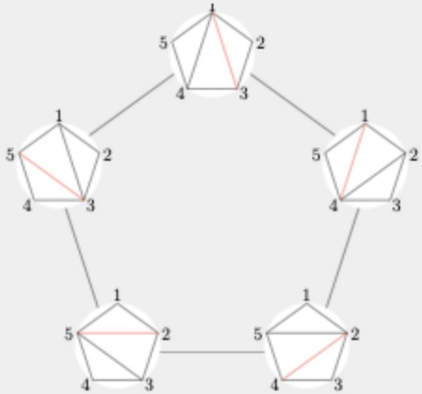
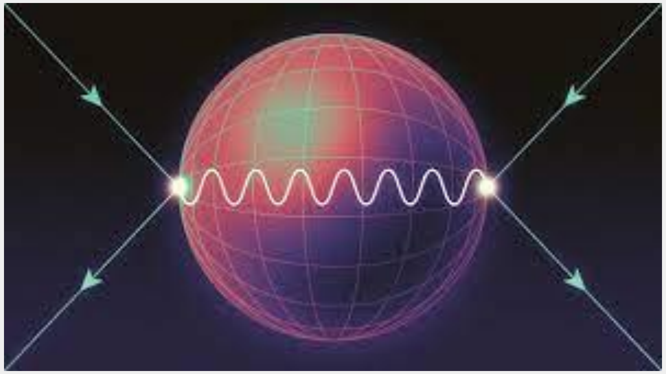
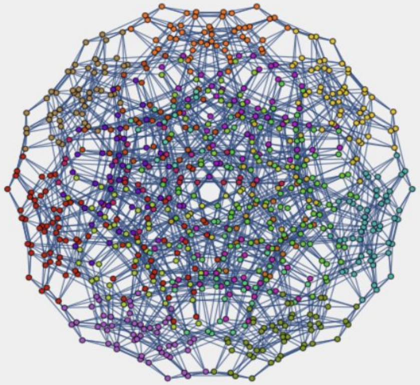
# Mathematical Structures
# in
# Scattering Amplitudes

**Anastasia Volovich**
**Brown University**

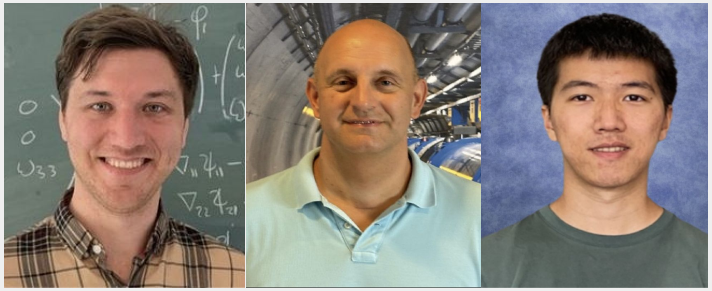






# Outline

- **Introduction**
- **Status and Methods: Feynman integrals and N=4 Yang-Mills amplitudes**
- **Mathematical Structures: Cluster Algebras**
- **Conclusion**



Pokraka, Spradlin, Weng 2506.11895
Bossinger, Drummond, Glew, Gürdoğan, Li 2507.01015

# Scattering Amplitudes

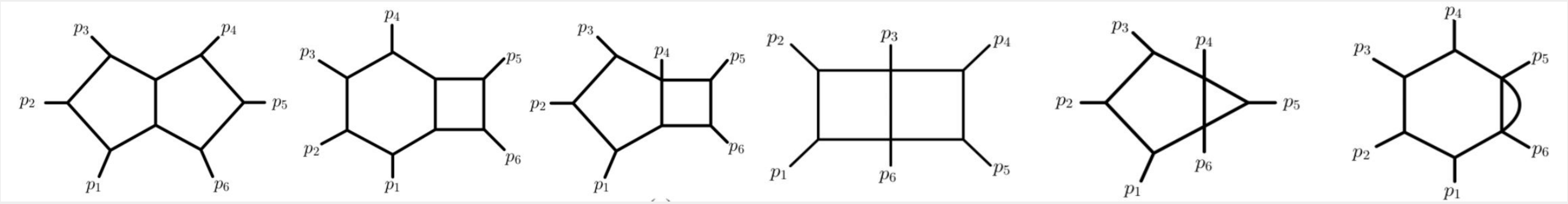
Amplitudes are indexed by

- n = 4, 5, … = number of particles (4-point, 5-point, etc)
- k = 0, 1, …, n - 4 = helicity sector (MHV, NMHV, etc)
- L = 0, 1, 2, 3, … = loop order (tree, one-loop, etc)

“the L-loop $N^k$MHV n-particle amplitude” is, in principle, computed by summing all Feynman diagrams with n external edges and L internal loops, and then integrating over all loops.

