Frontiers of Multi-loop Multi-leg Feynman integrals

International Workshop
on New Opportunities for Particle Physics 2025
2025.07.19

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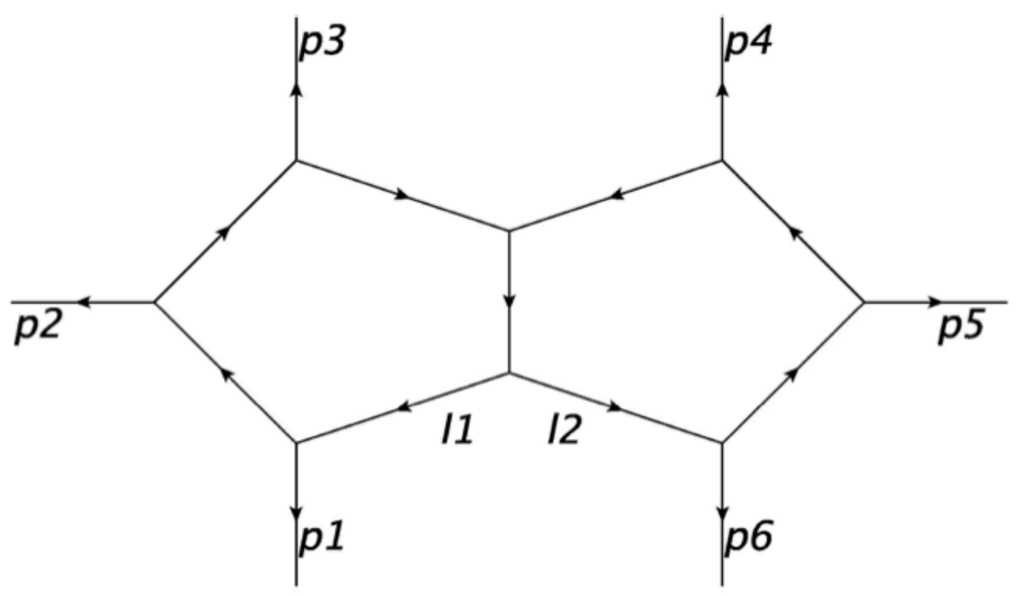
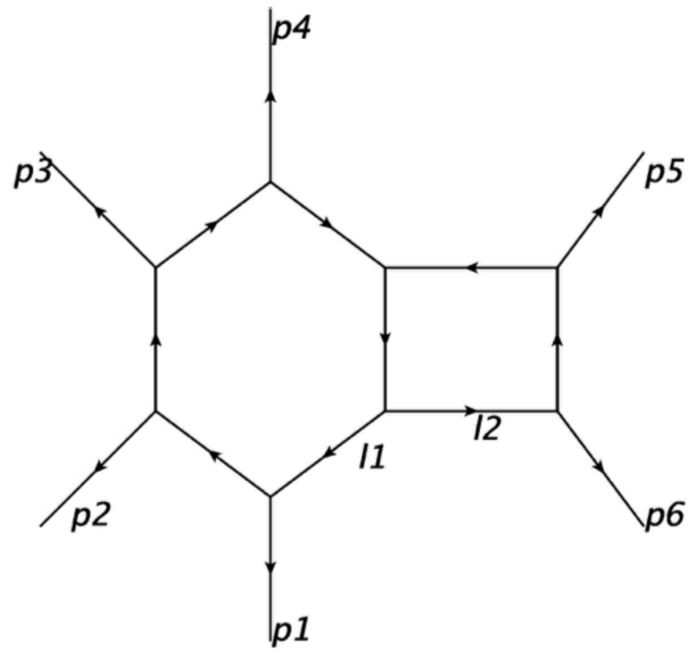
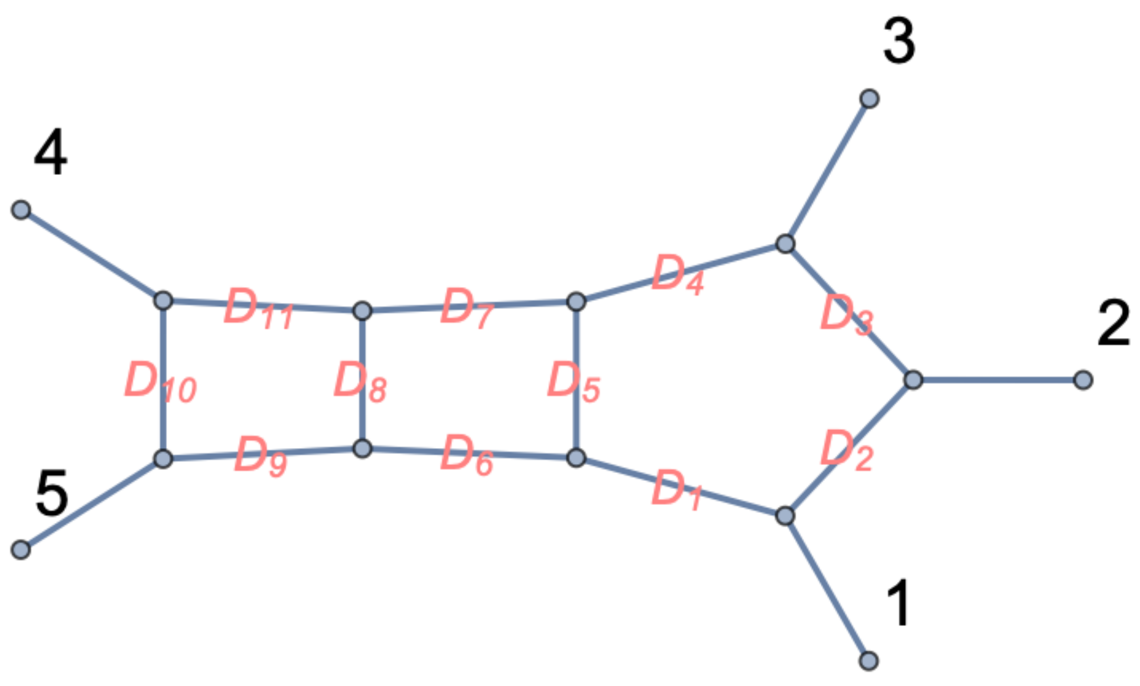
Based on

