Chiral Representation of the Nucleon Mass at Leading Two-loop Order

De-Liang YAO

Hunan University

With Z.-R. Liang(梁泽锐), H.-X. Chen(陈寒雪), F.-K. Guo(郭奉坤), Z.-H. Guo(郭志辉)

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Outline

I. Introduction

II. Leading two-loop result of the nucleon mass

III. Chiral extrapolation of lattice QCD data

IV. Summary and outlook

1. Introduction

Origin of the nucleon mass

- Decomposition of the nucleon mass (classical view)

$$m_N = \langle N(p)|T^{\mu}_{\ \mu}|N(p)\rangle$$

$$\sim \langle N(p)|\underbrace{\frac{\beta}{2g}G^a_{\mu\nu}G^{\mu\nu}_a}_{\text{field energy}} + \underbrace{m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s}_{\text{Higgs}}|N(p)\rangle$$

➡ How large is the trace anomaly term?

➡ Sigma terms: $\sigma_{\pi N} \sim \hat{m}(\bar{u}u + \bar{d}d), \sigma_s = m_s\bar{s}s$

- Feynman-Hellman Theorem

$$\sigma_{\pi N} = \frac{\hat{m}}{\partial \hat{m}} m_N = M_\pi^2 \frac{\partial m_N}{\partial M_\pi^2}, \quad \hat{m} = (m_u + m_d)/2 .$$

➡ three-point to two-point problem

➡ High-precision determination of the quark mass dependence of the nucleon mass is crucial.

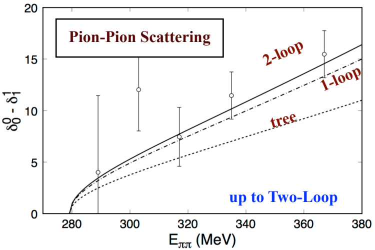
Chiral perturbation theory beyond one-loop is necessary!

Status of ChPT beyond one-loop

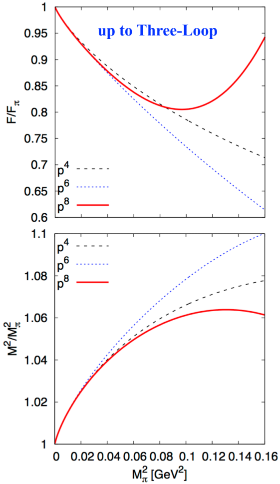
- Pure meson sector: a great triumph has been achieved.

  ➡ Two-loop calculations become standard and good convergence can be seen.

  - $\pi\,\pi$ scattering
  - $\pi\,K$ scattering
  - $K_{\ell 4}$
  - Electromagnetic form factor
  - Pion decay constant,
  - Pion mass
  - ...



[Bijnens, Colangelo, Ecker, Gasser, Sainio, PLB374 (1996) 210-216]



[J. Bijnens and N. H. Truedsson, JHEP 1711 (2017) 181]

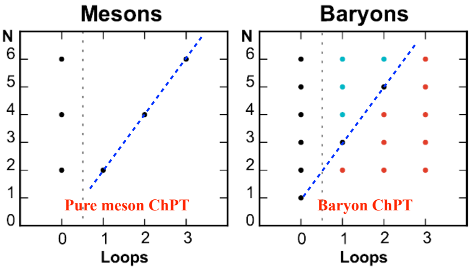
Status of ChPT beyond one-loop

- Baryon sector: multi-loop calculation is much more complex than pure meson sector

  ✓ Dirac algebra & axial coupling between baryons and Goldstone bosons
  $\longrightarrow$ more diagrams and master integrals

  ✓ Unequal mass
  $\longrightarrow$ Hard to compute multi-loop integrals (solved by AMflow)

  ✓ The introduction of non-zero baryon mass in the chiral limit
  $\longrightarrow$ the notable **Power Counting Breaking (PCB)** problem!



Naive power counting rule !!!

red dots denote PCB terms!!!

