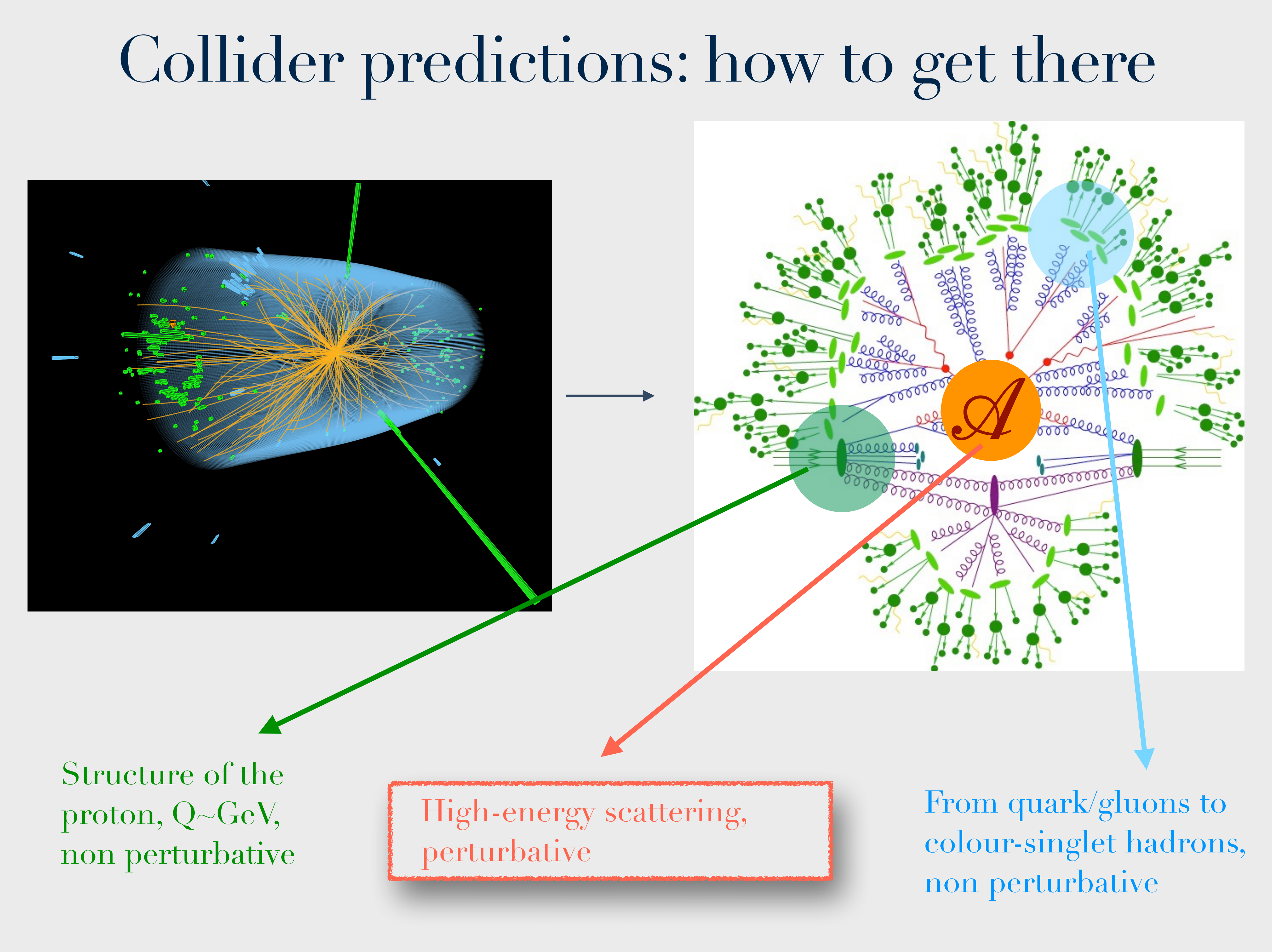
universe+

Bootstrapping hexagonal Wilson loops

Johannes M. Henn (Max Planck Institute for Physics)

Plenary talk at International Workshop on Opportunities for Particle Physics, IHEP, July 19, 2025

Scattering amplitudes connect theory and experiment



From Fabrizio Caola's talk at Amplitudes 2023, CERN]

‚Scattering Amplitudes: the most perfect microscopic structures in the Universe [Lance Dixon, arXiv:1105.0771]

‘Les Houches wishlist’ gives an idea of what is needed from an experimental viewpoint

NNLO QCD and NLO EW Les Houches Wishlist

Wishlist part 1 - Higgs (V=W,Z)