Feynman integral Integral evaluation





analytic computation of all 2loop 6point massless planar integrals is done

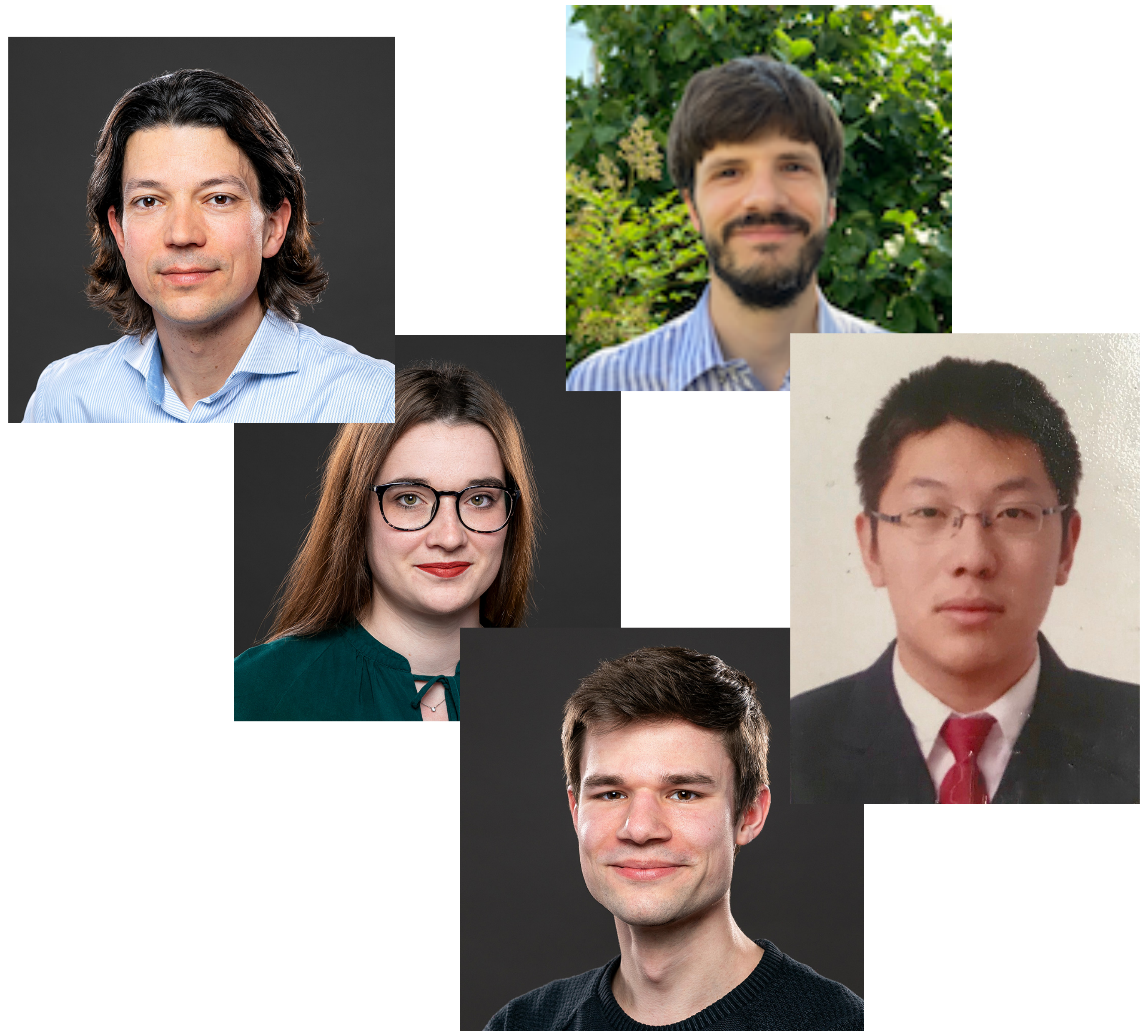


The first analytic computation of 3loop 5-point Feynman integral family

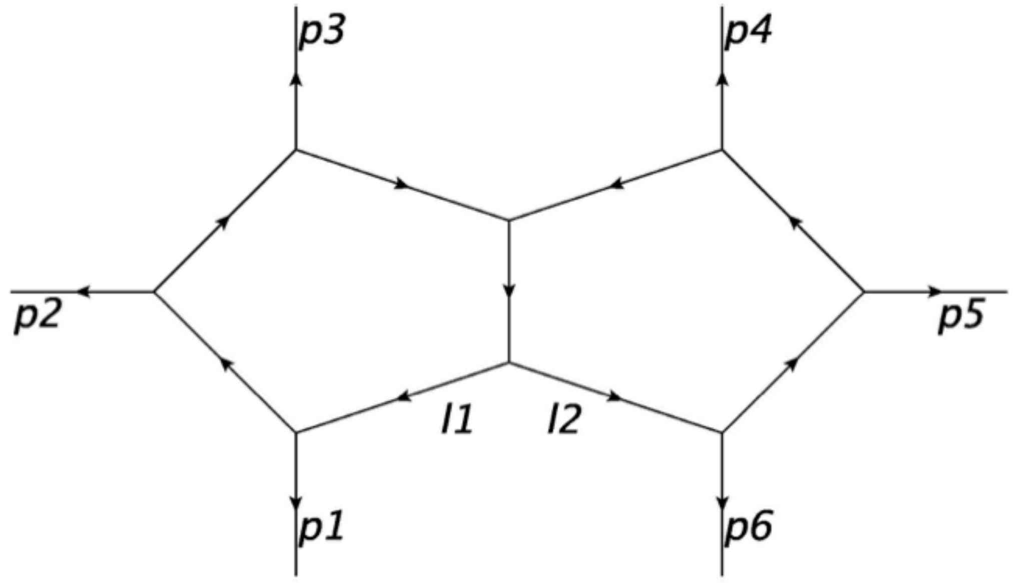
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135, 031601

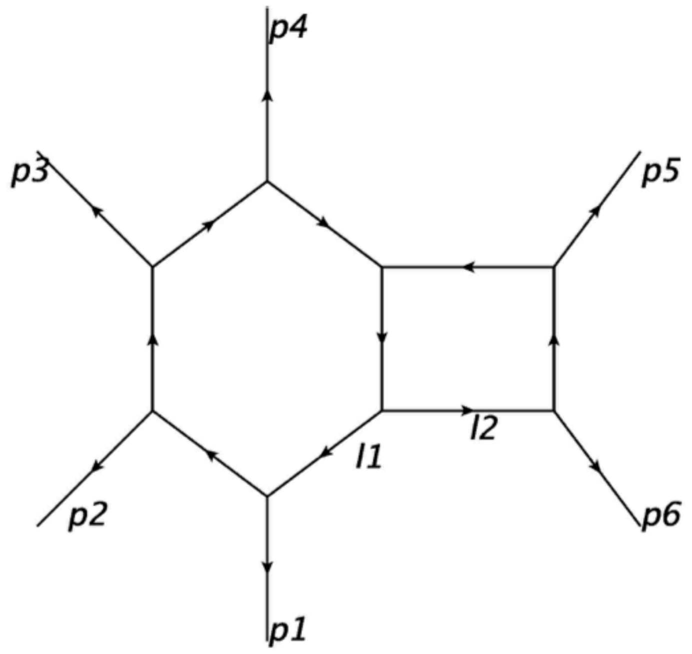
Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697 (Phys. Rev. D Editors' Suggestion)

