# Generic spinning binaries from the scattering amplitude perspective

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Bern, Kosmopoulos, Luna, Roiban, **FT**, 2203.06202
Bern, Kosmopoulos, Luna, Roiban, Scheopner, **FT**, Vines, 2308.14176
Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, **FT**, 2407.10928, 2503.03739

# Gravitational wave: new window to probe our Universe

New physics!

- Probe dynamics of black holes
- Test general relativity
- Black hole formation
- Early universe

Future ground based observatories

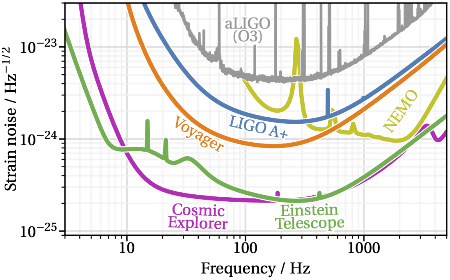
- Advanced LIGO
- Einstein Telescope
- Cosmic Explorer

Future space based observatories

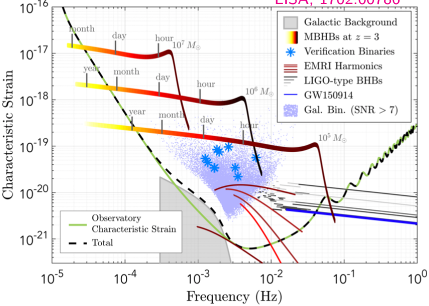
- LISA
- TaiJi
- TianQin

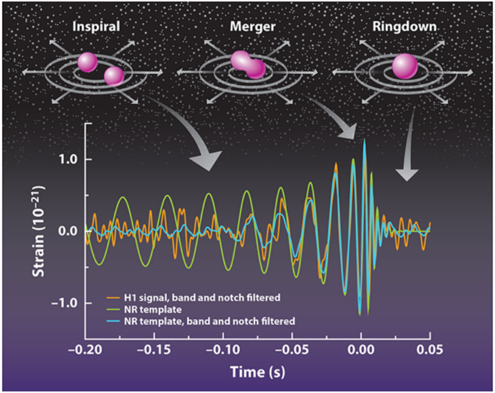
Require accurate theoretical prediction

cosmicexplorer.org




LISA, 1702.00786






Accurate theoretical prediction of the GW production puts challenges on the understanding of its source

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{G m_1 m_2}{|\mathbf{r}|} + \underbrace{\left[ \# \frac{G^2 m_1 m_2 (m_1 + m_2)}{\mathbf{r}^2} + \dots \right]}_{\text{(corrections from Relativity)}}$$

I will focus on long range interactions in the conservative sector

# How to organize perturbations?

- Post-Newtonian (PN) expansion

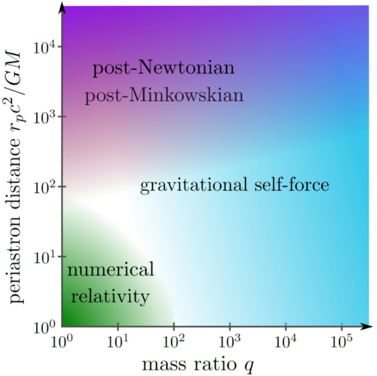
$$v^2 \sim \frac{Gm}{r} \ll 1$$

- Post-Minkowskian (PM) expansion

$$\frac{Gm}{r} \ll v^2 \sim 1$$

- Self-force expansion

$$\frac{Gm}{r} \sim v^2 \sim 1\,, \quad \frac{m_1}{m_2} \ll 1$$



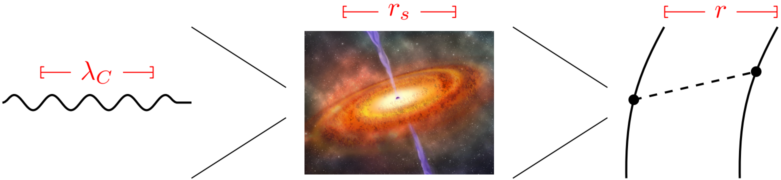

Khalil, Buonanno, Steinhoff, Vines, 2204.05047

Amplitude-based methods naturally lead to PM expansion

PM expansion is relevant to bound orbits with large eccentricity and scattering process

# EFT matching using amplitudes

Cheung, Rothstein, Solon, 1808.02489




Full theory: $S_{\text{full}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x \sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right) + \mathcal{O}(R^2\phi^2)$

Implemented by method of regions
Beneke, Smirnov, hep-ph/9711391

Classical limit $(q, \ell, G) \to (\hbar q, \hbar\ell, \hbar^{-1}G)$

Integrate out soft gravitons

Observables ← Eikonal formula ← $\mathcal{M}_{\text{QFT}} = \mathcal{M}_{\text{EFT}}$