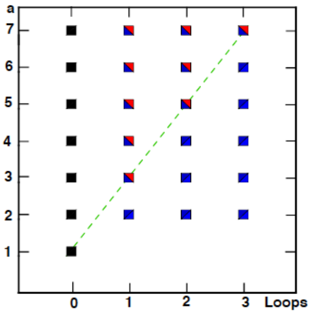
Solutions to the PCB problem

- Various approaches

  ✓ Heavy Baryon (HB) formalism [Jenkins and Manohar,PLB255'91]

  ✓ Infrared Regularisation (IR) prescription [T.Becher and H.Leutwyler,Eur.Phys.J.C9'99]

  ✓ **Extended-On-Mass-Shell (EOMS) scheme**
  [T.Fuchs,J.Gegelia,G.Japaridze and S.Scherer,PRD68'03]



philosophy

HB & IR: throwing away

EOMS: reorganisation

Status of the two-loop nucleon mass

- Previous works on the nucleon mass in ChPT

  ➡ 1999: HB formalism up to $O(p^5)$ [McGovern & Birse, PLB446(1999)]

  ➡ 2007/2008: IR prescription up to $O(p^6)$ [expanded]

  [Schindeler, Djukanovic, Gegelia & Scherer, PLB649(2007), NPA803(2008)]

  ➡ 2024: EOMS up to $O(p^6)$ [expanded]

  [Conrad, Gasparyan & Epelbaum, PoS CD2021, 074 (2024), EPJ Web Conf. 303, 02001(2024)]

  [Chen, Hu, Mo & Jia, arXiv:2406.040124 [hep-ph]]

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5$$
$$+ k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6 .$$

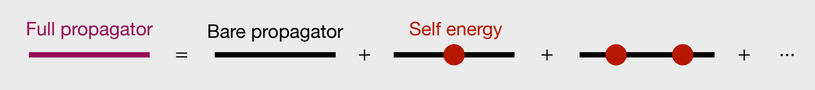
- Our work: **full EOMS** chiral result up to $O(p^5)$

  ➡ All the necessary analytical two-loop structure
  ➡ Avoid too many unknown LECs

2. Leading two-loop ChPT result

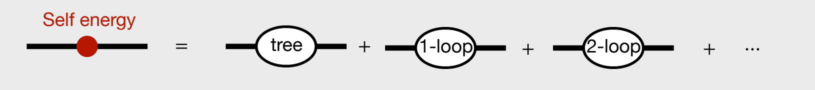
Nucleon self-energy

- The dressed propagator of the nucleon



$$i\,S_N(p) = \frac{i}{p\cdot\gamma - m - \Sigma_N(p\cdot\gamma)} = \frac{i\,Z_N}{p\cdot\gamma - m_N} + \text{non-pole piece}$$

- Physical mass as the pole: $m_N - m - \Sigma(m_N, m) = 0$



$$m_N = m_4 + \hbar\,\Delta m_N^{(1)} + \hbar^2\,\Delta m_N^{(2)}$$

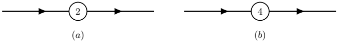
Mass insertion contribution by setting:

$$m \to m_4 = m + m_{\text{tree}}$$

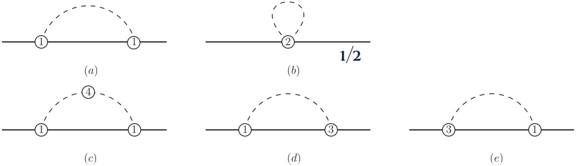
$$\Delta m_N^{(1)} = \Sigma_N^{(1)}(m_4, m_4)$$

$$\Delta m_N^{(2)} = \Sigma_N^{(1)}(m_4, m_4)\,\Sigma_N^{(1)\prime}(m_4, m_4) + \Sigma_N^{(2)}(m_4, m_4)$$

- Tree and one-loop diagrams up to $O(p^5)$



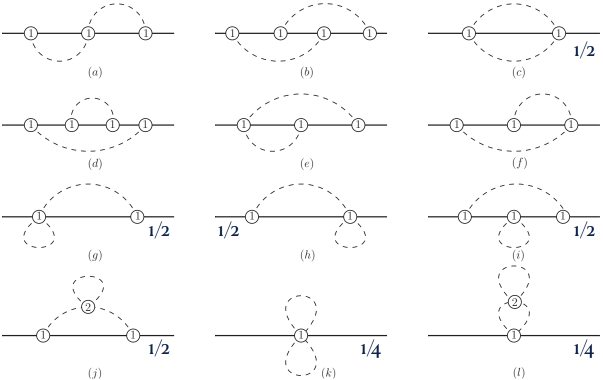
$$m_{\text{tree}} = -4c_1M^2 - 2e_mM^4$$



$$\Delta m_N^{(1)} = m_N^{(1a)} + m_N^{(1b)} + m_N^{(1c)} + 2m_N^{(1d)}$$

Two-loop diagrams

- Two-loop diagrams at $O(p^5)$



$$\Delta m_N^{(2)} = m_N^{(2a)} + m_N^{(2b)} + m_N^{(2c)} + m_N^{(2d)} + 2m_N^{(2e)} + 2m_N^{(2g)} + m_N^{(2j)}$$

