Generating entangled pairs of vortex photons via induced emission

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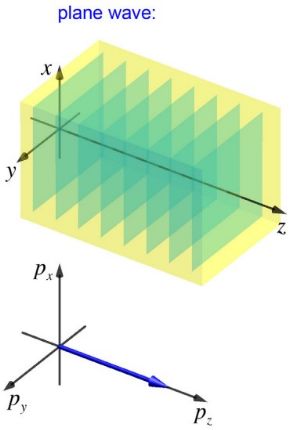
Plane waves



Eigenstates of the momentum operator

$$\hat{\boldsymbol{p}}|\Psi\rangle = \boldsymbol{p}_0|\Psi\rangle. \quad (1)$$

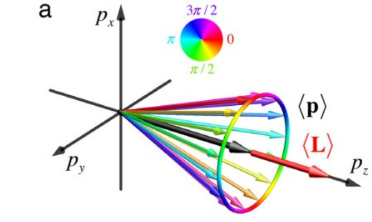
Plane wave photons with a definite helicity λ:

$$\hat{\Lambda}|\Psi\rangle = \lambda|\Psi\rangle, \ \hat{\Lambda} = \frac{\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{p}}}{|\boldsymbol{p}|} \quad (2)$$

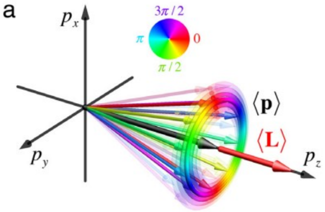
Wave function in momentum space

$$\langle \boldsymbol{p}, s|\Psi\rangle = \delta(\boldsymbol{p} - \boldsymbol{p}_0)\delta_{s,\lambda} \quad (3)$$

Wave packets



a) Bessel wave-packet



b) Laguerre-Gaussian wave-packet

Arbitrary wave packet states can be assembled as a superposition of plane waves

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} a(\boldsymbol{p})|\boldsymbol{p}\rangle \tag{4}$$

Vortex photons

Bessel photon wave function in momentum representation

$$\langle \boldsymbol{p}, s | \kappa, k_z, m, \lambda \rangle = \frac{2\pi}{\kappa} \delta(p_\perp - \kappa) \delta(p_z - k_z) e^{im\varphi_p} \delta_{s,\lambda} \tag{5}$$

Bessel photons posses definite total angular momentum m, definite absolute value of transverse momentum κ, definite longitudinal momentum k_z and definite helicity λ

Disadvantage — non-normalizable in the transverse plane :

$$\begin{aligned} &\langle \kappa, k_z, m, \lambda | \kappa, k_z, m, \lambda \rangle \to \infty, \\ &\langle \kappa', k_z', m', \lambda' | \kappa, k_z, m, \lambda \rangle \propto \delta(\kappa - \kappa') \delta(k_z - k_z') \delta_{m,m'} \delta_{\lambda,\lambda'} \end{aligned} \tag{6}$$

Bessel - Gaussian photons

A normalized vortex photons state is obtained by introducing a finite dispersion of the momentum

$$|\mathrm{BG}\rangle = \int e^{-(\kappa-\kappa_{\mathrm{c}})^2/2\sigma_\kappa^2} e^{-(k_z-k_{\mathrm{c}})^2/2\sigma_z^2} |\kappa, k_z, m, \lambda\rangle \frac{\kappa d\kappa}{(2\pi)^2} \frac{dk_z}{(2\pi)} \quad (7)$$

The probability density of a Bessel-Gaussian vortex photon is peaked at average transverse and longitudinal momentum $\kappa_{\mathrm{c}}, k_{\mathrm{c}}$. A finite width of the peaks results in a non vanishing momentum dispersion

$$\begin{aligned} \langle k_\perp^2\rangle - \langle k_\perp\rangle^2 &= \sigma_{\kappa^2}, \\ \langle k_z^2\rangle - \langle k_z\rangle^2 &= \sigma_{z^2} \end{aligned} \quad (8)$$

Interaction of an atom with EM field

Hamiltonian of the full system:

$$\hat{\mathcal{H}} = \hat{H}_{\mathsf{atom}} + \hat{\mathcal{H}}_{\mathsf{field}} + \hat{\mathcal{V}} \tag{9}$$