$V_{\text{PM}}$ given by an ansatz

Solve $V_{\text{PM}}$ by matching amplitudes

EOM

Effective theory: $S_{\text{eff}} = \int dt\left[m_1\sqrt{1-\mathbf{v}_1^2} + m_2\sqrt{1-\mathbf{v}_2^2} - V_{\text{PM}}\right]$

# EFT matching Cheung, Rothstein, Solon, 1808.02489

- Full theory: Schwarzschild black hole $\Longrightarrow$ scalar field $\phi$

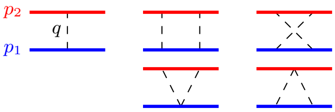
$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{16\pi G}R + \frac{1}{2}\sum_{i=1,2}(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2)\right] + \mathcal{O}(R^2\phi^2)$$

- Effective theory: potential $V(\mathbf{k},\mathbf{k}')$ given by an ansatz

$$L = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\sum_{i=1,2} a_i^\dagger(-\mathbf{k})\left(i\partial_t + \sqrt{\mathbf{k}^2 + m_i^2}\right)a_i(\mathbf{k}) - \int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} V(\mathbf{k},\mathbf{k}')a_1^\dagger(\mathbf{k}')a_1(\mathbf{k})a_2^\dagger(-\mathbf{k}')a_2(\mathbf{k})\right]$$

- Solve the EFT potential by matching the full theory and EFT amplitudes order-by-order in $G$ in the classical limit

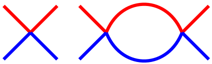
$$\mathcal{M}^{(L)} \xrightarrow{\text{classical limit}} \mathcal{M}^{(L)}_{\text{EFT}}$$




$q \to \hbar q$

$\ell \to \hbar \ell$

$G \to \hbar^{-1} G$



no massless loops (quantum)
no massive contact vertices (delta-function potential)

$$\mathcal{M}^{(L)}_{\text{classical}} \sim \frac{Gm^4}{\hbar^3 q^2}(Gmq)^L$$

Cheung, Rothstein, Solon, 1808.02489
Bern, Cheung, Roiban, Solon, Shen, Zeng, 1901.04424
Bern, Parra-Martinez, Roiban, Ruf, Solon, Shen, Zeng, 2112.10750
Bern, Herrmann, Roiban, Ruf, Smirnov, 2406.01554

$$E_1 = \sqrt{\mathbf{p}^2 + m_1^2} \qquad E_2 = \sqrt{\mathbf{p}^2 + m_2^2}$$
$$\gamma = \frac{E_1 + E_2}{m_1 + m_2} \qquad \xi = \frac{E_1 E_2}{(E_1+E_2)^2}$$
$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \qquad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

**Hamiltonian:** $H = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{|\mathbf{r}|^n} c_{n\mathrm{PM}}(\mathbf{p}^2)$

$$c_{1\mathrm{PM}} = -\frac{\nu^2(m_1+m_2)^2}{\gamma^2\xi}(2\sigma^2-1)$$

Westpfahl and Goller 1979

$$c_{2\mathrm{PM}} = -\frac{\nu^2(m_1+m_2)^3}{\gamma^2\xi}\left[\frac{3(5\sigma^2-1)}{4} - \frac{4\nu\sigma(2\sigma^2-1)}{\gamma\xi} + \frac{\nu^2(1-\xi)(2\sigma^2-1)^2}{2\gamma^3\xi^2}\right]$$

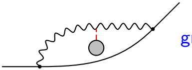
Bel, Damour, Deruelle, Ibanez, Martin, 1981
Westpfahl 1985

$$c_{3\mathrm{PM}} = \frac{\nu^2 m^4}{\gamma^2\xi}\left[\frac{3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3}{12} - \frac{4\nu(3+12\sigma^2-4\sigma^4)\,\mathrm{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$-\frac{3\nu\gamma(2\sigma^2-1)(5\sigma^2-1)}{2(1+\gamma)(1+\sigma)} + \frac{3\nu\sigma(20\sigma^2-7)}{2\gamma\xi} + \frac{\nu^2(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2)(2\sigma^2-1)}{4\gamma^3\xi^2}$$
$$\left.+\frac{2\nu^3(3-4\xi)\sigma(2\sigma^2-1)^2}{\gamma^4\xi^3} - \frac{\nu^4(1-2\xi)(2\sigma^2-1)^3}{2\gamma^6\xi^4}\right]$$

State-of-the-art: $c_{4\mathrm{PM}}^{\mathrm{hyp}}$ and $c_{5\mathrm{PM}}^{\mathrm{hyp}\ 1\mathrm{SF}}$

- $c_{3\mathrm{PM}}$ is not known to general relativists before computed this way
- $c_{5\mathrm{PM}}$ for GR is obtained using the amplitude-worldline hybrid method

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch, 2403.07781
Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch, 2411.11846

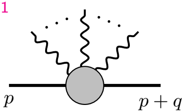


gravitational tails depend on the trajectory

Black holes and neutron stars can carry spin

How to incorporate spin into the EFT?