Two-loop Feynman integrals

- Two-loop integrals

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \iint \frac{\mathrm{d}^d \ell_1}{(i\pi^{d/2})} \frac{\mathrm{d}^d \ell_2}{(i\pi^{d/2})} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5}}$$

➡ ν_i integers; scalar integrals: all $\nu_i \geq 0$; tensor integrals: $\exists \nu_i \geq 0$

➡ Number of irreducible scalar product: $N = L \times E + L(L+1)/2$

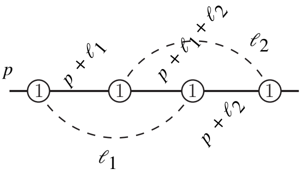
$$\mathcal{D}_1 = \ell_1^2 - M^2$$

$$\mathcal{D}_2 = \ell_2^2 - M^2$$

$$\mathcal{D}_3 = (p + \ell_1 + \ell_2)^2 - m^2$$

$$\mathcal{D}_4 = (p + \ell_1)^2 - m^2$$

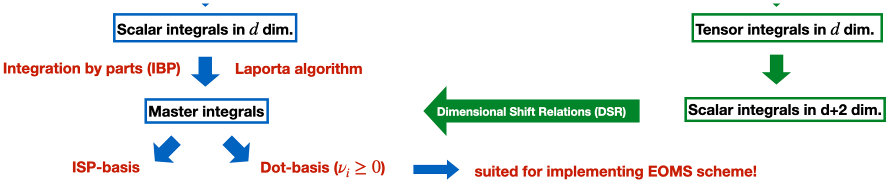
$$\mathcal{D}_5 = (p + \ell_2)^2 - m^2$$



IBP reduction and master integrals

- IBP reduction: e.g. diagram (2c)

$$m_N^{(2c)} = \frac{-3\kappa^2}{32 m F^4}\left[2M^2\left(2M^2 I_{11100} + 2I_{01100} + 2I_{10100} - I_{11000}\right) + I_{110(-1)0} + I_{1100(-1)} - 4I_{111(-1)(-1)}\right]$$



- Master integrals: 3 one-loop + 10 two-loop (dot-basis)

$$J^{\text{1-loop}}_{\text{master}} = \{J_{10},\, J_{01},\, J_{11}\}\ .$$

$$I^{\text{2-loop}}_{\text{master}} = \{I_{11111},\, I_{11110} \leftrightarrow I_{11101},\, I_{10111} \leftrightarrow I_{01111},\, I_{21100} \leftrightarrow I_{12100},\, I_{11100},\, I_{00111},\\ I_{10101} \leftrightarrow I_{01110},\, I_{10110} \leftrightarrow I_{01101},\, I_{10100} \leftrightarrow I_{01100},\, I_{00110} \leftrightarrow I_{00101}\}\ .$$

➡ Differential equation method [Lotikov, PLB254 (1991), PLB267(1991)] [Henn, PRL110(2013)]

- **Pedagogic example:** $g_A = 0$

$$m_N = m + 4c_1 M^2 - e_m M^4 + m_N^{\text{1-loop}} + m_N^{\text{2-loop}}$$

$$\mathrm{UV}[m_N^{\text{2-loop}}] = \frac{3\left(6M^4 m + 8M^2 m^3 - 3m^5\right)}{32F^4\Lambda^2\epsilon^2} + \frac{132M^4 m - 16M^2 m^3 - 9m^5}{128F^4\Lambda^2\epsilon} + \frac{3m^3\left(3m^2 - 8M^2\right)}{16F^4\Lambda^2\epsilon}\log\frac{m^2}{\mu^2} - \frac{9M^4 m}{8F^4\Lambda^2\epsilon}\log\frac{M^2}{\mu^2}$$

Non-local UV divergence

$$\mathrm{UV}[m_N^{\text{1-loop}}] = -\frac{3(d(c_3 - 2c_1) + c_2)M^4}{F^2 d}\left[-\frac{1}{\Lambda\epsilon} + \frac{1}{\Lambda}\log\frac{M^2}{\mu^2} + \mathcal{O}(\epsilon)\right]$$

➡ Local UV div. →tree-loop counter term

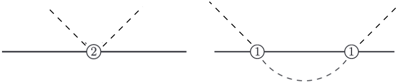
➡ Non-local UV div. →one-loop diagram as the sub-diagram of two-loop diagram