In a two-level model, the Hamiltonian of the atom is

$$\hat{\mathcal{H}}_{\mathsf{atom}} = \varepsilon_{\mathsf{e}}|\mathrm{e}\rangle\langle\mathrm{e}| + \varepsilon_{\mathsf{g}}|\mathrm{g}\rangle\langle\mathrm{g}|, \tag{10}$$

the Hamiltonian of the quantized electromagnetic field

$$\hat{\mathcal{H}}_{\mathsf{field}} = \sum_{\nu} \omega_{\nu} \hat{a}^{\dagger}_{\nu} \hat{a}_{\nu}, \tag{11}$$

where $\nu = \{\boldsymbol{k}_{\nu}, s_{\nu}\}$ is a multi-index with photon momentum and polarization

Interaction of an atom with EM field

The interaction is commonly described in the interaction picture

$$i\partial_t|\psi(t)\rangle = \hat{\mathcal{V}}_{\text{int}}|\psi(t)\rangle, \tag{12}$$

where the interaction in the rotating wave approximation (RWA) reads

$$\hat{\mathcal{V}}_{\text{int}} = \sum_{\nu,n}\left[g^*_{\nu,n}\hat{\sigma}_+\hat{a}_\nu e^{i\Delta_\nu t} + \text{h.c.}\right], \tag{13}$$

and here

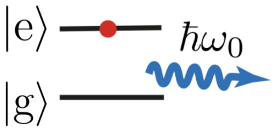
$$\begin{aligned}\hat{\sigma}_+ &= |\mathrm{e}\rangle\langle\mathrm{g}|, \hat{\sigma}_- = \hat{\sigma}_+^\dagger = |\mathrm{g}\rangle\langle\mathrm{e}|,\\ g_{\nu,n} &= -\frac{e}{m}\sqrt{\frac{2\pi}{V\omega_\nu}}\langle g|\boldsymbol{e}_{\boldsymbol{k},s}\cdot\hat{\boldsymbol{p}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|e\rangle\end{aligned} \tag{14}$$

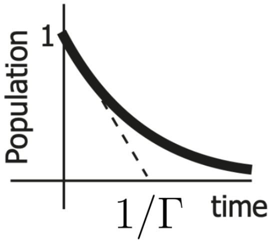
$\Delta_\nu = \omega - \omega_\nu$, ω - atomic resonance frequency.

No dipole approximation is assumed!

Spontaneous emission — Weisskopf - Wigner model

Weisskopf-Wigner model describes the spontaneous decay of an excited atom





The inital condition for the state of the system is

$$|\psi(0)\rangle = |\mathrm{e}\rangle|0\rangle \tag{15}$$

Spontaneous emission — Weisskopf - Wigner model

The Weisskopf-Wigner model makes use of the Markov approximation which can be stated as

$$\sum_n |g_{\nu,n}|^2 e^{i\Delta_\nu(t-\tau)} = \Gamma\delta(t-\tau) \tag{16}$$

This implies that the change of atomic level population at a given moment of time only depends on its value at this moment of time System without memory.

Markov approximation works really well for realistic systems: because typically $\Gamma \ll \omega$

Spontaneous emission — Weisskopf - Wigner model

The interaction changes the number of quanta of light only by ± 1, therefore the natural ansatz for the state of the system is

$$|\psi(t)\rangle = C_e(t)|\mathrm{e}\rangle|0\rangle + \sum_n C_{g,\nu}(t)|\mathrm{g}\rangle|\nu\rangle \tag{17}$$

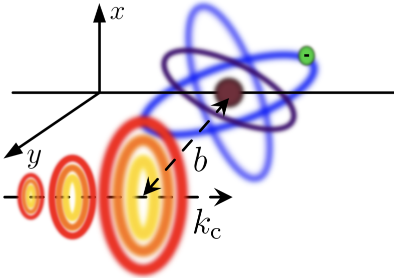
In the original work the state of the system was obtained in a dipole approximation, where the interaction constant is