Process	known	desired	motivation
H	d\sigma @ NNLO QCD d\sigma @ NLO EW finite quark mass effects @ NLO	d\sigma @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H+j	d\sigma @ NNLO QCD (g only) d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H+2j	\sigma_tot(VBF) @ NNLO(DIS) QCD d\sigma(gg) @ NLO QCD d\sigma(VBF) @ NLO EW	d\sigma @ NNLO QCD + NLO EW	H couplings
H+V	d\sigma(V decays) @ NNLO QCD d\sigma @ NLO EW	with H→bb @ same accuracy	H couplings
t\bar tH	d\sigma(stable tops) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	d\sigma @ LO QCD finite quark mass effects d\sigma @ NLO QCD large m_t limit	d\sigma @ NLO QCD finite quark mass effects d\sigma @ NNLO QCD	Higgs self coupling

Scattering amplitudes at next-to-next-to-leading-order (NNLO) and even beyond are needed to match the experimental precision.

State of the art Feynman integrals and amplitudes

Dramatic progress for massless 5-particle scattering

Analytic results for all Feynman integrals

[Gehrmann, Henn, Lo Presti 2015;
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018;
Abreu, Page, Zeng 2018; Chicherin, Henn, Mitev 2018;
Abreu, Dixon, Herrmann, Page, Zeng 2018;
Chicherin, Gehrmann, Henn, Wasser, Zhang, **SZ** 2018]

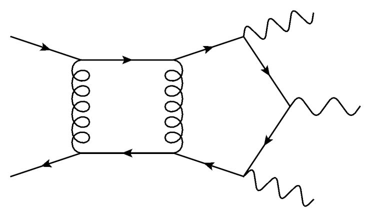
Special function basis

[Gehrmann, Henn, Lo Presti ’18; Chicherin, Sotnikov ’20]

[see M. Marcoli’s talk]

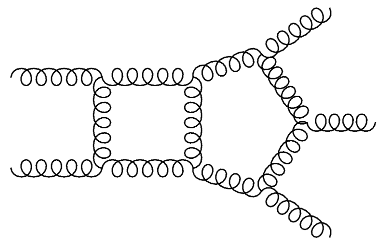
Analytic results for scattering amplitudes

3γ planar



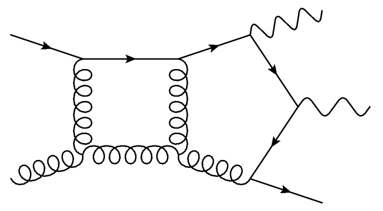
[Abreu, Page, Pascual, Sotnikov 2020; Chawdhry, Czakon, Mitov, Poncelet 2021]

$3j$ planar

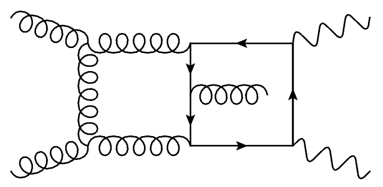


[Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021; Badger, Brönnum-Hansen, Bayu Hartanto, Peraro, Moodie, **SZ**, *to appear*]

$2\gamma + j$ full colour



[Agarwal, Buccioni, von Manteuffel, Tancredi 2021 x2; Chawdhry, Czakon, Mitov, Poncelet 2021]



[Badger, Brönnum-Hansen, Chicherin, Gehrmann, B. Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]

$d\sigma$ **@NNLO QCD:**
- $pp \to 3\gamma$ [Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]
- $pp \to 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]
- $pp \to 3j$ [Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]

[slide from S. Zoia, LoopFest 2022]

This talk: towards six-particle scattering amplitudes at NNLO.

Challenge: Proliferation of rational coefficients

Challenge

- Rationality of integral coefficients in momenta

$$\mathcal{N}_i(\vec{p}) = \sum_{\vec{\alpha}} n_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots \right), \quad \mathcal{D}_i(\vec{p}) = \sum_{\vec{\alpha}} d_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots \right)$$

$$r_i(\vec{p}) = \frac{\mathcal{N}_i(\vec{p})}{\mathcal{D}_i(\vec{p})}, \quad s_{ij} = (p_i + p_j)^2$$

$\hookrightarrow$ linear in numerical coefficients $n_{i,\vec{\alpha}} \in \mathbb{Q}$

- Linear systems constructed from multiple numerical computations of $r_i(\vec{p})$

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \dots\} \in \mathbb{Q}$$

$\hookrightarrow$ linear system for required coefficients $n_{i,\vec{\alpha}}$ and $d_{i,\vec{\alpha}}$

- Bottleneck:
  - Run times
  - Complexity of coefficients = number of unknowns $n_{i,\vec{\alpha}}$