- **UV renormalisation at two-loop level**

➡ **Step 1**: split the bare LECs in one-loop diagram

$$\mathrm{UV}[m_N^{\text{1-loop}}] = -\frac{3(d(c_3-2c_1)+c_2)M^4}{F^2 d}\left[-\frac{1}{\Lambda\epsilon}+\frac{1}{\Lambda}\log\frac{M^2}{\mu^2}+\mathcal{O}(\epsilon)\right]$$

$$X = -\frac{\beta_X^{(1)}}{\epsilon\Lambda} + X^r + \mathcal{O}(\epsilon)\,, \quad X \in \{c_1, c_2, c_3\} \qquad \beta_{c_1}^{(1)} = 0\,, \quad \beta_{c_2}^{(1)} = \frac{m}{2F^2}\,, \quad \beta_{c_3}^{(1)} = \frac{m}{4F^2}$$



Pion-nucleon scattering up to $\mathcal{O}(\epsilon)$

➡ **Step 2**: split the bare LECs in the tree diagrams

$$m_N = \underbrace{m + 4c_1M^2 - e_m M^4}_{\text{tree}} + m_N^{\text{1-loop}} + m_N^{\text{2-loop}}$$

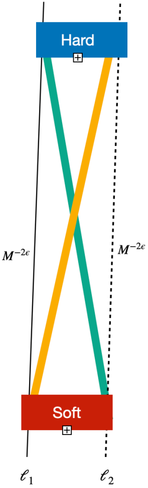
$$X = \frac{\beta_X^{(22)}}{\epsilon^2\Lambda^2} - \frac{\beta_X^{(21)}}{\epsilon\Lambda^2} - \frac{\beta_X^{(11)}}{\epsilon\Lambda} + X^r\,, \quad X \in \{m, c_1, e_m\}$$

Two-loop UV renormalised parameters!

Chiral expansion of loop integrals

- **Dimensional counting analysis**

i.e. Equivalent to strategy of regions [Gegelia, Japaridze and Turashvili, Theor. Math. Phys. 101(1994)] [Beneke and Smirnov, NPB522(1998)]



$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \int\int \frac{\mathrm{d}^d \ell_1}{(i\,\pi^{d/2})} \frac{\mathrm{d}^d \ell_2}{(i\,\pi^{d/2})} \frac{1}{\mathscr{D}_1^{\nu_1}\mathscr{D}_2^{\nu_2}\mathscr{D}_3^{\nu_3}\mathscr{D}_4^{\nu_4}\mathscr{D}_5^{\nu_5}}$$

$\ell_i \longrightarrow M^{\alpha_i}\,\tilde{\ell}_i$ α_i non-negative real numbers

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \sum_{\alpha_1,\alpha_2} M^{\varphi(\alpha_1,\alpha_2)} \times I^{(\alpha_1,\alpha_2)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}$$

$(\alpha_1, \alpha_2) \in \{(0{,}0), (0{,}1), (1{,}0), (1{,}1)\}$

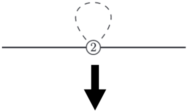
$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = I^{(0,0)}_{\dots} + M^{-2\epsilon} I^{(0,1)}_{\dots} + M^{-2\epsilon} I^{(1,0)}_{\dots} + M^{-4\epsilon} I^{(1,1)}_{\dots}$$

Pure infrared regular part	**infrared regular-singular part**	**Pure infrared singular part**
Local PCB term	**Non-local PCB term**	**PCR-conserving term (IR)**

PCB renormalisation

- **EOMS scheme**: absorption of PCB terms by LECs

One-loop



$$X^r = \widetilde{X} + \delta_X^{(1)}\,, \quad X \in \{c_1, c_2, c_3\}$$

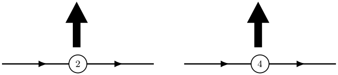
1-loop local PCB term

non-local PCB term

Two-loop

2-loop local PCB term

$$X^r = \widetilde{X} + \delta_X^{(21)} + \delta_X^{(22)}\,, \quad X \in \{m\,, c_1, e_m\}$$