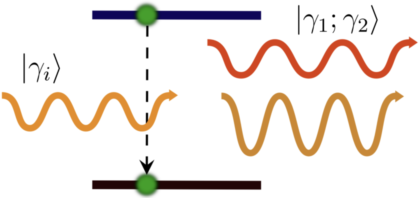
$$g_\nu = -\sqrt{\frac{2\pi\omega_\nu}{V}}\langle g|\hat{\boldsymbol{d}}|e\rangle \cdot \boldsymbol{e}_\nu, \tag{18}$$

and the excited and ground state probability amplitudes read

$$C_e(t) = e^{-\frac{\Gamma}{2}t},\ C_{g,\nu}(t) = -ig_\nu \int_0^t C_e(\tau)e^{-i\Delta_\nu\tau}d\tau \tag{19}$$

Induced emission





<u>What changes:</u>

$$|\psi(0)\rangle = |\mathrm{e}\rangle|0\rangle \Rightarrow |\psi(0)\rangle = |\mathrm{e}\rangle|\gamma\rangle \tag{20}$$

Plane wave inducing field

Arbitrary photon field cna be decomposed in terms of plane waves

$$|\gamma\rangle = \sum_{\nu_0} \gamma(\nu_0)|\nu_0\rangle, \quad \nu_0 = \{\boldsymbol{k}_{\nu_0}, s_{\nu_0}\} \tag{21}$$

And a response of the system to an arbitrary incident photon can be constructed from responses to plane waves

$$|\psi(t)\rangle = \sum_{\nu_0} \gamma(\nu_0)|\psi^{\mathrm{PW}}(t)\rangle. \tag{22}$$

The state of the system corresponding to a plane wave incident photon satisfies the initial condition

Ansatz for the state of the system

To describe the induced emission processes a natural ansatz for the state of the system is of the form

$$|\psi(t)\rangle = \sum_{\nu} C^{\mathrm{PW}}_{e,\nu,n}(t)|\mathrm{e}_n\rangle|\nu\rangle + \sum_{\nu_1,\nu_2} C^{\mathrm{PW}}_{g,\nu_1,\nu_2}(t)|\mathrm{g}\rangle|\nu_1,\nu_2\rangle. \tag{23}$$

Here n is the index, that numerates the magnetic quantum number of the excited state. It is crucial to account for it to later describe the interaction with a vortex incident photon.

The initial condition yields

$$C^{\mathrm{PW}}_{e,\nu,n}(0) = \delta_{n,m_e}\delta_{\nu,\nu_0}, \quad C^{\mathrm{PW}}_{g,\nu_1,\nu_2}(0) = 0. \tag{24}$$

Equations for population amplitudes

The Schrödinger equation can be rewritten as a set of equations for population amplitudes

$$
\begin{aligned}
i\dot{C}^{\mathrm{PW}}_{e,\rho,n}(t) &= \sum_{\nu} \left[C^{\mathrm{PW}}_{g,\nu,\rho}(t) + C^{\mathrm{PW}}_{g,\rho,\nu}(t) \right] g^{*}_{\nu,n} e^{-i\Delta_\nu t}, \\
i\dot{C}^{\mathrm{PW}}_{g,\nu,\rho}(t) &= \frac{1}{2} \sum_{n} C^{\mathrm{PW}}_{e,\nu,n}(t) g_{\rho,n} e^{-i\Delta_\rho t} + C^{\mathrm{PW}}_{e,\rho,n}(t) g_{\nu,n} e^{-i\Delta_\nu t}.
\end{aligned}
\tag{25}
$$

which can be rewritten as a single integro-differential equation

$$
\begin{aligned}
\dot{C}^{\mathrm{PW}}_{e,\rho,n}(t) = -\sum_{\nu,n'} \int_0^t d\tau \big[C^{\mathrm{PW}}_{e,\rho,n'}(\tau) g_{\nu,n'} e^{-i\Delta_\nu t} + \\
C^{\mathrm{PW}}_{e,\nu,n'}(t) g_{\rho,n'} e^{-i\Delta_\rho t} \big] g^{*}_{\nu,n} e^{i\Delta_\nu t}.
\end{aligned}
\tag{26}
$$

First term is exactly what one has for the spontaneous emission, the second one is inherent to the induced emission process.