# On-shell description of spin Bern, Luna, Roiban, Shen, Zeng, 2005.03071




- On-shell spin-$s$ states are symmetric traceless and transverse

$$\varepsilon_{a_1 a_2 \ldots a_s} = \varepsilon_{(a_1 a_2 \ldots a_s)} \qquad p^{a_1} \varepsilon_{a_1 a_2 \ldots a_s} = \eta^{a_1 a_2} \varepsilon_{a_1 a_2 \ldots a_s} = 0$$

- Classical limit $\Longrightarrow$ spin coherent state $\varepsilon^s_{a_1 a_2 \ldots a_s} = \varepsilon^+_{a_1} \varepsilon^+_{a_2} \ldots \varepsilon^+_{a_s}$ with large $s$

$$\varepsilon^s_p \cdot M^{ab} \cdot \varepsilon^s_{p+q} \sim S^{ab} \qquad (M^{ab})_{c(s)}{}^{d(s)} = -2is\delta^{[a}_{(c_1} \eta^{b](d_1} \delta^{d_2}_{c_2} \ldots \delta^{d_s)}_{c_s)}$$

$$\varepsilon^s_p \cdot \{M^{ab} M^{cd}\} \cdot \varepsilon^s_{p+q} \sim S^{ab} S^{cd} \qquad S^{ab} = (1/m) \varepsilon^{abcd} p_c S_d$$

- The spin tensor satisfy covariant spin supplementary condition (SSC)

$$S^{ab} p_b = 0 \qquad (S^{ab} \text{ is boosted from rest frame } S^{ij})$$

- Transversality and covariant SSC are related
- Spin magnitude is conserved: $S^{ab} S_{ab} \sim S^a S_a \sim \boldsymbol{S}^2 = \text{const}$

# How to describe interactions?

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab} \phi_s$$
$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd} \nabla_b$$

Higher spin quantum field theory ($\phi_s \equiv \phi_{a_1 a_2 \ldots a_s}$)

$$\mathcal{L} = -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8}R_{abcd}\phi_s M^{ab} M^{cd}\phi_s - \frac{C_2}{2m^2} R_{a f_1 b f_2} \nabla^a \phi_s \mathbb{S}^{(f_1} \mathbb{S}^{f_2)} \nabla^b \phi_s$$
$$+ \frac{D_2}{2m^2} R_{abcd} \nabla_i \phi_s \{M^{ai} M^{cd}\}\phi_s + \frac{E_2 - 2D_2}{2m^4} R_{abcd} \nabla^{(a}\nabla^{i)}\phi_s \{M^b{}_i M^d{}_j\} \nabla^{(c}\nabla^{j)}\phi_s + \mathcal{O}(M_{ab}^3)$$

We prefer to use a formalism that is uniform in $s$:

- Contractions of $\phi_s$ facilitated by $M^{ab}$ only
- Propagator uniform in $s$: $i\delta^{b(s)}_{a(s)}/(p^2 - m^2)$
- Classical and large spin limit is straightforward
- There are additional lower spin ($s' < s$) states in the spectrum

Problematic? Not in the classical limit:

- Ghost nature easily cured by an analytic continuation on classical variables
- We get a more generic non-rigid spinning object (more internal DOFs)
- Conventional rigid spinning objects correspond to special Wilson coefficients

Bern, Luna, Roiban, Shen, Zeng, 2005.03071
Bern, Kosmopoulos, Luna, Roiban, **FT**, 2203.06202
Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

# Generalized spin coherent state

Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176

The external state now contains lower spin components. Consider the coherent sum

$$\mathcal{E}_{\mu_1\ldots\mu_s} = \varepsilon^{(s)}_{\mu_1\ldots\mu_s} + u_{(\mu_1}\varepsilon^{(s-1)}_{\mu_2\ldots\mu_s)} + \ldots$$

Similar coherent sum was also considered in Aoude, Ochirov, 2108.01649, etc

Classical limit

$$\mathcal{E}_p \cdot M^{ab} \cdot \mathcal{E}_{p+q} \sim \mathsf{S}^{ab} \qquad \mathsf{S}^{ab} = S^{ab} + (i/m)(p^a K^b - p^b K^b)$$

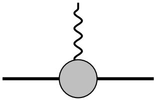
$$\mathcal{E}_p \cdot \{M^{ab} M^{cd}\} \cdot \mathcal{E}_{p+q} \sim \mathsf{S}^{ab}\mathsf{S}^{cd}$$

where $K^a$ is identified as the boost generator, and $S^{ab}p_b = K^a p_a = 0$

- $K^a$ emerges from the transition between spin $s$ and lower spin states
- Consequently, $\mathsf{S}^{ab}\mathsf{S}_{ab} \sim \boldsymbol{S}^2 - \boldsymbol{K}^2$ is a still constant but $\boldsymbol{S}^2$ is not