Tree

✓ $\delta_X^{(21)}$: cancel 1-loop local PCB term

✓ $\delta_X^{(22)}$: cancel 2-loop local PCB term

Two-loop chiral expression

- **The nucleon mass up to $O(p^5)$**

$$m_N = \widetilde{m} - \underbrace{4\tilde{c}_1 M^2}_{\mathcal{O}(p^2)} + \underbrace{\bar{m}_N^{(1a)}}_{\mathcal{O}(p^3)} \underbrace{- 2\tilde{e}_m M^4 + \bar{m}_N^{(1b)}}_{\mathcal{O}(p^4)} + \underbrace{\bar{m}_N^{(1c)} + 2\bar{m}_N^{(1d)} + \bar{m}_N^{\text{2-loop}} + \bar{m}_N^{\text{sub-diag.}}}_{\mathcal{O}(p^5)}$$

$$m_N^{\text{sub-diag.}} \sim \sum_X \left\{ \left(- \frac{\beta_X^{(1)}}{\epsilon\Lambda} \right) \times \left[\text{linear term in } \epsilon \text{ from one loops} \right] \right\}$$

- **Merit:**
  - ➡ Faithful structure: respect original analytic property
  - ➡ Controllable accuracy: possess correct power counting rule
  - ➡ Renormalisation scale independent

Two-loop chiral expression

- EOMS truncated at $O(p^5)$

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 \, .$$

$$k_1^{\text{EOMS}} = -4c_1 \, ,$$

$$k_2^{\text{EOMS}} = -\frac{3g^2\pi}{2F^2\Lambda} \, ,$$

$$k_3^{\text{EOMS}} = -\frac{3g^2}{2F^2 m \Lambda} + \frac{3(8c_1 - c_2 - 4c_3)}{2F^2\Lambda} \, ,$$

$$k_4^{\text{EOMS}} = -2e_m + \frac{3g^2}{2F^2 m \Lambda} + \frac{3c_2}{8F^2\Lambda} \, ,$$

$$k_5^{\text{EOMS}} = \frac{3g^2(16g^2 - 3)\pi}{4F^4\Lambda^2} \, ,$$

$$k_6^{\text{EOMS}} = \frac{19\pi g^4}{4F^4\Lambda^2} + \frac{6\pi g\,(d_{18} - 2d_{16})}{F^2\Lambda} + g^2 \left[\frac{6\pi}{F^4\Lambda^2} + \frac{3\pi \left(F^2 + 8m^2(2\ell_4 - 3\ell_3)\right)}{16F^4\Lambda m^2} \right] .$$

Caution: the chiral expansion unavoidably enforces a truncation at a certain order, which changes the analytic property (as a function of M) of the original expression.

Two-loop chiral expression

- IR truncated at $O(p^5)$

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 \,.$$

$$k_1^{\rm IR} = -4c_1\,,$$

$$k_2^{\rm IR} = -\frac{3g^2\pi}{2F^2\Lambda}\,,$$

$$k_3^{\rm IR} = -\frac{3g^2}{2F^2 m\Lambda} + \frac{3(8c_1 - c_2 - 4c_3)}{2F^2\Lambda}\,,$$

$$k_4^{\rm IR} = -2e_m - \frac{3g^2}{4F^2 m\Lambda} + \frac{3c_2}{8F^2\Lambda}\,,$$

$$k_5^{\rm IR} = \frac{3g^2(16g^2-3)\pi}{4F^4\Lambda^2}\,,$$

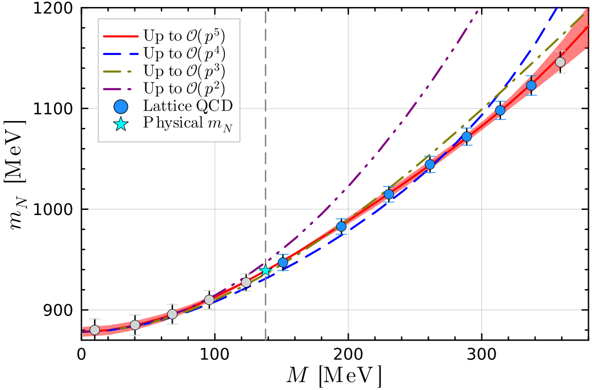
$$k_6^{\rm IR} = \frac{3\pi g^4}{F^4\Lambda^2} + \frac{6\pi g(d_{18} - 2d_{16})}{F^2\Lambda} + \frac{3\pi g^2\left(F^2 + 8m^2(2\ell_4 - 3\ell_3)\right)}{16F^4\Lambda m^2}\,.$$

➡ Non-analytic terms in quark mass: (k_2, k_3, k_5) same, k_6 different;

➡ Scheme independent terms: k_3 & k_5, guaranteed by the RGE method.

3. Chiral extrapolation

- Full EOMS



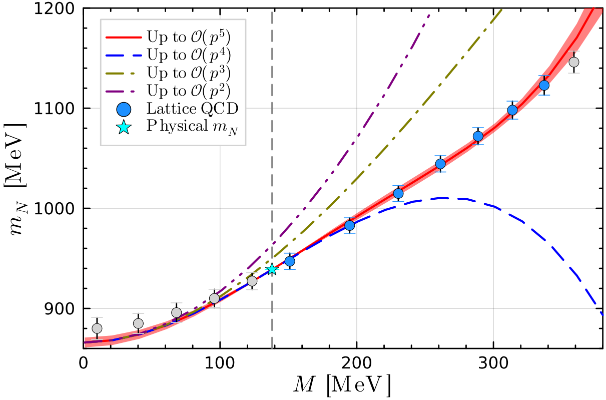
✓ Original analyticity guarantees the successfulness of chiral extrapolation

✓ Relativistic renormalisation scheme possesses good convergency

m	878.2(5.0) MeV
c_1	-0.90(0.07) GeV^{-1}
e_m	-7.41(1.71) GeV^{-2}

Chiral extrapolation

- EOMS truncated at order p^5

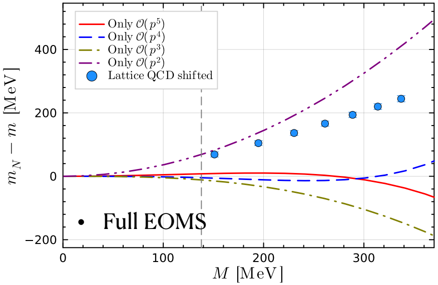


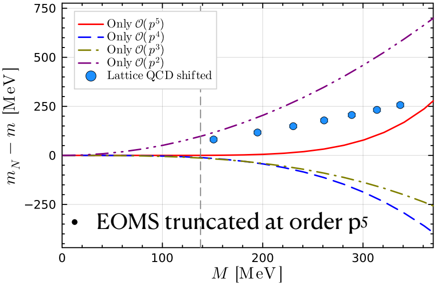
✓ Agree well in the fitting range, but deviate beyond;

✓ Weird behaviour of the pion mass dependence.

m	878.2(5.0) MeV
c_1	-0.90(0.07) GeV^{-1}
e_m	-7.41(1.71) GeV^{-2}

- Assess the convergency of the chiral expansion





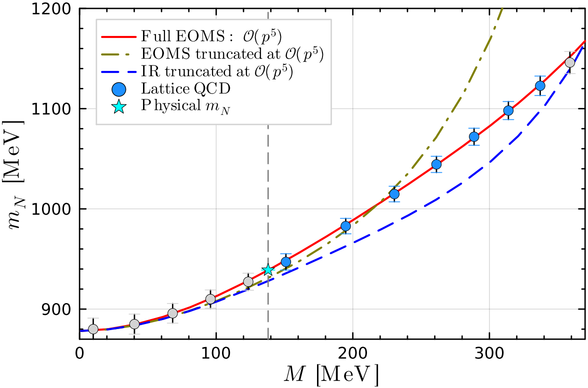
➡ Two-loop contribution is small, approximately 10 MeV, for pion mass < 300 MeV;

➡ At physical point:

$$m_N = \{878.2 + \underbrace{68.8}_{\mathcal{O}(p^2)} + \underbrace{[-11.4]}_{\mathcal{O}(p^3)} + \underbrace{[-4.6]}_{\mathcal{O}(p^4)} + \underbrace{7.9}_{\mathcal{O}(p^5)}\}\ \text{MeV} = 938.9\ \text{MeV}$$

Comparison

- Full EOMS, EOMS truncated, IR truncated



➡ IR-truncated: the discarded terms due to the O(p^5) truncation contribute sizeably.

➡ EOMS-truncated: starts to deviate at pion mass ~220MeV.

4. Summary and outlook

Summary and outlook

❖ Summary

1. The applicability of EOMS at two-loop order is verified explicitly.
2. The notable PCB issue can be addressed by using the dimensional counting analysis approach.
3. The O(p5) contribution from the full EOMS is small around 10 MeV, implying that the chiral series in the full EOMS scheme converges very well.
4. The EOMS-truncated and IR-truncated results change the analytic structure.

❖ Outlook

Step into a promising era of two-loop BChPT, where the nucleon property can be explored with high precision!

Thanks!