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027



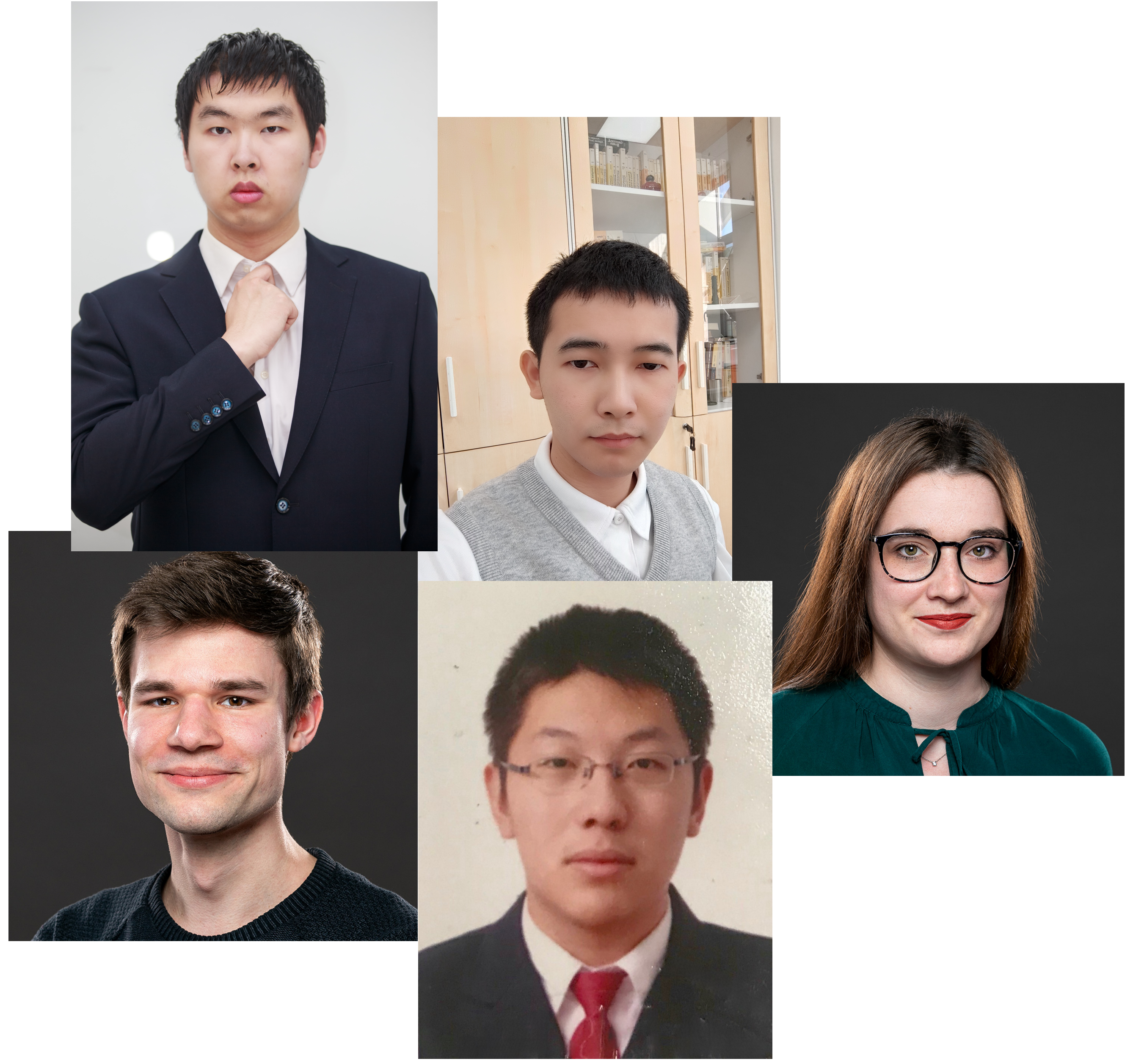
2loop 6point collaborators



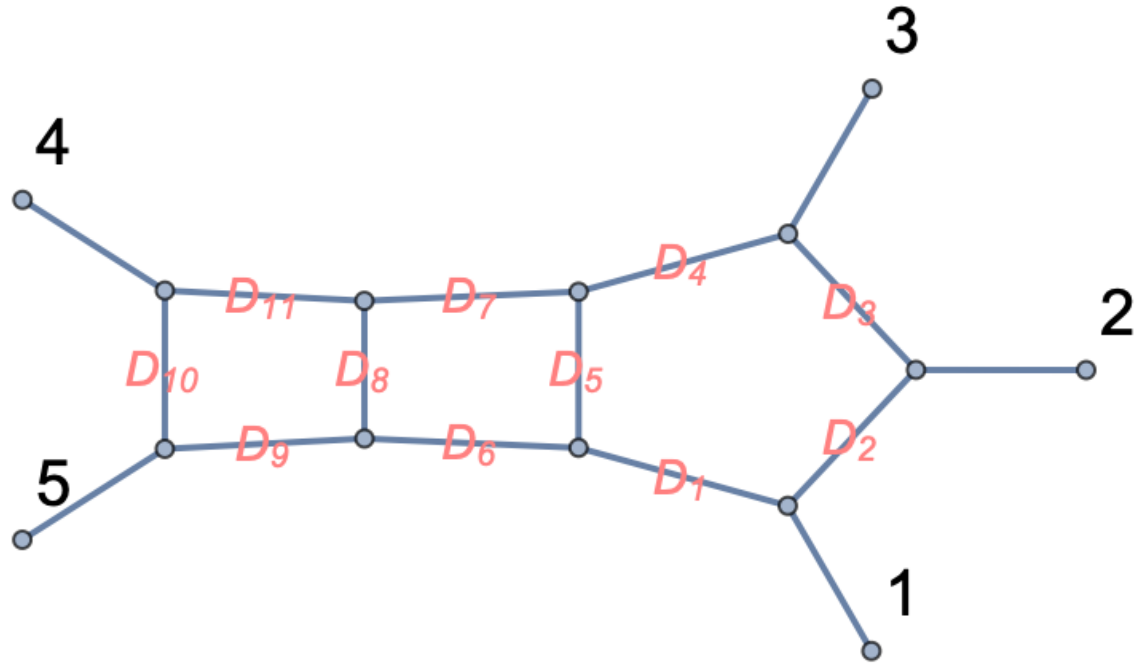


Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135, 031601

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027



3loop 5point collaborators



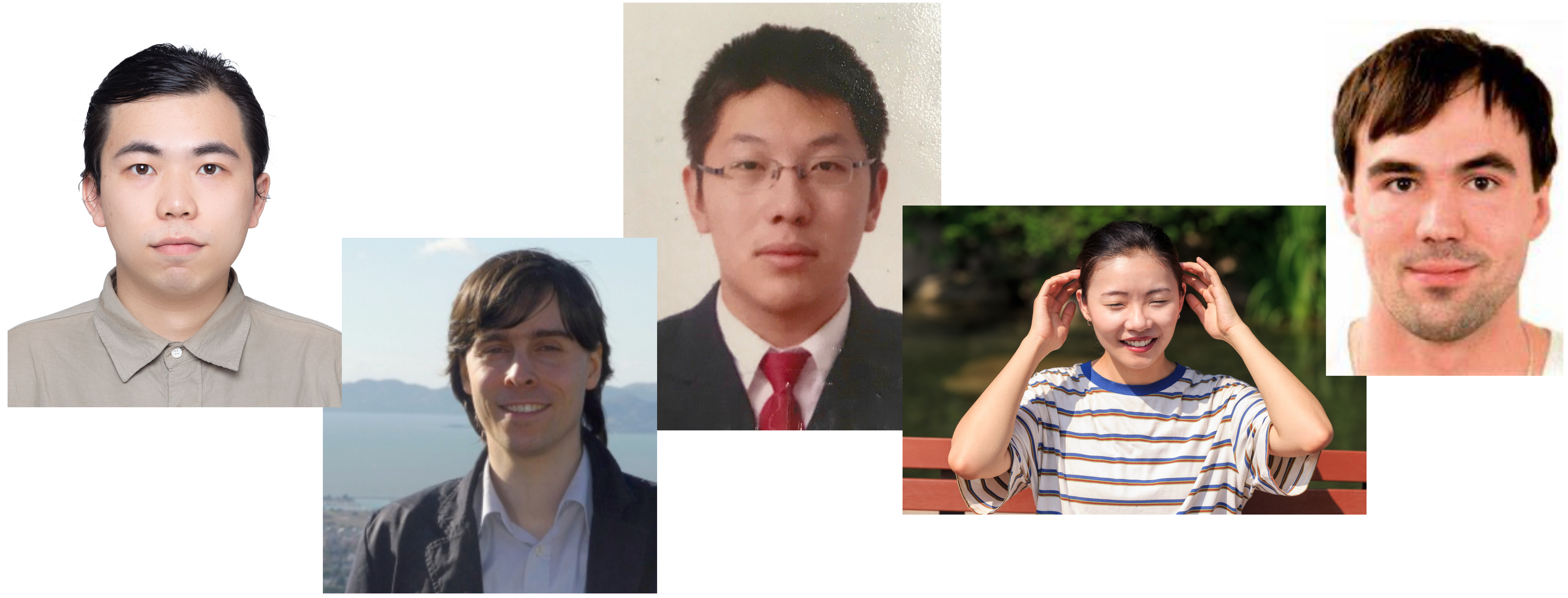
Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697

PRD editors' suggestion

Based on package development

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, *arXiv: 2502.20778*



NeatIBP **collaborators**

Outline

Introduction

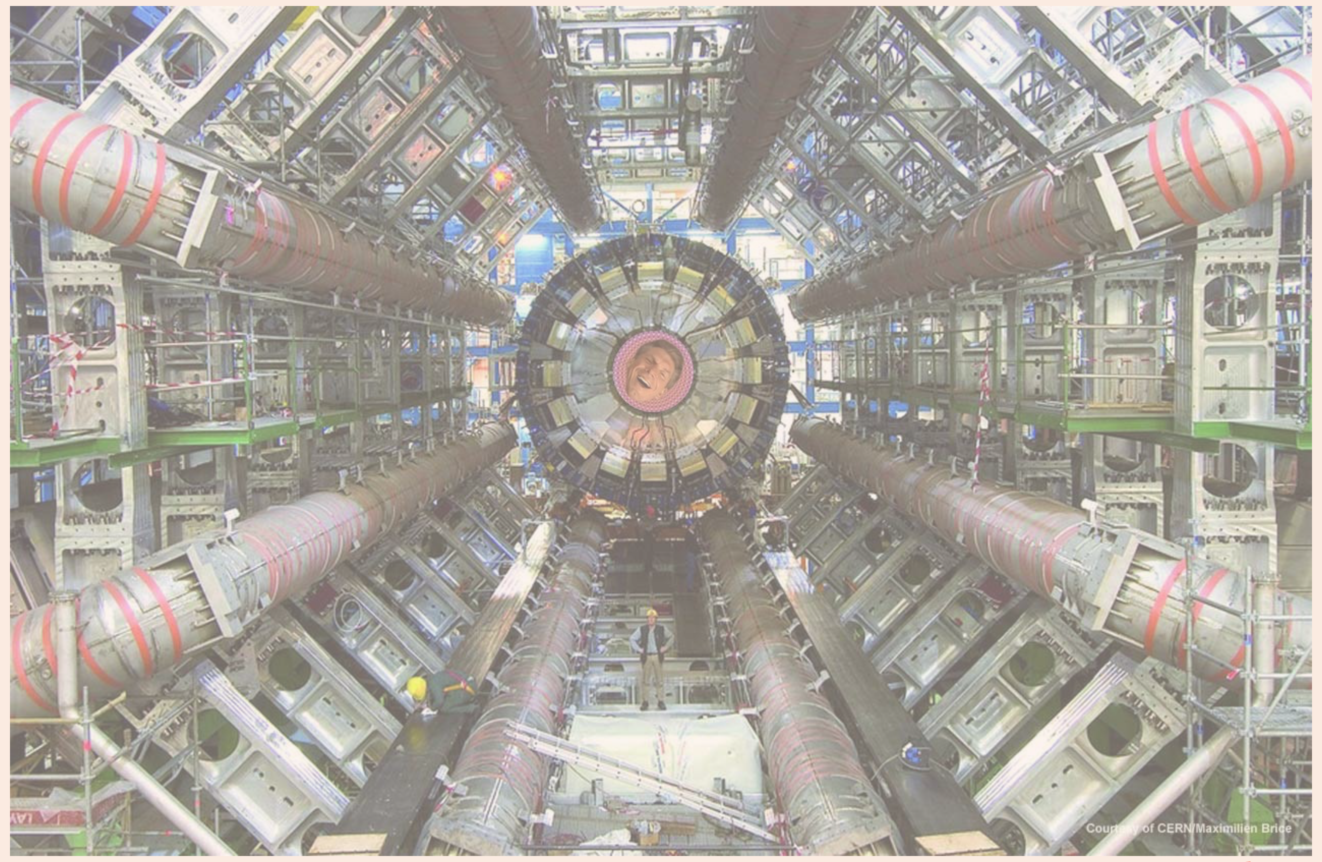
Methodology

Analytic Feynman integral frontiers

- Case 1: 2loop 6point Feynman integrals
- Case 2: 3loop 5point Feynman integrals