# Classical Compton amplitudes

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739



Three-point amplitude (metric):

$$\mathcal{M}_3 = -(\varepsilon_1 \cdot p)^2 + \frac{(\varepsilon_1 \cdot p)\tilde{f}_1^{\mu\nu} p_\mu S_\nu}{m} - \frac{(1+C_2)(\varepsilon_1 \cdot p)^2 (k_1 \cdot S)^2}{2m^2} - \frac{D_2 (k_1 \cdot K)(\varepsilon_1 \cdot p)\tilde{f}_1^{\mu\nu} p_\mu S_\nu}{m^2} - \frac{E_2 (k_1 \cdot K)^2 (\varepsilon_1 \cdot p)^2}{2m^2}$$

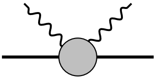
- A stationary metric source by $K$-dependent multipole moments
- The presence of $K$ does NOT modify the spin-induced dipole contribution
- LO matched to Rasheed-Larsen black hole Rasheed, hep-th/9505038, Larsen, hep-th/9909102
- $K$ drops out of the amplitude when $D_2 = E_2 = 0$

$S^{\mu\nu} = (1/m)\epsilon^{\mu\nu\rho\sigma} p_\rho S_\sigma$

$f_{\mu\nu} = k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu$ and $\tilde{f}^{\mu\nu} = (i/2)\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}$

# Classical Compton amplitudes

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739



The same property holds at four points

$$s = 2p\cdot k_1,\ t = 2k_1\cdot k_2,\ u = 2p\cdot k_2$$

$$\mathcal{M}_4 = \frac{4}{stu}\Big[\alpha^2 - \alpha\,\mathcal{O}_{(1)} + \frac{1}{2}\mathcal{O}_{(1)}^2 + C_2\,\alpha\,\mathcal{O}_{(2)} + D_2\,\alpha\left(\mathcal{O}_{(1)}\frac{(k_1+k_2)\cdot K}{m} - \mathcal{K}_{(1,1)}\right) + E_2\left(\alpha\,\mathcal{O}_{(2)}\Big|_{S\to K}\right)\Big]$$

$$\alpha = p\cdot f_1\cdot f_2\cdot p$$

$$\mathcal{O}_{(1)} = \frac{1}{m}\Big[f_2(p,k_1)\tilde{f}_1(p,S) + \frac{s}{2}\,\tilde{f}_{12}(p,S) + (1\leftrightarrow 2)\Big]$$

$$\mathcal{O}_{(2)} = \frac{1}{2m^2}\Big[t f_1(p,S) f_2(p,S) + \alpha(k_1\cdot S + k_2\cdot S)^2\Big]$$

$$\mathcal{K}_{(1,1)} = \frac{t}{2m^2}\Big[f_2(p,K)\tilde{f}_1(p,S) + f_1(p,K)\tilde{f}_2(p,S)\Big]$$

When $D_2 = E_2 = 0$, the additional dynamical freedom drops out automatically

# Non-minimal interactions up to $\mathcal{O}(M_{ab}^2)$

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab} \phi_s$$
$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd} \nabla_b$$

$$\mathcal{L} = -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8} R_{abcd}\phi_s M^{ab} M^{cd}\phi_s - \frac{C_2}{2m^2} R_{af_1bf_2}\nabla^a\phi_s \mathbb{S}^{(f_1}\mathbb{S}^{f_2)}\nabla^b\phi_s$$
$$+ \frac{D_2}{2m^2} R_{abcd}\nabla_i\phi_s\{M^{ai}M^{cd}\}\phi_s + \frac{E_2 - 2D_2}{2m^4} R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_i M^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s$$

- The $C_2$-operator has an origin in the world-line formalism for neutron stars
  Porto, 0511061; Levi, Steinhoff, 1501.04956
- It is the only independent operator assuming that rest frame spin is the only dynamical degree of freedom
- The $D_2$- and $E_2$-operators supply additional $\mathcal{O}(SK)$ and $\mathcal{O}(K^2)$ interactions

$D_2 = E_2 = 0 \quad \Longrightarrow \quad$ Conventional compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S})$

$C_2 = D_2 = E_2 = 0 \quad \Longrightarrow \quad$ Kerr black hole

Generic values: generic compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S}, \mathbf{K})$