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Five-gluon amplitudes

[De Laurentis, Ita, Klinkert, Sotnikov '23]

Helicity remainder	dim(basis)	ansatz size
$R^{(2),(2,0)}_{+++--}$	31	21,910
$R^{(2),(2,0)}_{++-+-}$	54	54,148
$R^{(2),(1,0)}_{+++--}$	274	163,635
$R^{(2),(1,0)}_{+-++-}$	270	241,156
$R^{(2),(1,0)}_{--+++}$	203	82,180
$R^{(2),(1,1)}_{+++--}$	31	21,910
$R^{(2),(1,1)}_{++-+-}$	54	54,148
$R^{(2),(0,1)}_{+++--}$	226	118,880
$R^{(2),(0,1)}_{+-++-}$	240	209,018
$R^{(2),(0,1)}_{--+++}$	157	76,845
$R^{(2),(-1,1)}_{+++--}$	25	5,320
$R^{(2),(-1,1)}_{++-+-}$	35	9,384

[slide from H. Ita, LoopFest 2025]

This talk: insights on coefficients from N=4 super Yang-Mills.

Perturbative structure in quantum field theory

$$f = \sum_{i,j} c_{i,j} \, r_{n,i} \, g_j$$

Constants (kinematic-independent) → $c_{i,j}$

Coefficients / Leading singularities (rational, algebraic) → $r_{n,i}$

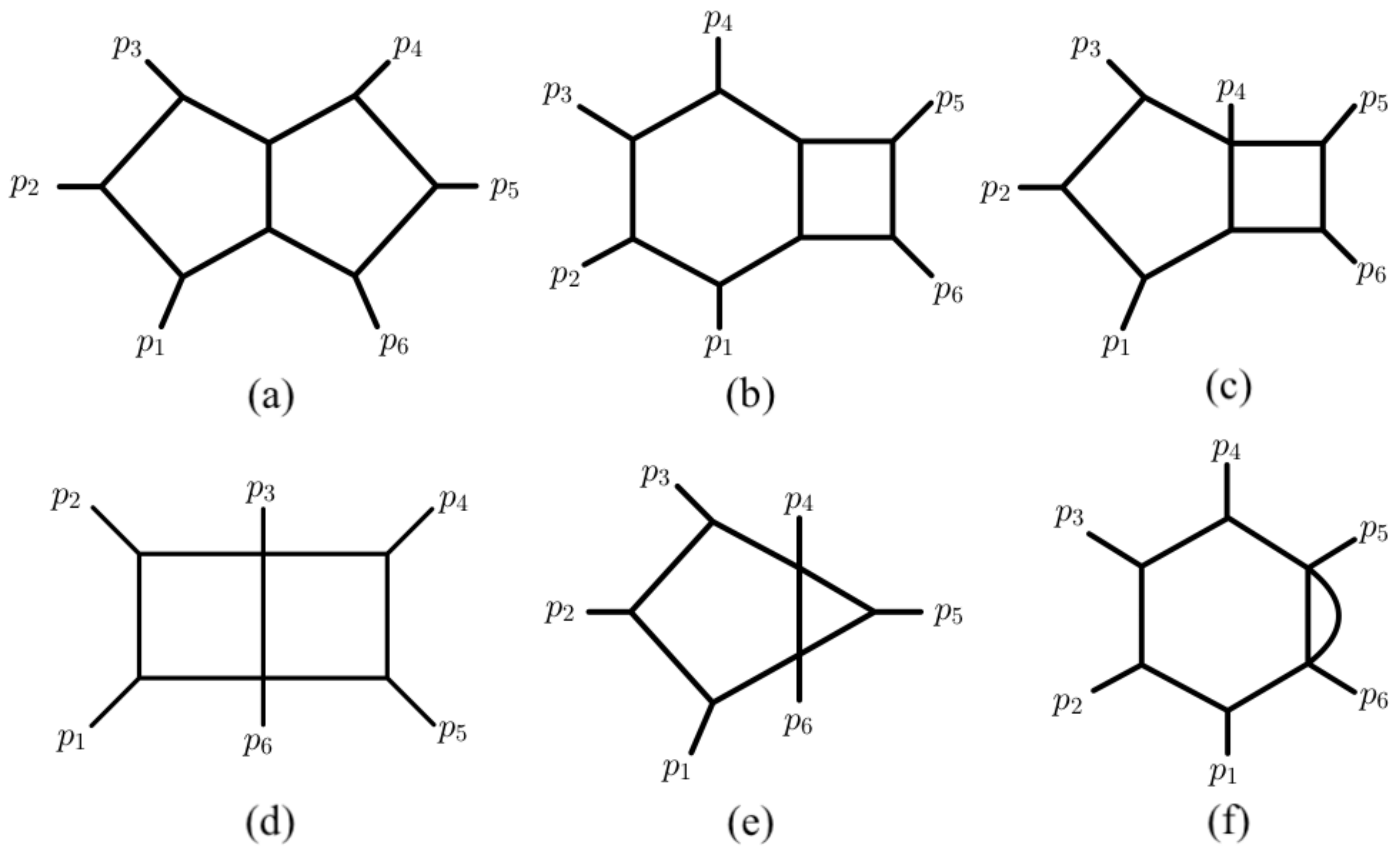
Special functions → g_j

Challenges:

- Complicated two-loop six-particle master integrals
- Proliferation of leading singularities

All massless planar two-loop six-particle Feynman integrals computed

- Differential equations method used
- Most complicated integrals not needed in D=4