Plane wave solution

To solve the integro-differential equation one need to consider an auxilary function

$$\mathcal{C}_{m,n}(t,t') = \sum_{\nu} g_n^{\nu *} e^{i(\omega_0-\omega_\nu)t'} C^{\mathrm{PW}}_{e,\nu,m}(t) \tag{27}$$

It satisfies the following equation

$$\partial_t \mathcal{C}_{n,n'}(t,t'>t) = -\frac{\Gamma}{2}\mathcal{C}_{n,n'}(t,t'>t). \tag{28}$$

And the solution for the excited state population amplitude is obtained to be

$$\begin{aligned} C^{\mathrm{PW}}_{e,\nu,n}(t) =& \delta_{\nu,\nu_0}\delta_{n,m_e} e^{-\frac{\Gamma}{2}t} - g_{\nu,m_e} g^{*}_{\nu_0,n} e^{-\frac{\Gamma}{2}t} \times \\ & \int\limits_0^t dt_2 \int\limits_0^{t_2} e^{\frac{\Gamma}{2}t_2 + i\Delta_{\nu_0}t_2 - \frac{\Gamma}{2}t_1 - i\Delta_\nu t_1} dt_1. \end{aligned} \tag{29}$$

Interaction with a vortex incident photon

To derive the state of the system which corresponds to an incident vortex photon via integrate the excited and ground state population amplitudes with

$$\gamma(\nu_0) = N \int e^{-(\kappa-\kappa_c)^2/2\sigma_\kappa^2} e^{-(k_z-k_c)^2/2\sigma_z^2} e^{i\kappa b \cos(\varphi_q-\varphi_b)} \langle \nu_0 | \kappa, k_z, m, \lambda \rangle \frac{\kappa d\kappa}{(2\pi)^2} \frac{dk_z}{(2\pi)}. \tag{30}$$

We introduces an impact parameter $\boldsymbol{b}$ to account for the relative distance between the atom and the incident photon

Average TAM and variation

At large enough times the population density of the excited state vanishes and the state of the system is described by

$$|\psi(t)\rangle = |\mathrm{g}\rangle|\gamma_{\mathrm{f}}\rangle,$$
$$|\gamma_{\mathrm{f}}\rangle = \sum_{\nu_1,\nu_2} C_{g,\nu_1,\nu_2}(t)|\nu_1;\nu_2\rangle \tag{31}$$

One of the main interest of the research is to analyze the transfer of TAM from the incident photon to the entangled pair of photons

$$\langle\gamma_f|\hat{J}_z|\gamma_f\rangle = (m_\gamma + m_e)\langle\gamma_f|\gamma_f\rangle +$$
$$\sum_n (n - m_\gamma) J^2_{m_\gamma - n}(\kappa_c b)\mathcal{I}_{n,\lambda}(\Gamma, \omega_{\mathrm{c}}, \kappa\mathrm{c}, \sigma, t),$$
$$\langle\gamma_f|\hat{J}^2_z|\gamma_f\rangle = (m_\gamma + m_e)\langle\gamma_f|\hat{J}_z|\gamma_f\rangle + \tag{32}$$
$$\sum_n (n - m_\gamma)(n + m_e) J^2_{m_\gamma - n}(\kappa_{\mathrm{c}} b)\mathcal{I}_{n,\lambda}(\Gamma, \omega_{\mathrm{c}}, \kappa\mathrm{c}, \sigma, t),$$

Mesoscopic target

In an experiment the position of the atom cannot be controlled with an arbitrary precision. Instead the incident photon interacts with a localized mesoscopic atomic target described by some distribution function $n(\boldsymbol{b})$. For the quantitative analysis of the results we will assume a Guassian distribution

$$n(\boldsymbol{b}) = \frac{1}{\pi\sigma_{\mathrm{b}}^2} \exp\left(-\frac{b^2}{\sigma_{\mathrm{b}^2}}\right) \tag{33}$$

centered on the the z-axis with the width σ_{b}.