# World-line Lagrangian with $K$

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

The above Compton amplitudes can be reproduced by the following world-line model

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}(p^2 - \mathrm{M}^2) \qquad \mathsf{S}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma + \hat{p}^\mu \mathrm{K}^\nu - \hat{p}^\nu \mathrm{K}^\mu$$

$$\mathrm{M}^2 = m^2 + \left[\frac{1+C_2}{4}R_{\hat{p}S\hat{p}S} + \frac{1+D_2}{2}\widetilde{R}_{\hat{p}S\hat{p}\mathrm{K}} + \frac{1+E_2}{4}R_{\hat{p}\mathrm{K}\hat{p}\mathrm{K}} + \mathcal{O}(\mathsf{S}^3)\right] \qquad \mathrm{K}^\mu = -\mathsf{S}^{\mu\nu}\hat{p}_\nu$$

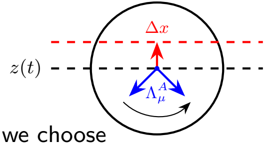
- Notably, NO SSC (for example, $\mathsf{S}^{\mu\nu}p_\nu = 0$) is imposed
- The classical Compton amplitude is identified as the ratio between the amplitude of the outgoing spherical wave and incoming plane wave Saketh, Vines, 2208.03170

$$h^{\mu\nu} = e^{-ik\cdot x}\varepsilon^\mu \varepsilon^\nu + \frac{e^{ikr - i\omega t}}{4\pi r}\mathcal{M}^{\mu\nu,\rho\sigma}_{\mathrm{Comp}}\varepsilon_\rho \varepsilon_\sigma$$

- The matching requires an identification $iK^a \equiv \mathrm{K}^a$
- Self-consistent world-line theory involving both $S^{ab}$ and $\mathrm{K}^a$ exists

d'Ambrosi, Kumar, van Holten, 1501.04879

# Generic spinning body with $K$




- The vector $\mathrm{K}^a$ is the displacement between the world-line we choose (center-of-spin) and the actual center-of-mass

$$J^{\mu\nu} = z^\mu p^\nu - z^\nu p^\mu + \mathsf{S}^{\mu\nu} = (z^\mu - \mathrm{K}^\mu/|p|)p^\nu - (z^\nu - \mathrm{K}^\nu/|p|) + \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma$$

- When $D_2 = E_2 = 0$, $\mathrm{K}$ drops out of the EOM under the redefinition of world-line

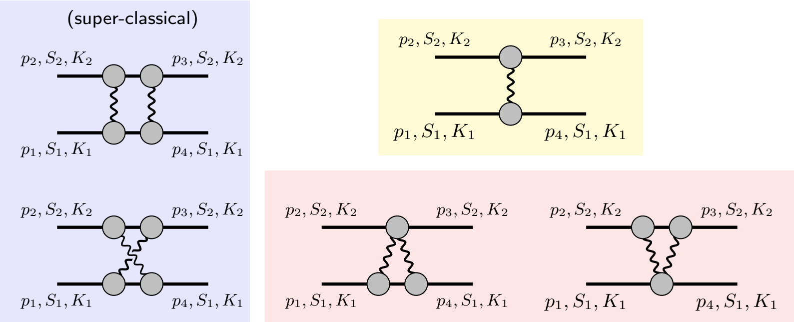
$$z'^\mu = z^\mu - \mathrm{K}^\mu/|p|$$

- One can show that the EOM is the same as that with the covariant SSC

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}\left(p^2 - m^2 + \frac{C_2}{4}R_{\mu\nu\rho\sigma}\hat{p}^\mu S^\nu \hat{p}^\rho S^\sigma\right) + \chi_\mu \mathsf{S}^{\mu\nu}\hat{p}_\nu + \zeta_\mu(\Lambda_0^\mu - \hat{p}^\mu)$$

- Emergence of spin gauge symmetry when $D_2 = E_2 = 0$
- For generic $D_2$ and $E_2$, $\mathrm{K}^\mu$ is a genuine dynamical variable that contributes at the quadrupole level

# Two-body amplitudes

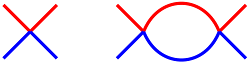



$$
\begin{aligned}
\mathcal{M}^{2\ \text{body}} = {} & A_0 + A_1 \boldsymbol{L}\cdot\boldsymbol{S} + A_{2,1}\boldsymbol{S}^2 + A_{2,2}\boldsymbol{K}^2 + A_{2,3}\boldsymbol{S}\cdot\boldsymbol{K} + A_{2,4}(\boldsymbol{b}\cdot\boldsymbol{S})^2 \\
& + A_{2,5}(\boldsymbol{p}\cdot\boldsymbol{S})^2 + A_{2,6}(\boldsymbol{b}\cdot\boldsymbol{K})^2 + A_{2,7}(\boldsymbol{p}\cdot\boldsymbol{K})^2 + A_{2,8}(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{p}\cdot\boldsymbol{S}) \\
& + A_{2,9}(\boldsymbol{L}\cdot\boldsymbol{S})(\boldsymbol{b}\cdot\boldsymbol{K}) + A_{2,10}(\boldsymbol{L}\cdot\boldsymbol{S})(\boldsymbol{p}\cdot\boldsymbol{K}) + A_{2,11}(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{L}\cdot\boldsymbol{K}) \\
& + A_{2,12}(\boldsymbol{p}\cdot\boldsymbol{S})(\boldsymbol{L}\cdot\boldsymbol{K}) + A_{2,13}(\boldsymbol{b}\cdot\boldsymbol{K})(\boldsymbol{p}\cdot\boldsymbol{K})
\end{aligned}
$$

Note: $\mathcal{M}_{\text{classical}}$ is a generating function (radial action) for classical observables

# Effective Hamiltonian through matching

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739



Consider canonical spin in the COM frame

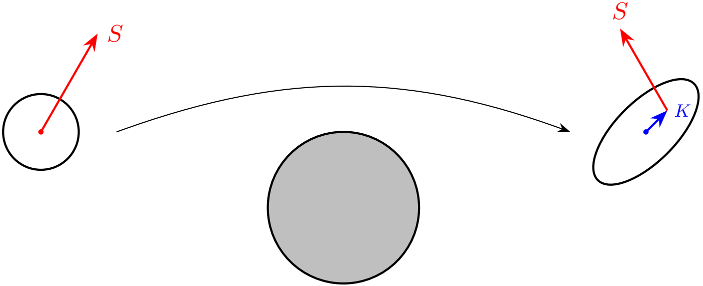
$$H = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + \sum_a \sum_{n=1}^{\infty} \left( \frac{G}{|\mathbf{r}|} \right)^n c_n^a(\mathbf{p}^2) \Sigma_a$$

where the operators $\Sigma_a$ takes value in

$$\begin{array}{ccc} \mathbf{1} & \left( (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S} \right) / \mathbf{r}^2 & \left( \mathbf{r} \cdot \mathbf{K} \right) / \mathbf{r}^2 \\ \left( \mathbf{r} \cdot \mathbf{S} \right)^2 / \mathbf{r}^4 & \left( \mathbf{r} \cdot \mathbf{K} \right) \left( (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S} \right) / \mathbf{r}^4 & \left( \mathbf{r} \cdot \mathbf{K} \right)^2 / \mathbf{r}^4 \\ \mathbf{S}^2 / \mathbf{r}^2 & \left( \mathbf{K} \cdot (\mathbf{p} \times \mathbf{S}) \right) / \mathbf{r}^2 & \mathbf{K}^2 / \mathbf{r}^2 \\ \left( \mathbf{p} \cdot \mathbf{S} \right)^2 / \mathbf{r}^2 & \left( \mathbf{r} \cdot \mathbf{S} \right) \left( (\mathbf{r} \times \mathbf{K}) \cdot \mathbf{p} \right) / \mathbf{r}^4 & \left( \mathbf{p} \cdot \mathbf{K} \right)^2 / \mathbf{r}^2 \end{array}$$

- $c_0^a$ matches to the tree level amplitude at $\mathcal{O}(G)$
- Iteration of $c_0^a$ should agree exactly with the super-classical box coefficients at $\mathcal{O}(G^2)$
- $c_1^a$ matches to the triangle coefficients at $\mathcal{O}(G^2)$ order of $V$
- The coefficient of $\left( \mathbf{r} \cdot \mathbf{K} \right) / \mathbf{r}^2$ vanishes identically
- All the $c_n^a$ coefficients are local in $\mathbf{p}^2$

# Generic spinning body with $K$

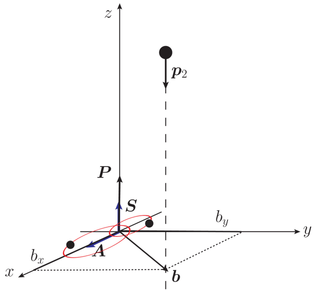



An additional conservative gapless degree of freedom

# Scattering off a Newtonian bound state

$$H = \frac{\boldsymbol{p}_2^2}{2m_2} + \frac{\boldsymbol{P}^2}{2m_1} + H_0(\boldsymbol{p}, \boldsymbol{r}) - \frac{G m_{B1} m_2}{|\boldsymbol{R} + \frac{m_{B2}}{m_1}\boldsymbol{r}|} - \frac{G m_{B2} m_2}{|\boldsymbol{R} - \frac{m_{B1}}{m_1}\boldsymbol{r}|}$$
$$= \frac{\boldsymbol{p}_2^2}{2m_2} + \frac{\boldsymbol{P}^2}{2m_1} + H_0(\boldsymbol{p}, \boldsymbol{r}) - \frac{G m_1 m_2}{|\boldsymbol{R}|} - \frac{3G\mu_B m_2}{2|\boldsymbol{R}|^5} \underbrace{\left( (\boldsymbol{r}\cdot\boldsymbol{R})^2 - \frac{1}{3}|\boldsymbol{r}|^2|\boldsymbol{R}|^2 \right)}_{Q_{ij}(\boldsymbol{r})Q^{ij}(\boldsymbol{R})} + \ldots$$

where $m_1 = m_{B1} + m_{B2}$ and $\mu_B = m_{B1}m_{B2}/m_1$



# Scattering off a Newtonian bound state

$$\begin{aligned}\mathcal{A}_{\mathrm{i}\to\mathrm{f}} &= \int_{-\infty}^{+\infty} dt e^{i(\mathsf{E}_\mathrm{f}^B - \mathsf{E}_\mathrm{i}^B)t} \left\langle \mathrm{i} \left| \frac{3G\mu_B m_2}{2|\boldsymbol{R}|^5} \left( (\boldsymbol{r}\cdot\boldsymbol{R})^2 - \frac{1}{3}|\boldsymbol{r}|^2|\boldsymbol{R}|^2 \right) \right| \mathrm{f} \right\rangle \\ &= \frac{3G\mu_B m_2 r_{\mathrm{cl},n}^2}{2|\boldsymbol{b}|^2 v_0} \left[ \frac{2(\boldsymbol{b}\cdot\boldsymbol{A})^2}{|\boldsymbol{b}|^2} - |\boldsymbol{A}|^2 \right]\end{aligned}$$

- Trajectory: $\boldsymbol{R} = (b_x, b_y, -v_0 t)$
- Initial and final state have the same energy; otherwise exponentially suppressed
- Use elliptical orbit coherent state with $bv_0^2 \gg r_{\mathrm{cl},n}$ Bhaumik, Dutta-Roy, Ghosh, 1986

$$\begin{aligned}\langle\alpha|x|\alpha\rangle &= r_{\mathrm{cl},n}\Big[\cos(2\omega_{\mathrm{cl}}t) + \sin(2\chi)\Big] \\ \langle\alpha|y|\alpha\rangle &= r_{\mathrm{cl},n}\sin(2\omega_{\mathrm{cl}}t)\cos(2\chi) \\ \langle\alpha|z|\alpha\rangle &= 0\end{aligned}$$

- Laplace-Runge-Lenz vector $\boldsymbol{A} = \sin(2\chi)\hat{\boldsymbol{x}}$

# Scattering off a Newtonian bound state

$$\mathcal{M}^{2\text{ body}} \sim A_{2,1}\boldsymbol{K}^2 + A_{2,6}(\boldsymbol{b}\cdot\boldsymbol{K})^2$$

Match to the field theory amplitude:

- Spin ⇔ bound system total orbital angular momentum
- Due to the geometric configuration, the spin does not appear in $\mathcal{A}_{\mathrm{i}\to\mathrm{f}}$
- $\boldsymbol{K}$-vector ⇔ Laplace-Runge-Lenz vector

$$\boldsymbol{K} = i\,Gm_1^2\frac{\mu_B}{m_1}\sqrt{\frac{\mu_B}{2|\mathsf{E}_{\mathrm{i}}^B|}}\,\boldsymbol{A}$$

- Wilson coefficient

$$E_2^{\text{bound 2-body}} = \frac{3|\mathsf{E}_{\mathrm{i}}^B|m_1}{\mu_B^2}(m_1 r_{\mathrm{cl},n})^2$$

$$\mathcal{L} \sim \frac{E_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_i M^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s$$

# Summary

- Framework for effective description of generic spinning binaries
  - Field theory: transition between fields with different $s$
  - World-line: introduce additional dynamical variables
  - Allow more Wilson coefficients compared to the conventional formalism
- Equivalence of the field-theory and world-line description
  - Consider a world-line model involving spin $S$ and another dynamical DOF $K$
  - Demonstrate by matching classical Compton amplitudes
  - Field theory and world-line agree at $\mathcal{O}(S^3)$
- The presence of $K$ does not affect spin-induced dipole moment
- $K$ drops out when additional Wilson coefficients take special values
  - No constraints needed [can use naive kinetic term $\phi_s(\nabla^2 + m^2)\phi_s$ for classical physics]
  - Simplify calculation [propagators and vertices uniform in $s$; straightforward large $s$ limit]

# Discussion

- Effective Hamiltonian at two-loop $\mathcal{O}(S^3)$ and beyond
- Efficient organization of loop integrands involving spin
- Better understanding of tidal operators at $S^4$ and beyond
- Phenomenology of hierarchical three-body system

# Amplitudes meet GW

**This talk: EFT matching**

Amplitudes $\longrightarrow$ Effective Hamiltonian $\longrightarrow$ EOM $\longrightarrow$ Observables

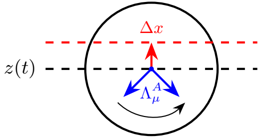
**Entry points for other methods (NOT a complete list of references):**

- Observable based formalism (KMOC) Kosower, Maybee, O'Connell, 1811.10950
- Eikonal scattering formalism Di Vecchia, Heissenberg, Russo, Veneziano, 2306.16488
- Self-force effective theory Cheung, Parra-Martinez, Rothstein, Shah, Wilson-Gerow, 2406.14770; Kosmopolous, Solon, 2308.15304
- World-line QFT Mogull, Plefka, Steinhoff, 2010.02865
- PM-EFT Kälin, Porto, 2006.01184

Thanks for listening!