- Analytic result, proof-of-concept numerical evaluation



[JMH, Antonela Matijašić, Julian Miczajka, Tiziano Peraro, Yingxuan Xu, Yang Zhang, ***Phys. Rev. Lett.* 135 (2025) 3, 031601**;

JHEP 08 (2024) 027; JHEP 01 (2023) 096

Also Samuel Abreu, Pier Monni, Johann Usovitsch, JHEP 03 (2025) 112]

Our result removes an important bottleneck for obtaining two-to-four scattering amplitudes at next-to-next-to-leading order.

[cf. Yang Zhang's talk at this conference]

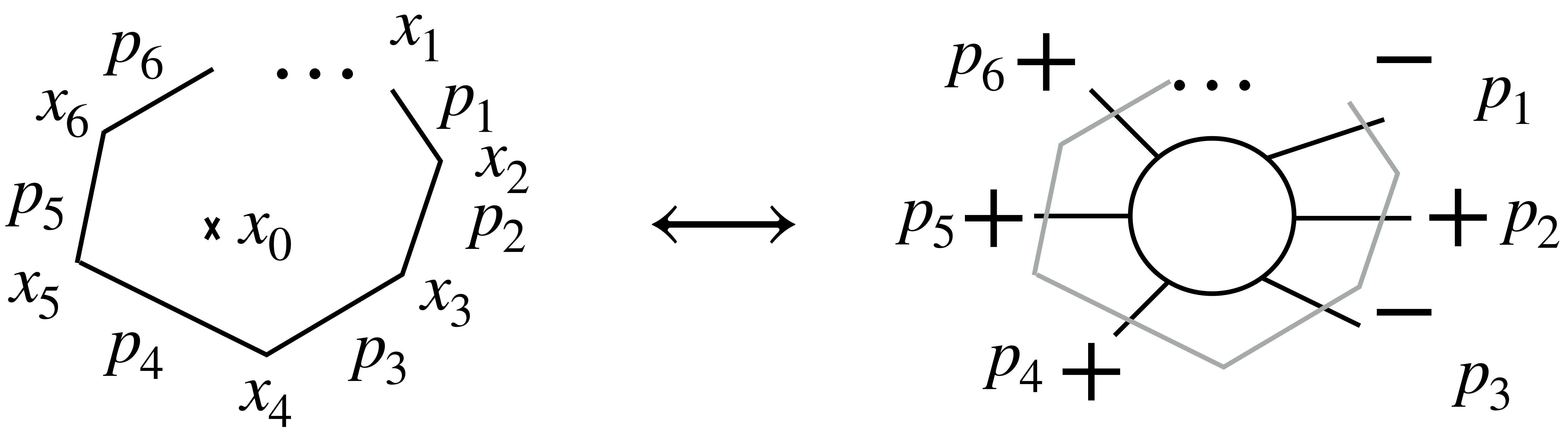
Plan for this talk.

In this talk, we are going to benefit from maximally supersymmetric Yang-Mills theory to learn something about six-particle scattering in pure Yang-Mills theory.

We benefit from a correspondence between scattering amplitudes and Wilson loops in this theory. This allows us to define interesting finite observables. We will use a bootstrap approach to determine the answer.

Wilson loop / scattering amplitudes duality in planar N=4 super Yang-Mills

[Alday, Maldacena; Drummond, Korchemsky, Sokatchev; +JMH; Brandhuber, Heslop, Travaglini]



Null Wilson loop

(MHV) scattering amplitude

Dual variables $x_{i+1} - x_i = p_i$

(Dual) conformal symmetry in x space.

Conformal symmetry in p space.

Why are Wilson loops with Lagrangian insertion (in planar, N=4 super Yang-Mills) interesting?

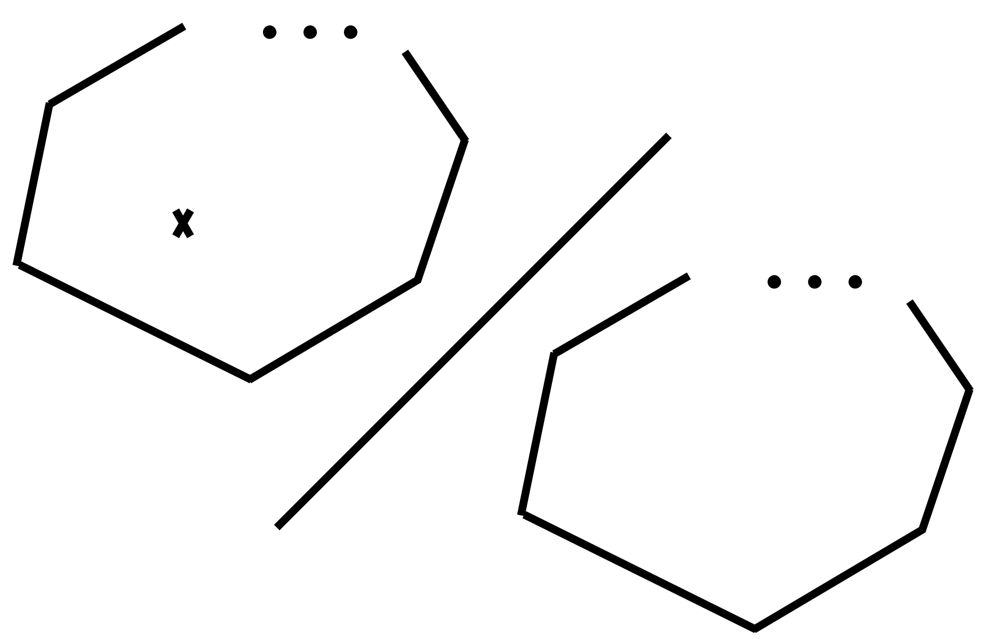
1. Well-defined, finite quantities, *similar to hard functions in QCD*. Integrand is described, to all loop orders, by the Amplituhedron.

2. Same kinematic space as Yang-Mills amplitudes. May shed light on *leading singularities and functions space* in QCD.

3. Many surprising features, such as conformal symmetry, positivity properties, and *duality to all-plus amplitudes*.

Definition and key properties of Wilson loop rat

$$F_n(x_i; x_0) = \frac{\langle W_n \mathscr{L}(x_0) \rangle}{\langle W_n \rangle} .$$



Divergences cancel in ratio. Dual conformal symmetry.

[Alday, Tseytlin, 2011; Alday, Buchbinder, Tseytlin, 2011]

Contains information about cusp anomalous dimension.

[Alday, JMH, Sikorowski, 2013; JMH, Korchemsky, Mistlberger, 2019; Arkani-Hamed, JMH, Trnka, 2021; Bargheer, Bercini, Gonçalves, Fernandes, Mann, 2024]

Same kinematics and function space as Yang-Mills amplitudes:

$$f_n(p_1, \ldots, p_n) = \lim_{x_0 \to \infty} (x_0^2)^4 \, F_n(p_1, \ldots, p_n; x_0)$$

F (equivalently, f) depends on (3n-10) dimensional variables.

Expected structure in perturbation theory

$$f_n^{(L)} = \sum_{i,j} c_{i,j}\, r_{n,i}\, g_j^{(2L)}$$

Constants (kinematic-independent)

Leading singularities (rational, algebraic)

Pure transcendental functions of weight 2L.

We benefit from two recent advances:

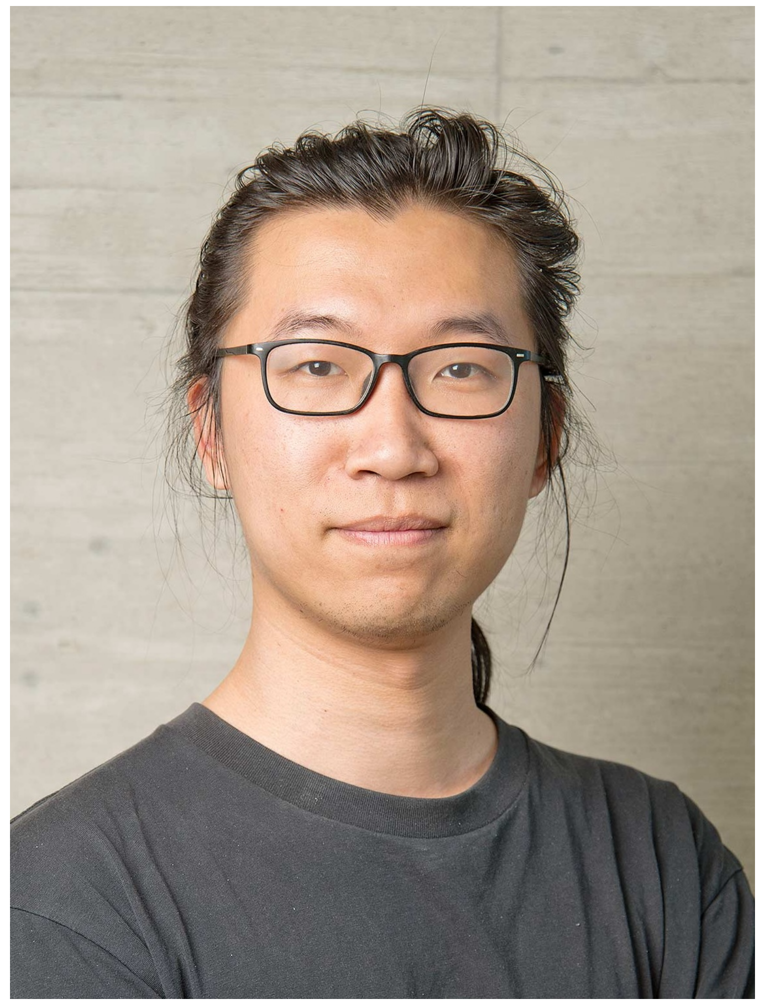
- All relevant two-loop six-particle master integrals evaluated
- All-loop classification of leading singularities

Bootstrapping the hexagonal Wilson loop with Lagrangian insertion

Based on the insights on leading singularities and on the two-loop function space, our goal is to ‚bootstrap' the answer.

[Sergio Carrôlo, Dima Chicherin, JMH, Qinglin Yang, Yang Zhang, 2505.01245]






What is known about the function space?

n	Number of variables	Variables	Known loop order	Alphabet letters	Function space
4	2	s, t	3	{s,t,s+t}	Harmonic polylogarithms [Gehrmann, Remiddi; Maître]
5	5	$s_{i,i+1}$	2	20 parity-even letters 5 parity-odd letters	Pentagon functions [Gerhmann, JMH; LoPresti; Chicherin, Sotnikov]
6	8	$s_{i,i+1}$; $s_{i,i+1,i+2}$ One Gram condition	2	245 letters	[JMH, Matijašić, Miczaijka, Peraro, Xu, Zhang; Abreu, Monni, Page, Usovitsch]

All-loop leading singularities from Amplituhedron

Proof of conjecture on form and number of leading singularities:

[Chicherin, JMH, 2022]

	n	5	6	7	8	9	10	11	12
$L=1$	$\frac{n(n-3)}{2}$	5	9	14	20	27	35	44	54
$L \geqslant 2$	$\frac{(n-1)(n-2)^2(n-3)}{12}$	6	20	50	105	196	336	540	825

Table 1. The number of linearly independent leading singularities of $F_n^{(L)}$, as conjectured in ref. [64], and proven in the present work.

$$[a_1b_1c_1;a_2b_2c_2] = \frac{\langle AB(a_1b_1c_1)\cap(a_2b_2c_2)\rangle^2}{\langle ABa_1b_1\rangle\langle ABb_1c_1\rangle\langle ABa_1c_1\rangle\langle ABa_2b_2\rangle\langle ABb_2c_2\rangle\langle ABa_2c_2\rangle}, \qquad (2.16)$$

Claim 1. All leading singularities of $F_n^{(L)}$ for $n \geqslant 4$ and $L \geqslant 1$ can be expressed as linear combinations of Kermit forms (2.16).

[Brown, JMH, Mazzucchelli, Trnka, 2503.1785]

Idea of the proof

[Brown, JMH, Mazzucchelli, Trnka, 2503.1785]

Leading singularities are maximal residues of the Amplituhedron form. We classify all *leading singularity configurations* that are allowed by the Amplituhedron geometry.

[cf. Dennen, Prlina, Spradlin, Stanojevich, Volovich, 1612.02708]

We evaluate the *residues (leading singularity values)* for all remaining configurations. Using triangulation identities, we show that a basis is given by certain Kermit forms.

For example, an explicit basis for the 20 leading singularities at n=6:

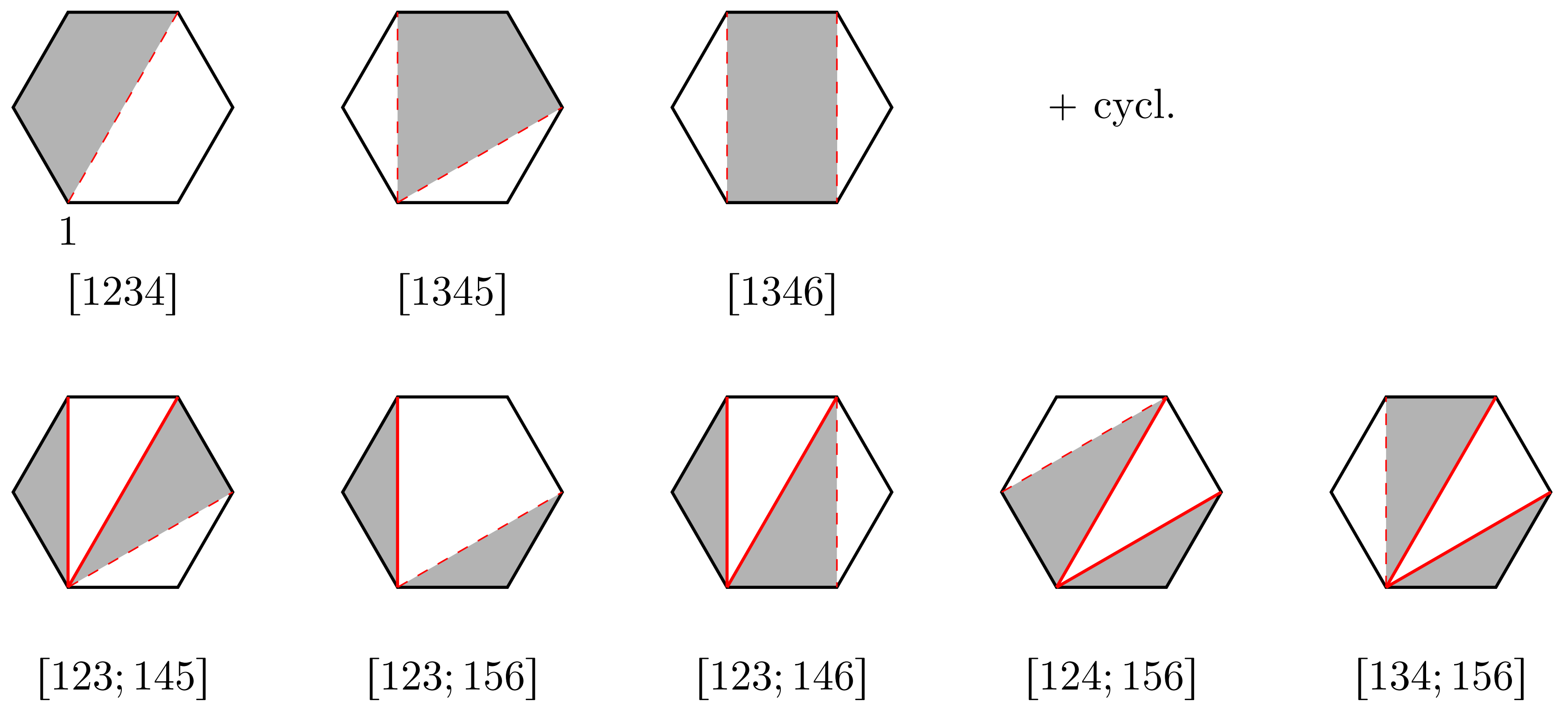


Figure 14. A LS basis for $F_6^{(L)}$ at $L \geqslant 2$. "+ cycl" in the first row means that there are 6 elements of [1234] type, 6 of [1345] type, 3 of [1346] type, so that together with the 5 elements from the second row we have 20 basis elements in total.

Symbol Bootstrap (1/2)

20 leading singularities. 945 weight-four symbols.

[Brown, JMH, Mazzucchelli, Trnka, 2503.1785]

Transcendental weight	1	2	3	4
# All symbols	9	62	319	945
# Two-loop six-point symbols	9	62	266	639
# Two-loop five-point one-mass symbols	9	59	263	594
# One-loop squared symbols	9	59	221	428
# Genuine two-loop six-point symbols	0	0	3	45

Table 2. Counting of independent symbols for two-loop six-point massless planar Feynman integrals, cf. also reference [74].

[JMH, Matijašić, Miczaijka, Peraro, Xu, Zhang, 2501.01847]

$$f_n^{(L)} = \sum_{i,j} c_{i,j}\, r_{n,i}\, g_j^{(2L)}$$

We make an ansatz with free coefficients. We then determine them from symmetry and physical requirements. [Carrôlo, Chicherin, JMH, Yang, Zhang, 2505.01245]

Symbol Bootstrap (2/2)

[Carrôlo, Chicherin, JMH, Yang, Zhang, 2505.01245]

Ansatz: $$f_n^{(L)} = \sum_{i,j} c_{i,j}\, r_{n,i}\, g_j^{(2L)}$$

Constraints:

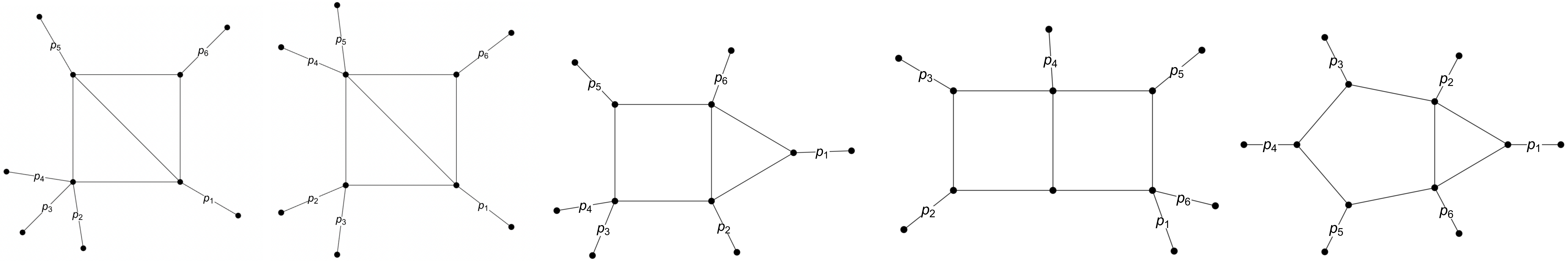
- Dihedral symmetry
- Scaling dimension
- Cancellation of spurious singularities
- Consistency with soft limit
- Consistency with (double and triple) collinear limit

weight	0	1	2	3	4
unknowns in dihedral ansatz	5	22	139	644	1892
genuine **unknowns**	4	20	125	585	1718
constraints:					
soft	3	20	116	515	1439
collinear	3	20	121	551	1539
spurious $s_{24} = 0$	1	12	76	360	1044
spurious $s_{25} = 0$	1	6	36	165	483
scaling dimension	0	4	20	125	585
triple collinear	1	5	31	134	353
total **constraints**	4	20	125	585	1718
unfixed **unknowns**	0	0	0	0	0

Table 4. Numbers of constraints following from each physical condition.

Discussion of the result

1) Only 137 of the 245 alphabet letters are needed. The two-loop letters that appear are associated to the following integral sectors:



(a) $\{W_{i+10}\}_{i=0,\cdots,5}$ (b) $\{W_{i+34}\}_{i=0,\cdots,11}$ (c) $\{W_{i+58}\}_{i=0,\cdots,11}$ (d) $\{W_{i+76}\}_{i=0,\cdots,11}$ (e) $\{W_{i+100}\}_{i=0,\cdots,5}$

2) Duality to all-plus Yang-Mills amplitudes implies nontrivial Steinmann relations:

$$\underset{s_{i,i+1,i+2}=0}{\text{Disc}}\ \underset{s_{i-1,i,i+1}=0}{\text{Disc}} \left(F_6^{(2)} + F_6^{(1)} \mathcal{H}_6^{(1)} \right) = 0\,.$$

3) Some leading singularity contributions are extremely simple:

$$G_{11}^{(2)} = \text{Pent}_{2,6} \times \text{Pent}_{3,5},\ G_{12}^{(2)} = \text{Pent}_{1,3} \times \text{Pent}_{4,6},\ G_{13}^{(2)} = \text{Pent}_{1,5} \times \text{Pent}_{2,4}\,.$$

Discussion and outlook

Our symbol result provides a first amplitude-type observable that uses the novel two-loop hexagon function space. Via the conjectured duality, this predicts the leading-weight terms of the *three-loop all-plus scattering amplitude*.

Interestingly, only 137 of the 245 alphabet letters are needed. Does this have an explanation in terms of *cluster algebras*?

[Cf. talks by Anastasia Volovich and Mark Spradlin]

Can one use the same method for bootstrapping pure Yang-Mills scattering amplitudes?

https://indico.mpp.mpg.de/e/symbology15

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Thank you!

henn@mpp.mpg.de

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