Summary and Outlook

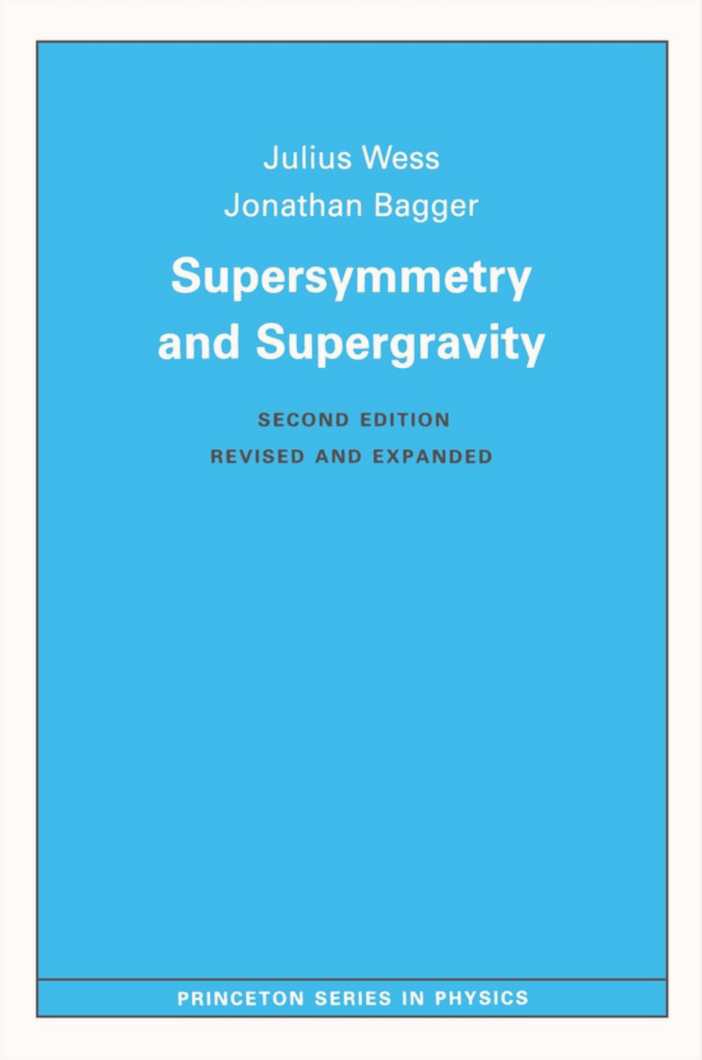
Introduction



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

Feynman integrals



Formal theory

N=8 supergravity UV finiteness

Gravitational wave template computations

for instance,
Driesse, Jakobsen , Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

Why *analytic*?

- Numeric method slow or not available yet for some multi-loop multi-leg Feynman integral

  3loop 5point Feynman integrals (numeric softwares do not finish the computing)

- Theoretical aspects of quantum field theory

  for examples: 2loop N=4 SYM theory spacelike splitting amplitude

  Henn, Ma, Xu, Yan, *YZ*, Zhu, arXiv 2406.14604

- Quantum field theory computation of gravitational wave

  analytic continuation/ Fourier transform is sometimes needed

Analytic Feynman integral computation and the methodology

“工欲善其事，必先利其器”

《论语．卫灵公》

A craftsman who wants to do his work well must first sharpen his tools.
— The Analects: Duke Ling of Wei

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
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Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601
JHEP 08(2024) 027, arXiv:2501.01847
Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
arXiv:2411.18697 PRD editors' suggestion

Goal of analyticity

Feynman integral

arguments related to
Letters, algebraic function of kinematics

$$I = \sum_{i=-2L} \epsilon^i \sum_{\alpha} c_\alpha G(W_{\alpha_1}, \ldots, W_{\alpha_{2L+i}}; z)$$

Goncharov polylogarithm function

Dimensional regularization parameter

$$G(\mathbf{0}_k; z) = \frac{1}{k!}(\log z)^k, \quad G(a_1, \ldots, a_k; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots a_k; t)$$

well studied function with Hopf algebra structure

For more complicated cases, iterative integral of elliptic functions, Calabi-Yau functions can appear

Canonical Differential Equation, new insights

Better Integration-by-parts (IBP) reduction

Blade, Guan, Liu, Ma, Wu 2024

Comput.Phys.Commun. 310 (2025) 109538

NeatIBP, Wu, Boehm, Ma, Xu, *YZ* 2023

Comput.Phys.Commun. 295 (2024) 108999

Alphabet searching

Effortless, Matijasic, Miczajka to appear

https://github.com/antonela-matijasic/Effortless

BaikovLetter, Jiang, Liu, Xu, Yang, 2401.07632

PLD, Fevola, Mizera, Telen

Comput. Phys. Commun. 303 (2024) 109278

SOFIA Correia, Giroux, Mizera

2503.16601

Solving differential equation

Novel representation of one-fold integration

Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697

Using NeatIBP

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, arXiv: 2502.20778

Using algebraic geometry (see Kosower's talk) to find short IBP system, 2 or 3 orders of magnitudes shorter than that from Laporta algorithm.

Studies that used NeatIBP	**Reference**
Differential Equations for Energy Correlators in Any Angle	arXiv:2506.02061
One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$ productions at the LHC through $O(\epsilon^2)$	arXiv:2505.10406
Two-loop Feynman integrals for leading color $Wt\bar{t}$ production	JHEP 07 (2025) 001
Two-loop QCD helicity amplitudes for $gg \to gt\bar{t}$ at leading color	JHEP 03 (2025) 070
Full-color double-virtual amplitudes for $q\bar{q} \to b\bar{b}H$	JHEP 03 (2025) 066
Three-loop five-point pentagon-box-box Feynman diagram	arXiv:2411.18697
Two-loop QCD corrections for $pp \to t\bar{t}j$	arXiv:2411.10856
Two-loop amplitudes for $W\gamma\gamma$ production at LHC	JHEP 12 (2025) 221
NLO corrections to $J/\Psi c\bar{c}$ photoproduction	Phys.Rev.D 110 (2024) 9, 094047
Two-loop five-point two-mass planar integrals	JHEP 10 (2024) 167
Two-loop integrals for $t\bar{t}j$ production at hadron colliders in the leading color approximation	JHEP 07 (2024) 073

phenomenology application of NeatIBP

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135, 031601*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

2loop Feynman integral: Scale frontier

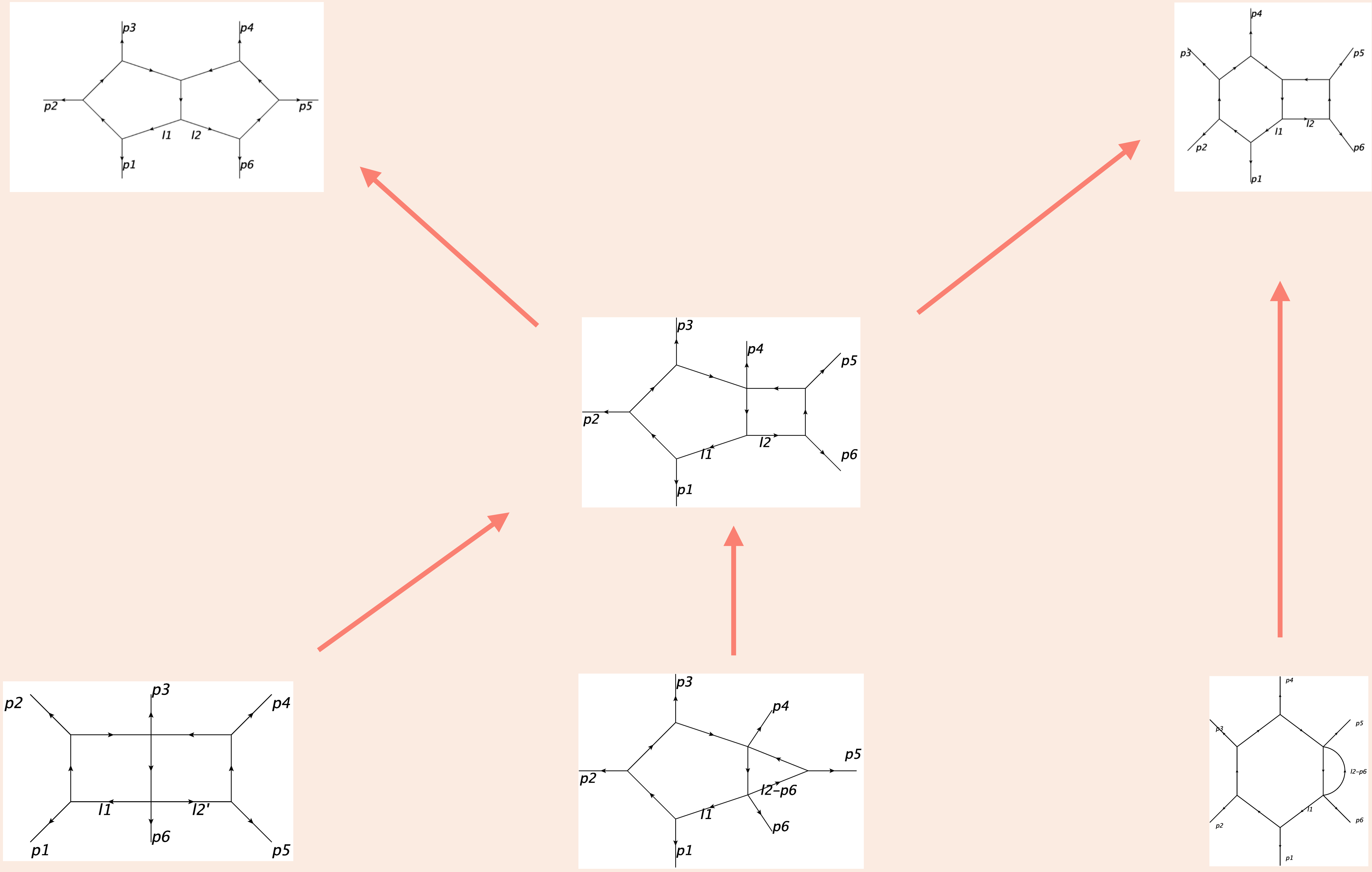
2loop 5point massless	*Gehrmann, Henn, Lo Presti 2015* *Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019*	5 scales
2loop 5point one-mass	*Papadopoulos, Tommasini, Wever 2019* *Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020* *Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023* *Jiang,Liu,Xu,Yang 2024* *Badger, Becchetti,Giraudo, Zoia 2024*	6 scales
2loop 5point two-mass	*Cordero, Figueiredo, Kraus, Page and Reina 2023* for leading-Color pp→ttH amplitudes with a light-quark loop	7 scales
2loop 6point massless	*Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2025* *for NNLO 4 jets production, 2 jets+ 2photons*	8 scales!

$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

All planar 2loop 6point integrals

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, PRL. 135, 031601

267
UT integrals



202
UT integrals

Momentum Twistor

external momenta $d=4$

$$p_i = x_{i+1} - x_i$$

dual coordinates

$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma}\lambda_{A,\gamma} = \mu_{A,\dot\beta}$$
$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma}\lambda_{B,\gamma} = \mu_{B,\dot\beta}$$

$$(Z_A, Z_B) \to x \qquad (Z_i, Z_{i+1}) \to x_{i+1}$$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1, \ldots 6$$

momentum twistor

$\langle Z_A Z_B Z_C Z_D \rangle$ is dual conformally invariant

$$\tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle\langle i-1,\, i\rangle},$$

$$(\lambda_{i,\alpha}, \tilde\lambda_{i,\dot\alpha})$$

spinor helicity

Momentum Twistor   external momenta $d=4$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1, \ldots 6 \qquad \tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle},$$

A particular parameterization   Badger, Frellesvig, YZ 2013

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_2 x_3 x_1} + \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_2 x_3 x_1} + \frac{1}{x_2 x_3 x_4 x_1} + \frac{1}{x_1} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_5}{x_2} & x_6 & 1 \\ 0 & 0 & 1 & 1 & x_7 & 1 - \frac{x_8}{x_5} \end{pmatrix}$$

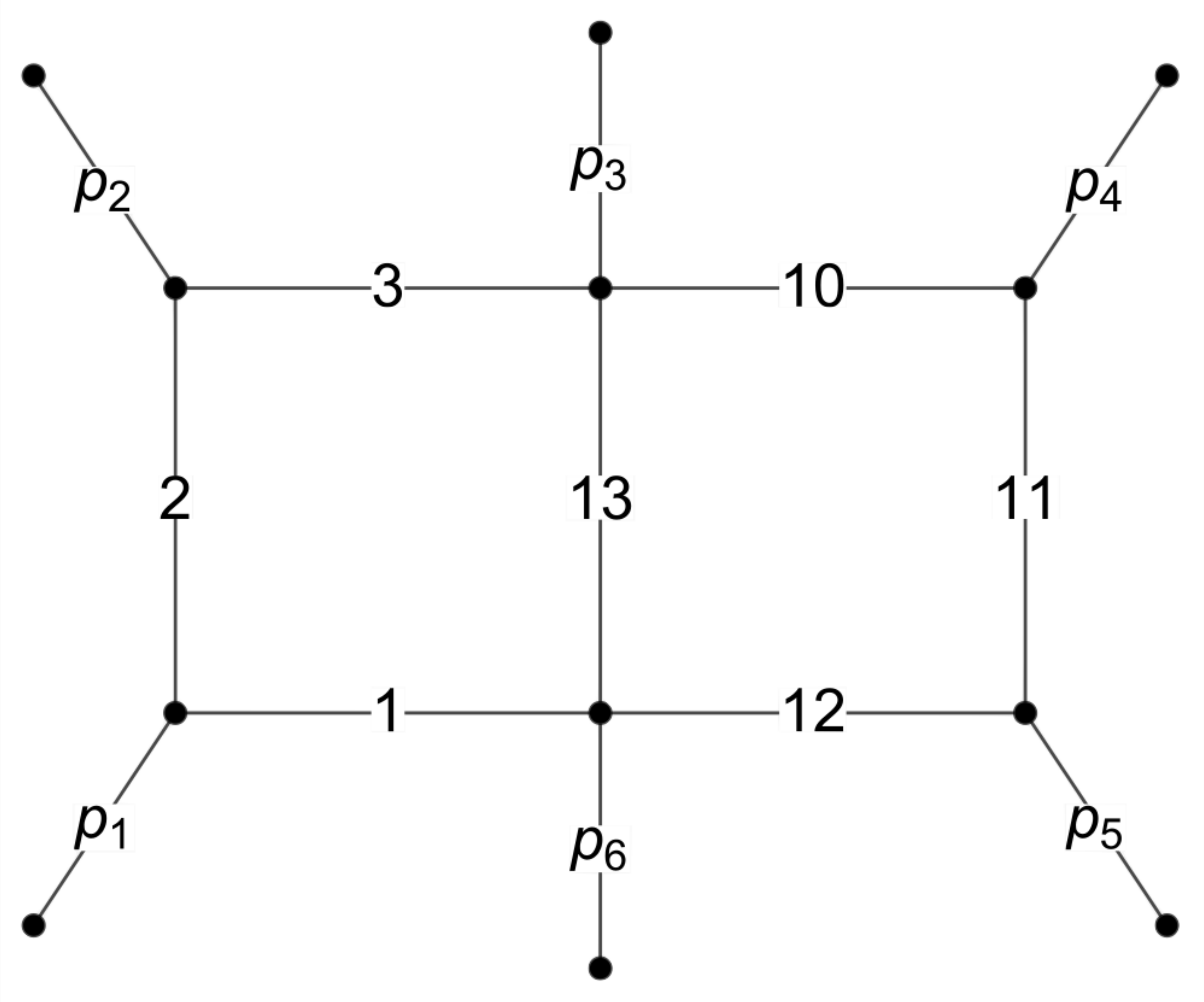
$$x_1 = s_{12}$$
$$x_2 = -\frac{\mathrm{Tr}_+(1234)}{2 s_{12} s_{34}}$$
$$x_3 = -\frac{\mathrm{Tr}_+(1345)}{2 s_{45} s_{13}}$$
$$x_4 = -\frac{\mathrm{Tr}_+(1456)}{2 s_{56} s_{14}}$$
$$x_5 = \frac{s_{23}}{s_{12}}$$
$$x_6 = -\frac{\mathrm{Tr}_+(1532) + \mathrm{Tr}_+(1542)}{2 s_{15} s_{12}}$$
$$x_7 = 1 + \frac{\mathrm{Tr}_+(1542) + \mathrm{Tr}_+(1543)}{2 s_{15} s_{23}}$$
$$x_8 = \frac{s_{123}}{s_{12}}$$

Momentum parametrization rationalizes all pseudo scalars

$$\epsilon_{ijkl} \equiv 4i\epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma , \quad \epsilon_{ijkl}^2 = G_{ijkl}$$

Uniformly transcendental (UT) basis determination

key step



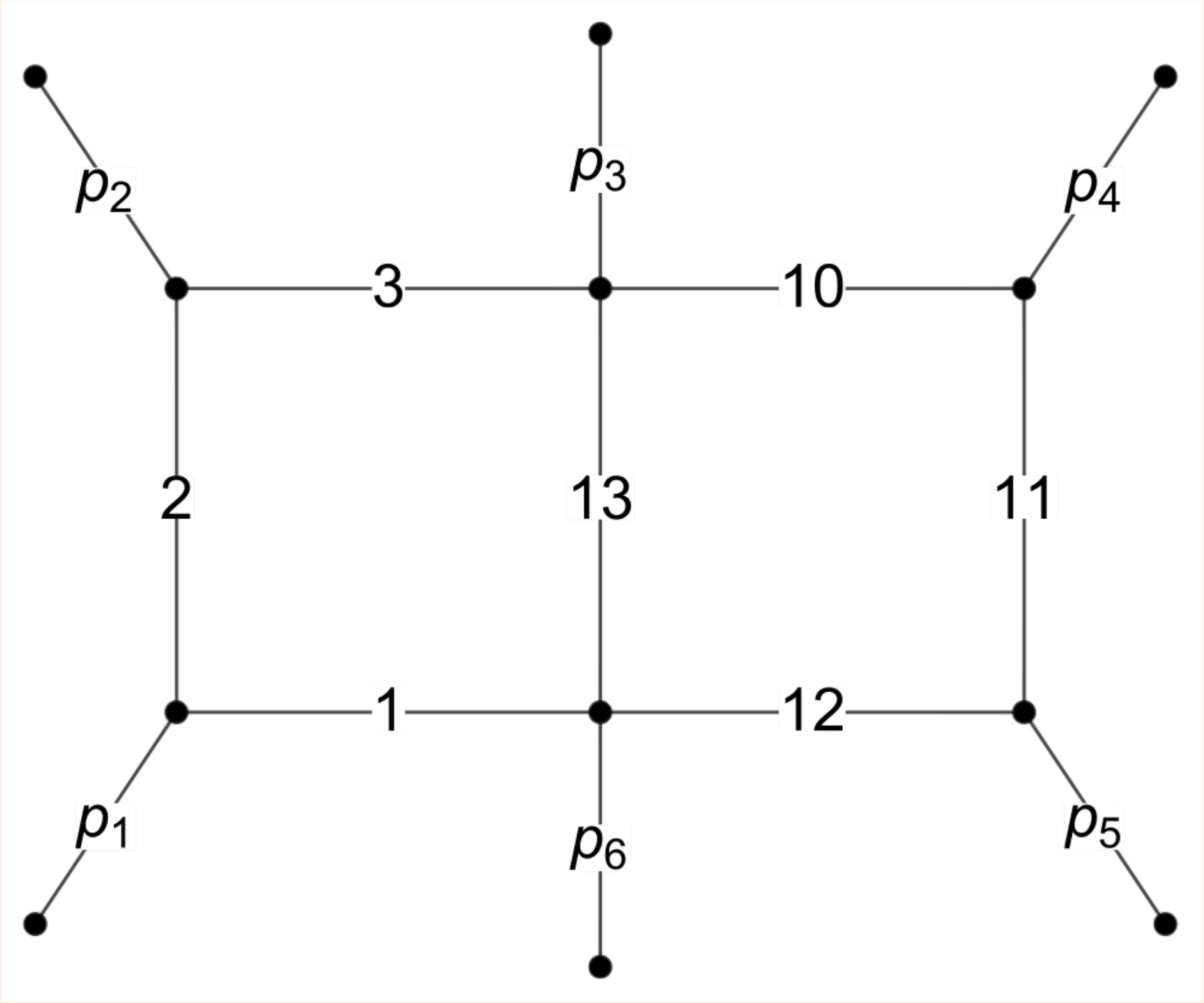
$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}, \qquad i = 1, \ldots, 7$$

$$\begin{aligned}
N_1 &= -s_{12} s_{45} s_{156}\,, \\
N_2 &= -s_{12} s_{45} (l_1 + p_5 + p_6)^2\,, \\
N_3 &= \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix}, \\
N_4 &= \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix}, \\
N_5 &= -\frac{1}{4} \frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix}, \\
N_6 &= \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11} (s_{123} + s_{126})}{8}, \\
N_7 &= -\frac{1}{2\epsilon} \frac{\Delta_6}{G(1,2,4,5) D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.
\end{aligned}$$

Chiral numerator
(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)
/ Gram determinant
correspondence

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

Chiral numerator to UT integral numerators



linear combination

$$\mathcal{N}_A = s_{45}\big(\langle 15\rangle[52] + \langle 16\rangle[62]\big) l_1 \cdot (\lambda_2 \tilde{\lambda}_1)\,,$$
$$\mathcal{N}_B = s_{45}\big([15]\langle 52\rangle + [16]\langle 62\rangle\big) l_1 \cdot (\lambda_1 \tilde{\lambda}_2)\,.$$

parity even

$$\mathcal{N}_A + \mathcal{N}_B = -\frac{1}{2} s_{12} s_{45} (l_1 + p_5 + p_6)^2 + \frac{1}{2} s_{12} s_{45} s_{156} + \ldots$$

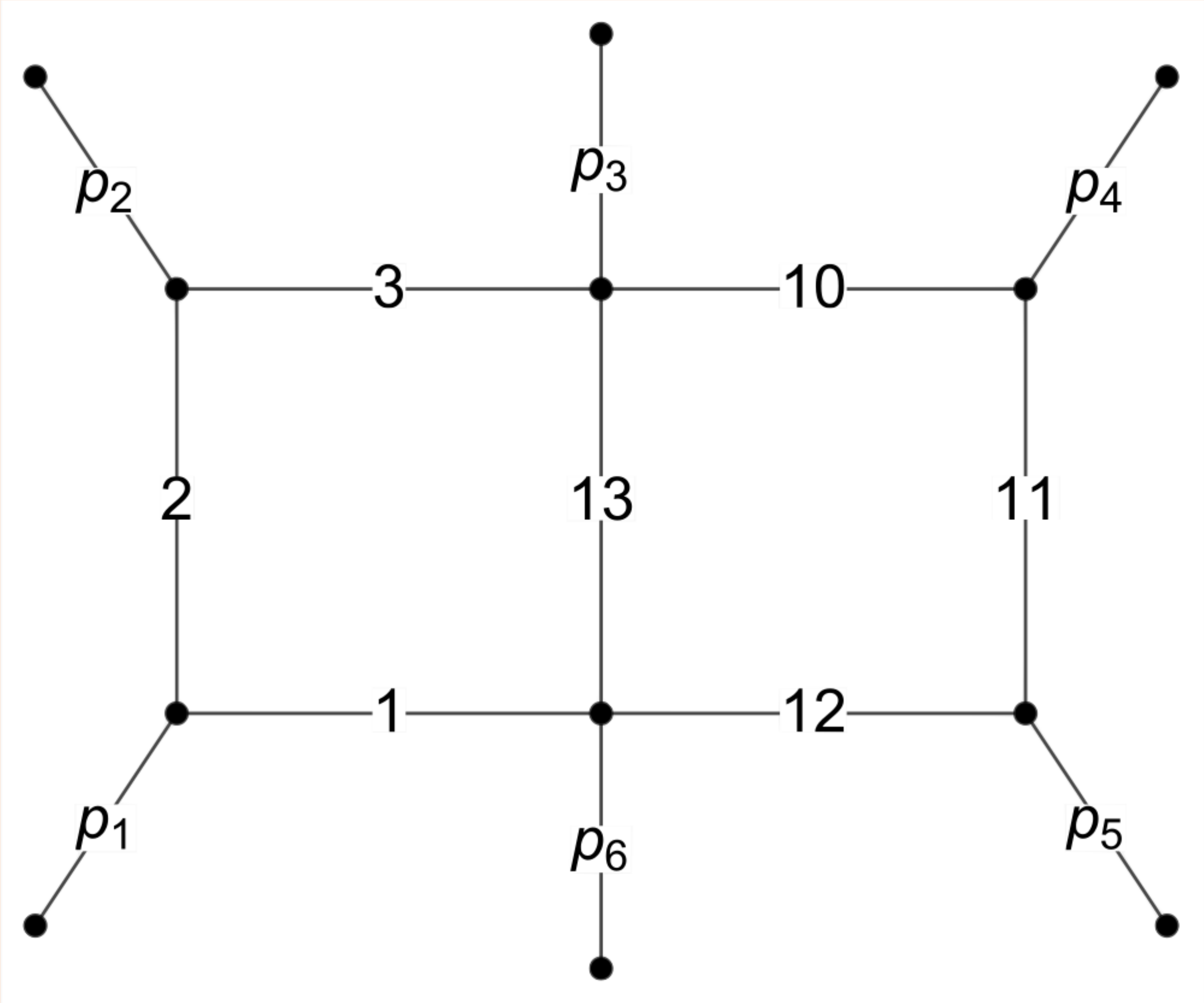
parity odd

$$\mathcal{N}_A - \mathcal{N}_B = \frac{-8 s_{45} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_5 & p_1 & p_2 & p_6 \end{pmatrix}}{\epsilon_{5126}},$$

Chiral numerator to UT integral numerators

quadratic combination

$$s_{24}\frac{\langle 15\rangle}{\langle 42\rangle}(l_1\cdot\lambda_2\tilde\lambda_1)(l_2'\cdot\lambda_4\tilde\lambda_5)$$



parity even

$$\frac{1}{8}G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5\end{pmatrix}+\frac{D_2D_{11}(s_{123}+s_{126})}{8}$$

additional term added
from the canonical DE construction

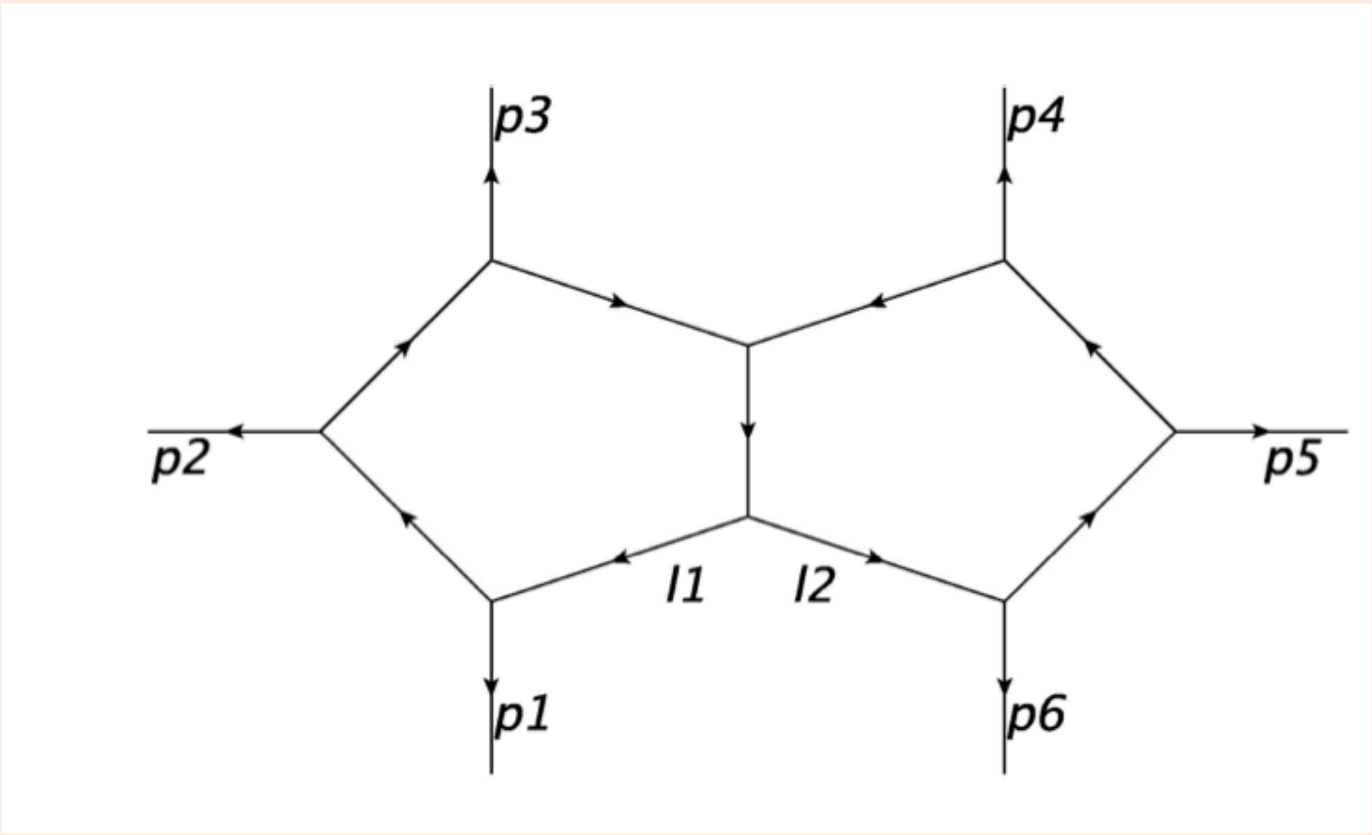
One-loop
hexagon leading singularity

parity odd

$$\Delta_6=\langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61]-\langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

$$-\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}}G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5\end{pmatrix}$$

2loop 6point top sector, UT integrals



J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, PRL 135, 031601

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9}$$ → evanescent

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9}$$ → evanescent

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9}$$ → evanescent, 6D weight-6 integral

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1 \dots D_9}$$

5 MIs (this sector)
267 MIs (whole family)

245 letters in total
except the 6D ones

"evanescent": vanishing up to ϵ^0

N_1, N_2, N_3 and N_4 are chiral numerators — Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

Complete canonical differential equation for 2l6p planar integrals

Use momentum twistor Variables

267 × 267 for double pentagon
202 × 202 for hexagon box

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

use alphabet to fit the canonical differential equation

Even letter, Odd letter and the more complicated ...

Even letter $F(s)$ a polynomial in Mandelstam variables or homogeneously linear in square roots

Conjecture: a Feynman integrals' even letters are all from Landau singularity?

Odd letter $\frac{P(s) - \sqrt{Q(s)}}{P(s) + \sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": ϵ_{ijkl}, Δ_6, , $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

ϵ_{ijkl} → pseudo scalar

Δ_6 → leading singularity hexagon

$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$ → Källin function from massive triangle diagrams

More complicated letter $\frac{P(s) - \sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s) + \sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

A new algorithm to search for odd letters

Odd letter $\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$

$$P^2 - Q = c\prod_i W_i^{e_i}, \quad c \in \mathbb{Q}, \quad e_i \in \mathbb{N}$$

Even letter

An observation (and conjecture) from Heller, von Manteuffel, Schabinger 2020

Matijasic, Miczajka, to appear

Algorithm to solve for e_i

Effortless

https://github.com/antonela-matijasic/Effortless

Even letter, Odd letter and the more complicated …

245 letters

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\ldots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\ldots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}$$
$$\ldots$$
$$\frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}},$$
$$\ldots \quad \ldots$$
$$\frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6}$$

Then the canonical DE is derived analytically after ~50 times of numeric IBP running

10 More complicated letters

$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \qquad \ldots$$

Boundary Values

Numeric boundary values

It is fine to use the package AMFlow to get ~100 digits as the boundary value for double-box, pentagon-triangle, hexagon-bubble diagrams

Liu, Wang, Ma, 2018
Liu, Ma 2022

Analytic boundary values

It is still possible to get *fully analytic* boundary values

$$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$$

Solve the canonical DE on a curve starting with X_0 and require the finite solution
Some known integrals' boundary values
} analytic boundary value

boundary value for a point in the physical region also obtained

Boundary Values

Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$\epsilon^4 I_{\mathrm{db},1}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$\epsilon^4 I_{\mathrm{db},2}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$I_{\mathrm{db},3}(X_0) = I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0,$$

$$\epsilon^4 I_{\mathrm{db},6}(X_0) = -\left(\frac{\pi^4}{540} + \frac{4}{3}\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$\epsilon^4 I_{\mathrm{db},7}(X_0) = 0.$$

from the ordinary differential equation

spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Solution of canonical DE

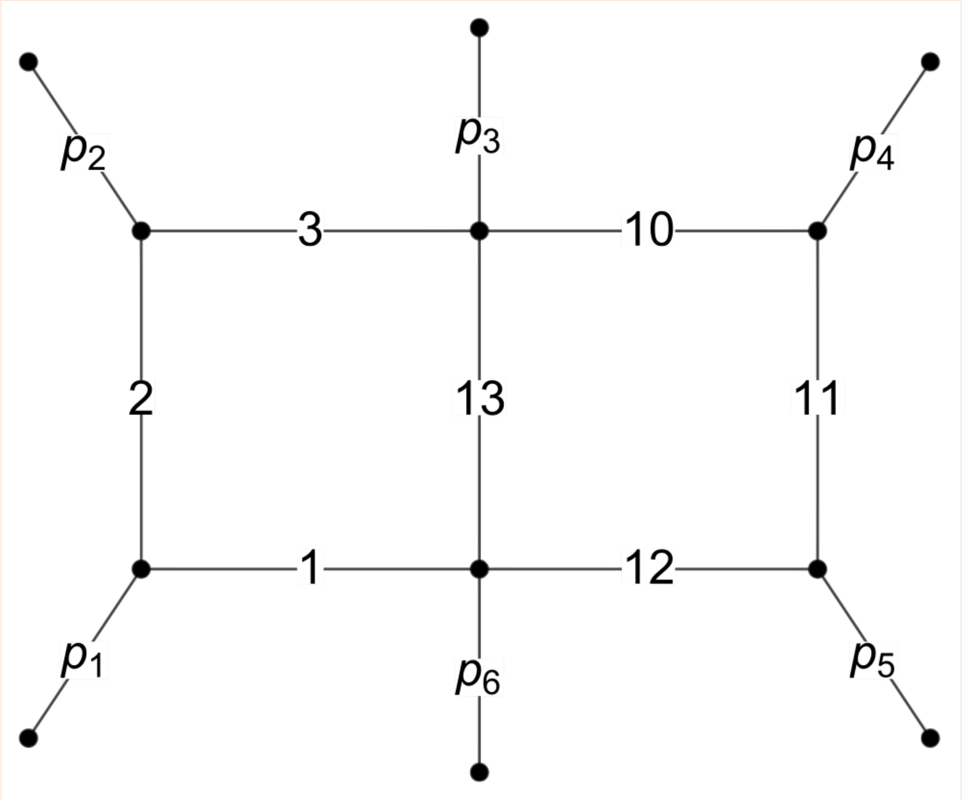
$$dI = \epsilon (d\tilde{A}) I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-1, weight-2

All in logarithm and classical poly-logarithm



$$\begin{aligned} I^{(2)}_{\mathrm{db},1} = {} & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\ & \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\ & \log(-v_2)\log(-v_4) - \mathrm{Li}_2\left(1 - \frac{v_2 v_5}{v_7 v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\ & 2\mathrm{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\ & \mathrm{Li}_2\left(1 - \frac{v_3 v_6}{v_8 v_9}\right) - 2\mathrm{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\ & \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\ & \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\mathrm{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\ & \log^2(-v_5) - 2\mathrm{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\ & \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\ & \frac{\pi^2}{6} \end{aligned}$$