Averaging of TAM and variation with the atomic distribution

To describe the transfer of TAM in a scenario where the incident photon interacts with a localized atomic target we average the observables over the impact parameter

$$\begin{aligned} J_z &= \int n(\boldsymbol{b}) \langle \hat{J}_z \rangle d^2 b, \\ (\Delta J_z)^2 &= \int n(\boldsymbol{b}) \left[\langle \hat{J}_z^2 \rangle - \langle \hat{J}_z \rangle^2 \right] d^2 b. \end{aligned} \tag{34}$$

We find the following final results

$$\begin{aligned} J_z &= m_\gamma + m_e + \sum_n \frac{(n - m_\gamma)}{(|m_\gamma - n|)!} \left(\frac{\kappa_c \sigma_b}{2} \right)^{2|m_\gamma - n|} \mathcal{I}_{n,\lambda}(\Gamma, \omega_c, \kappa_c, \sigma, t), \\ (\delta J_z)^2 &= \sum_n \frac{(n - m_\gamma)^2}{(|m_\gamma - n|)!} \left(\frac{\kappa_c \sigma_b}{2} \right)^{2|m_\gamma - n|} \mathcal{I}_{n,\lambda}(\Gamma, \omega_c, \kappa_c, \sigma, t) \end{aligned} \tag{35}$$

Results

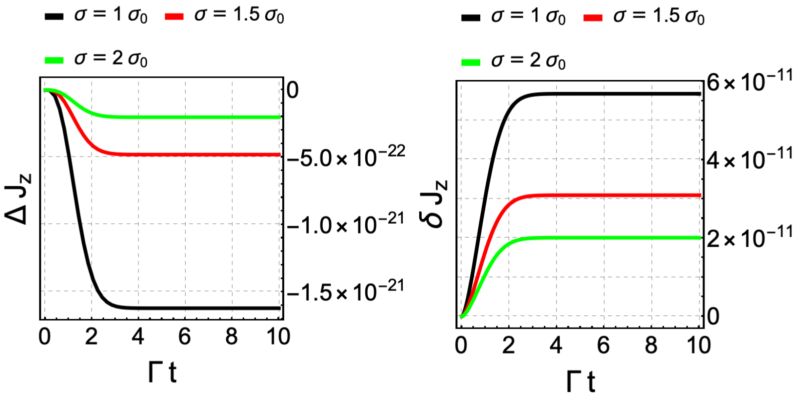


Figure 1: Average angular momentum and variation versus time for different dispersion of momentum of the incident photon $\sigma = 1\sigma_0(\text{black}), 1.5\sigma_0(\text{red}), 2\sigma_0(\text{green}), \sigma_0 = 10^{-2}\text{eV}, m_\gamma = 3, m_e = 1$

Results

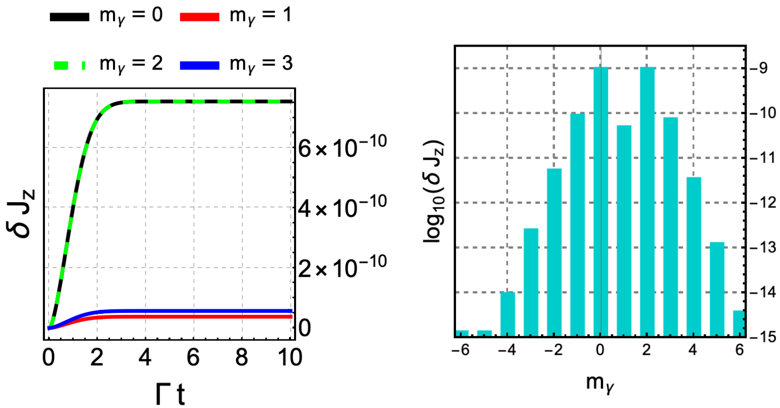


Figure 2: (Left): Variation of TAM versus time for different TAM of the incident photon $m_\gamma = 0$(black), 1(red), 2(green), 3(blue), $\sigma = 10^{-2}$eV. (Right): natural logarithm of variation of TAM versus TAM of the incident photon for $\sigma = 10^{-2}$eV, $t = 1/\Gamma$

Conclusion

- Interaction of a vortex photon with an exited localized atomic target $\Rightarrow$ entangled pair of photons with a definite TAM
- Variation of TAM drops with incident photon m_γ

$$\left(\frac{\kappa_c \sigma_b}{2}\right)^{m_\gamma}, \tag{36}$$

provided $\kappa_c \sigma_b < 1$.

ArXiv: