(E_\chi + E'_\chi)^2 - Q^2]}{4\mu^2_{\chi e} E_\chi E'_\chi v_\chi} \left(\frac{\alpha^2 m_e^2 + m^2_{A'}}{Q^2 - \omega^2 + m^2_{A'}}\right)^2 2\pi\delta\left(\omega_{pp'} - \omega\right) \\
&\times \sum_{i,f} \mathcal{P}_i \left\langle i \middle| \hat{\rho}_{\mathbf{Q}} \middle| f \right\rangle \left\langle f \middle| \hat{\rho}_{-\mathbf{Q}} \middle| i \right\rangle \delta\left(\omega + \varepsilon_i - \varepsilon_f\right) \\
&= V_{\text{target}} \int \mathrm{d}\omega \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \frac{\pi\bar{\sigma}_{\chi e}[(E_\chi + E'_\chi)^2 - Q^2]}{4\mu^2_{\chi e} E_\chi E'_\chi v_\chi} \left(\frac{\alpha^2 m_e^2 + m^2_{A'}}{Q^2 - \omega^2 + m^2_{A'}}\right)^2 \delta\left(\omega_{pp'} - \omega\right) S_{\hat{\rho}\hat{\rho}}(\mathbf{Q}, \omega) \\
&= 2V_{\text{target}} \int \mathrm{d}\omega \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \frac{\pi\bar{\sigma}_{\chi e}[(E_\chi + E'_\chi)^2 - Q^2]}{4\mu^2_{\chi e} E_\chi E'_\chi v_\chi} \left(\frac{\alpha^2 m_e^2 + m^2_{A'}}{Q^2 - \omega^2 + m^2_{A'}}\right)^2 \frac{\delta\left(\omega_{pp'} - \omega\right)}{V_{\text{Cou}}\left(\mathbf{Q}\right)} \mathrm{Im}\left[\frac{-1}{\epsilon\left(\mathbf{Q}, \omega\right)}\right],
\end{aligned}
$$

where $V_{\text{target}} = N_{\text{cell}} V_{\text{cell}}$ is the volume of the target material. In the third line, we insert two identities $\int \mathrm{d}\omega\, \delta\left(\omega + \varepsilon_i - \varepsilon_f\right)$ and $\int \mathrm{d}^3\mathbf{Q}\, \delta^{(3)}\left(\mathbf{Q} + \mathbf{p}'_\chi - \mathbf{p}_\chi\right)$ for the change of variables. In order to discuss the excitation process in the context of the response theory, we introduce the electron density operator $\hat{\rho}_e\left(\mathbf{x}\right) \equiv \hat{\psi}^\dagger_e\left(\mathbf{x}\right) \hat{\psi}_e\left(\mathbf{x}\right)$, where the nonrelativistic field operator $\hat{\psi}_e\left(\mathbf{x}\right) = \sum_j \hat{a}_j u_j\left(\mathbf{x}\right)$ is expressed in terms of the eigen wavefunctions $\{u_j\left(\mathbf{x}\right)\}$ and their corresponding creation (annihilation) operators $\left\{\hat{a}^\dagger_j\right\}$ $(\{\hat{a}_j\})$. In the momentum space, the density operator is rewritten as $\rho_{\mathbf{Q}} \equiv \int \mathrm{d}^3\mathbf{x} \hat{\rho}_e(\mathbf{x}) e^{-i\mathbf{Q}\cdot\hat{\mathbf{x}}}$, which describes the density fluctuation of the electron system [84]. The the dynamic structure factor is defined as

$$
\begin{aligned}
S_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\omega) &= \frac{2\pi}{V_{\text{target}}} \sum_{i,f} \mathcal{P}_i \left|\langle f|\hat{\rho}_{-\mathbf{Q}}|i\rangle\right|^2 \delta(\omega+\varepsilon_i-\varepsilon_f) \\
&= i\frac{\left[\chi_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\omega+i0^+)-\chi_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\omega-i0^+)\right]}{1-e^{-\beta\omega}} \\
&\simeq i\left[\chi_{\hat{\rho}\hat{\rho}}\left(\mathbf{Q},\,\omega+i0^+\right)-\chi_{\hat{\rho}\hat{\rho}}\left(\mathbf{Q},\,\omega-i0^+\right)\right]
\end{aligned}
$$

$$
= -2\,\mathrm{Im}\left[\chi^{\mathrm{r}}_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\,\omega)\right] \tag{S.4}
$$

$$
= \frac{2}{V_{\mathrm{Cou}}(\mathbf{Q})}\,\mathrm{Im}\left[\frac{-1}{\epsilon(\mathbf{Q},\omega)}\right], \tag{S.5}
$$

where $1-e^{-\beta\omega}$ depends on the inverse temperature $\beta = 1/k_B T$, which can be approximated as 1 for the semiconductor target. The master function $\chi_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\,z)$, representing the correlation function of density operator $\hat{\rho}\hat{\rho}$, can be estimated by the Matsubara Green's function, and $\chi^{\mathrm{r}}_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\,\omega) = \chi_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\,\omega+i0^+)$ and $\chi_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\,\omega-i0^+)$ are the corresponding retarded and advanced correlation function. For the last line, we use the relation between the dielectric function and retarded correlation function in the linear response theory, that is

$$
\frac{1}{\epsilon(\mathbf{Q},\omega)} = 1 + V_{\mathrm{Cou}}(\mathbf{Q})\,\chi^{\mathrm{r}}_{\hat{\rho}\hat{\rho}}(\mathbf{Q},\,\omega)\,. \tag{S.6}
$$

In the nonrelativistic limit, the cross section in Eq. (S.1) reduces to the expression in Ref. [53], which describes the scattering between a halo DM particle and the electrons in solids.