Solution of canonical DE

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-3, weight-4

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1}\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2}\vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{f}^{(2)}(t)\right).\end{aligned}$$

one-fold integration

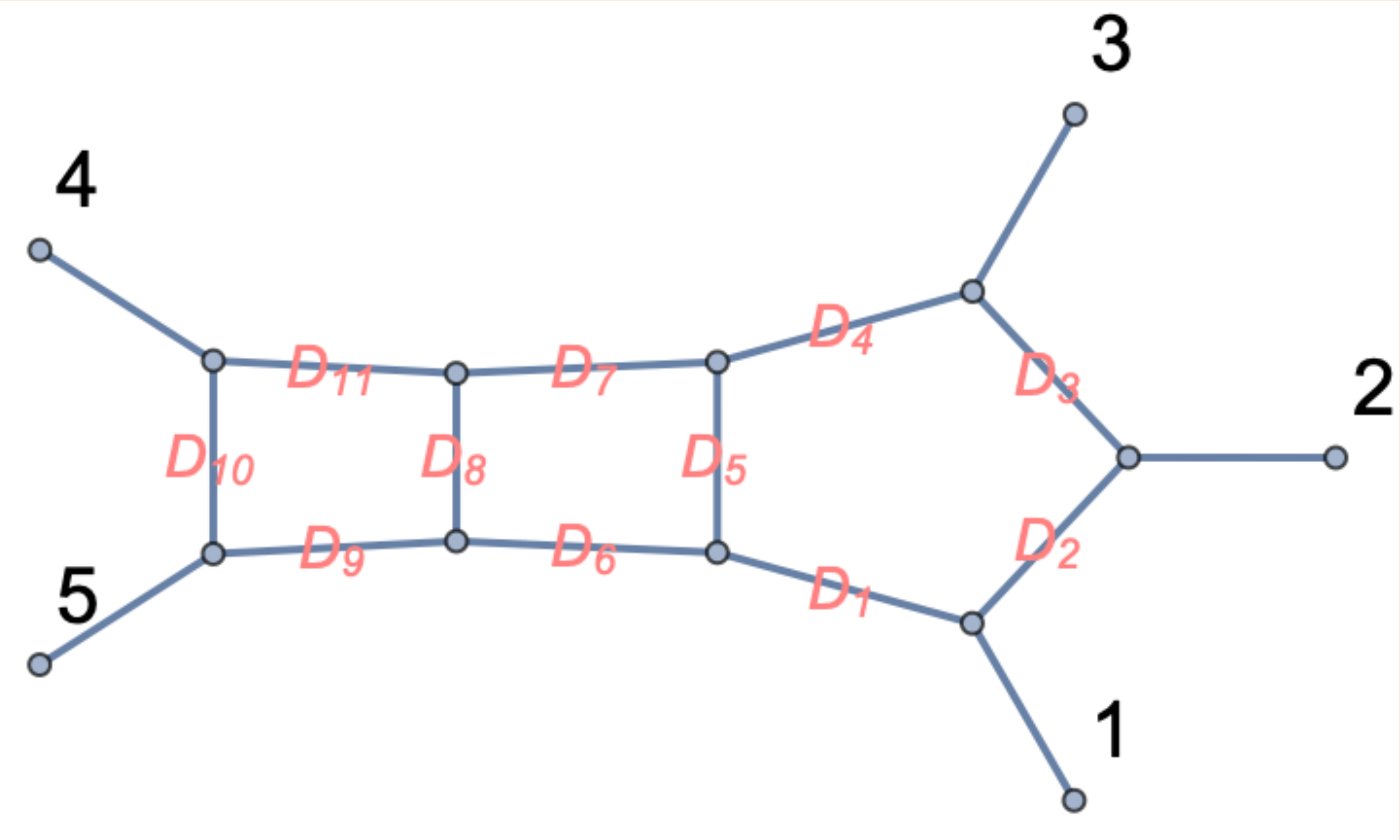
It takes minutes on a laptop to get 14 digits for all 2loop 6point integrals

from our solution in both Euclidean and Physical regions

3loop 5point Feynman integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, arXiv: 2411.18697 PRD editors' suggestion

3loop 5point planar family



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, arXiv: 2411.18697

✓ UT basis found!

Baikov analysis
Gram determinant

Canonical differential equation complicated ?

We use NeatIBP to derive the differential equation

~100 million IBPs → 85000 IBPs

hard to integrate to weight-6?

A novel one-fold representation

A novel representation of iterative integrals

$$
\begin{aligned}
\mathbf{I}^{(n+2)}(x) =& \mathbf{I}^{(n+2)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t} \mathbf{I}^{(n+1)}(x_0)\mathrm{d}t \\
&+ \int_0^1 (\tilde{A}(1) - \tilde{A}(t)) \frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t} \mathbf{I}^{(n)}(t)\mathrm{d}t.
\end{aligned}
$$

Weight $+2$

A novel formula

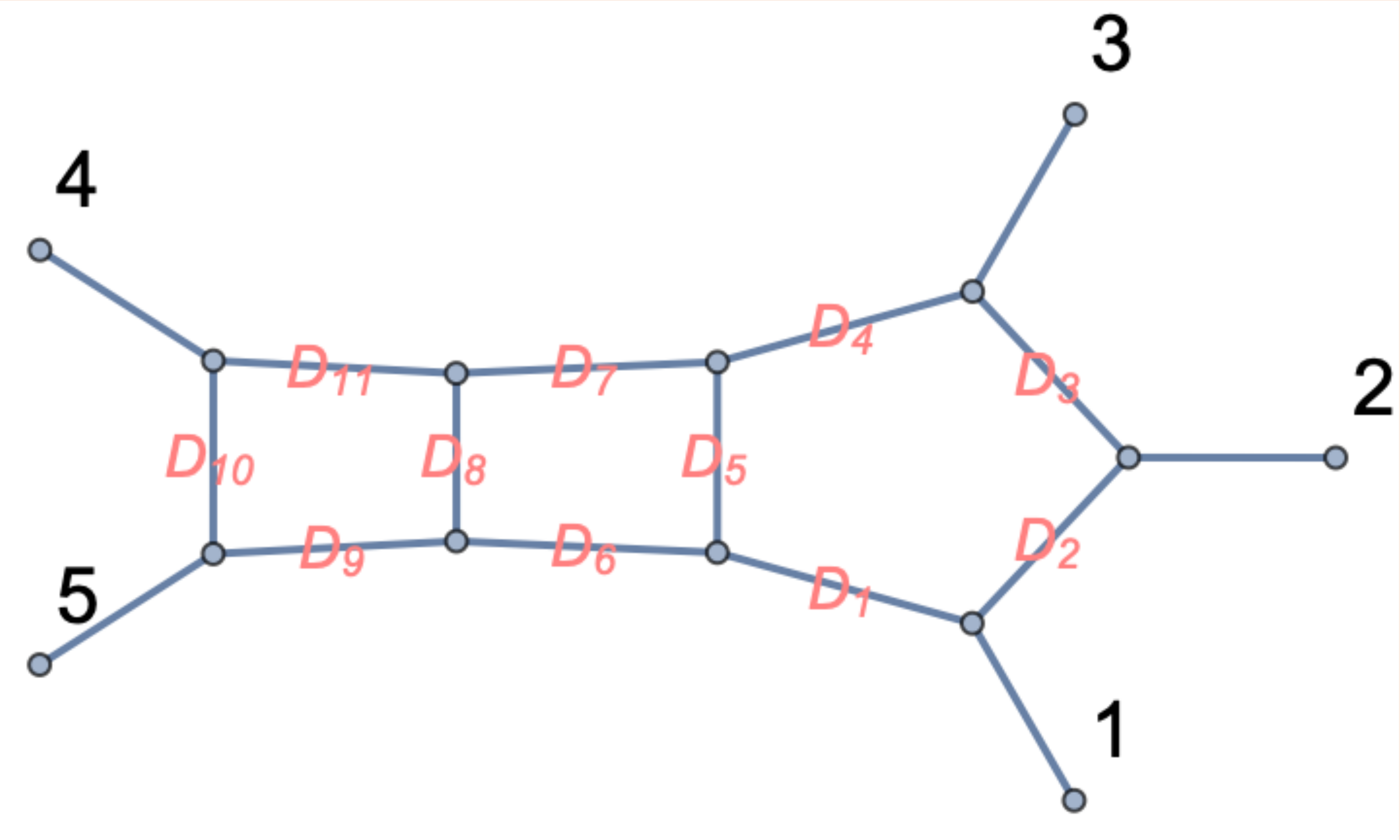
$$
\begin{aligned}
\mathbf{I}^{(n+3)}(x) =& \mathbf{I}^{(n+3)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n+2)}(x_0)\mathrm{d}t \\
&+ \int_0^1 \left(\tilde{A}(1) - \tilde{A}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n+1)}(x_0)\mathrm{d}t \\
&+ \int_0^1 \left(\tilde{A}(t) - \tilde{A}(1)\right) \tilde{A}(t) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n)}(t)\mathrm{d}t \\
&+ \int_0^1 \left(\tilde{B}(1) - \tilde{B}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \mathbf{I}^{(n)}(t)\mathrm{d}t.
\end{aligned}
$$

Weight $+3$

$$
\mathrm{d}\tilde{B} = (\mathrm{d}\tilde{A})\tilde{A},
$$

$\tilde{B}$ exists due to Poincare lemma

First family of 3loop 5point calculated



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

✔ UT basis found!

✔ Canonical differential equation found with NeatIBP

31 letters … All boundary values up to weight-6 are obtained by spurious pole analysis

✔ weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

Liu, Matijasic, Miczajka, Xu, Xu, YZ, PRD editors' suggestion

Summary and Outlook

今兵威已振，譬如破竹，数节之后，皆迎刃而解
《晋书 · 杜预传》

Now the situation is like splitting a bamboo stalk—
once you break through the first few sections, the rest will yield of itself.
— The Book of Jin: Biography of Du Yu

Analytic computation of all 2loop 6point planar massless integrals is done
The first computation on 3loop 5point family is done; all families' result is coming

NeatIBP, a powerful package for cutting-edge IBP reduction

a lot of future applications

Wilson loop bootstrap computation (see Henn's talk)

future

2-loop amplitude for four jets (photons) production
2-loop multi-collinear analysis
3-loop amplitude for three jets (photons) production
3-loop infrared structure studies
…

Multi-loop multi-leg Feynman integrals are no longer that difficult!