# Back-up slides

# World-line description of spin




Consider a rigid spinning body

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}\left(p^2 - m^2 + \frac{C_2}{4}R_{\mu\nu\rho\sigma}\hat{p}^\mu S^\nu \hat{p}^\rho S^\sigma + \mathcal{O}(S^3)\right)$$
$$+ \chi_\mu \mathsf{S}^{\mu\nu}\hat{p}_\nu + \zeta_\mu(\Lambda^\mu_0 - \hat{p}^\mu)$$

- We decompose the spin tensor into the rotation and boost components

$$\mathsf{S}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma + (\hat{p}^\mu \mathrm{K}^\nu - \hat{p}^\nu \mathrm{K}^\mu)$$

- The covariant spin supplementary condition (SSC) sets $\mathrm{K}^\mu = 0$
- Spin gauge freedom: the freedom to choose the time direction of the body-fixed frame, which also corresponds to the choice of worldline
- SSC fixes the spin gauge freedom
- Non-minimal interactions: one independent spin-induced multipole moment per order in spin

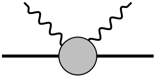
$C_2 R_{pSpS}$

$C_3 \nabla_S \tilde{R}_{pSpS}$

$C_4 \nabla_S \nabla_S R_{pSpS}\ \ldots$


Porto, gr-qc/0511061
Porto, Rothstein, gr-qc/0604099
Levi, Steinhoff, 1501.04956
Vines, Kunst, Steinhoff, Hinderer, 1601.07529

# How to understand? Consider a simple QED example



Consider the following two Lagrangians Kim, Steinhoff, 2302.01944

$$\mathcal{L}_1 = -\frac{1}{2}W_{\mu\nu}\overline{W}^{\mu\nu} + m^2 W_\mu \overline{W}^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i(g-1)F_{\mu\nu}\overline{W}^\mu W^\nu$$

$$\mathcal{L}_2 = \overline{W}^\mu (D^2 + m^2) W_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - igF_{\mu\nu}\overline{W}^\mu W^\nu$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- The Proca Lagrangian $\mathcal{L}_1$ contains a physical $s = 1$ W-boson, while $\mathcal{L}_2$ also contains a ghost-like $s = 0$ degree of freedom
- When $g = 2$, the two Lagrangians produce identical Compton amplitudes

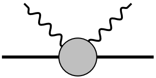
When $g = 2$, a hidden $SU(2)$ symmetry emerges

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) + \mathrm{Tr}(\nabla_\mu \boldsymbol{\Phi} \nabla^\mu \boldsymbol{\Phi})$$

and the mass is obtained through the $SU(2) \to U(1)$ symmetry breaking (unitary and Feynman-'t Hooft gauge)

- Contact terms neglected
- Goldstone decouples from Compton amplitudes of $W$

# Simple QED example



Now we consider one more non-minimal interaction:

$$\mathcal{L}_1' = \underbrace{-\frac{1}{2}W_{\mu\nu}\overline{W}^{\mu\nu} + m^2 W_\mu \overline{W}^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i(g-1)F_{\mu\nu}\overline{W}^\mu W^\nu}_{\mathcal{L}_1} + \frac{iD_1}{m^2}F_{\mu\nu}(D^\nu W^\mu D_\rho \overline{W}^\rho - \text{c.c.})$$

$$\mathcal{L}_2' = \underbrace{\overline{W}^\mu(D^2+m^2)W_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - igF_{\mu\nu}\overline{W}^\mu W^\nu}_{\mathcal{L}_2} + \frac{iD_1}{m^2}F_{\mu\nu}(D^\nu W^\mu D_\rho \overline{W}^\rho - \text{c.c.})$$

The additional interaction does not affect amplitudes of $\mathcal{L}_1$,

$$A^{\mathcal{L}_1}_{\text{Comp}} = A^{\mathcal{L}_1'}_{\text{Comp}}$$

It modifies amplitudes of $\mathcal{L}_2$, but in the classical limit we have

$$A^{\mathcal{L}_1}_{\text{Comp}} = A^{\mathcal{L}_2'}_{\text{Comp}}\Big|_{D_1=(g-2)/2}$$

The $s=0$ degree of freedom effectively decouples when $D_1 = (g-2)/2$

Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176

What is the emergent symmetry?