# Status: Feynman Integrals

- **n=4 L=3 one mass** Henn, Lim, Torres Bobadilla 2302.12776; Gehrmann, Henn, Jakubcik, Lim, Mella, Syrrakos, Torres Bobadilla 2410.19088
- **n=5 L=2 massless** Germann, Henn, Lo Presti 1511.05409; Chicherin, Henn, Mitev 1712.09610; Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia 1812.11160; Abreu, Dixon, Herrmann, Page, Zeng 1812.08941, 1902.08563
- **n=5 L=3 massless** Liu, Matijasic, Miczajka, Xu, Xu, Zhang 2411.18697
- **n=5 L=2 one mass** Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431
- **n=5 L=2 two masses** Abreu, Chicherin, Sotnikov, Zoia 2406.05201
- **n=6 L=2 massless** Henn, Matijasic, Miczajka, Peraro, Xu, Zhang 2403.19742 2501.01847; Abreu, Monni, Page, Usovitsch 2412.19884



# Status: N=4 Yang-Mills Amplitudes

- all n, all k L=0
- n<6 all L
- n=6 L=8
- n=7 L=4
- all n MHV L=2
- n=8 MHV L=3
- n=8, 9 NMHV L=2

Britto, Cachazo, Feng, Witten (BCFW) 0501052

Bern, Dixon, Smirnov (BDS) 0505205

Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
Dixon, Liu 2308.08199

Caron-Huot 1105.5606

Li, Zhang 2110.00350

He, Li, Zhang 1911.01290
2009.11471

# N=4 Yang-Mills

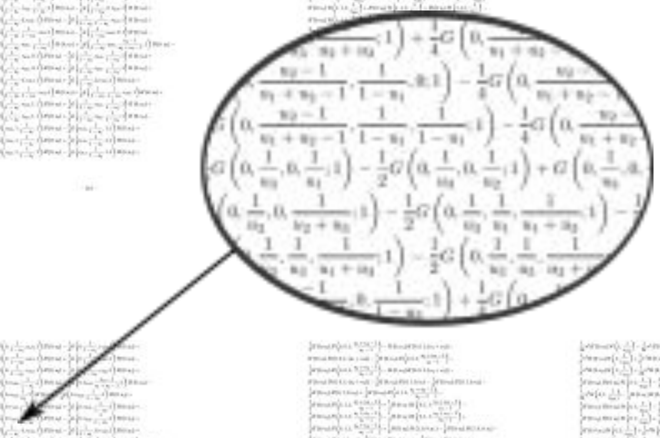
**Planar N=4 Yang-Mills amplitudes have the most amazing mathematical structures:**

- **Dual Conformal/Yangian Symmetry**
- **Amplituhedron**
- **Cluster Algebras**

**It is a simple model that (likely) can be solved exactly, and provides general tools and intuition.**

**Arkani-Hamed, Dixon, McLeod, Spradlin, Trnka, AV 2207.10636**

# 2-loop 6-point MHV in N=4 Yang-Mills



$$R_6^{(2)} = \sum_{\text{cyclic}} \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$$+\text{products of } \mathrm{Li}_k(-x) \text{functions of lower weight}$$

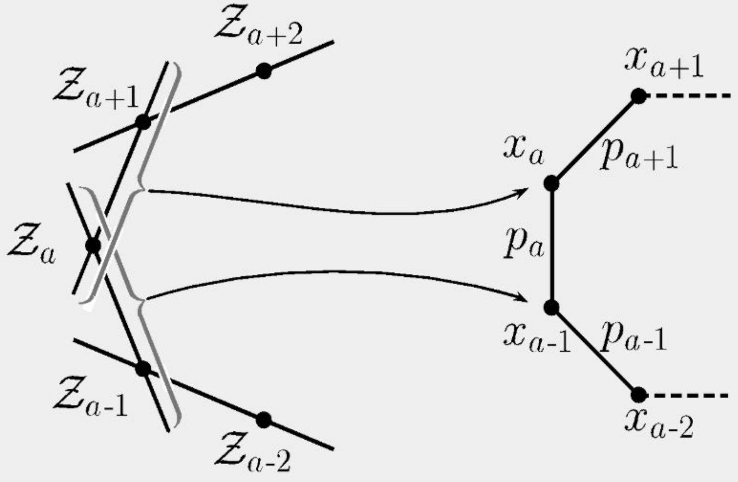
Goncharov, Spradlin, Vergu, AV 1006.5703

# Kinematic Variables

$$\mathbf{p}_{\mathbf{i}}^{\mu} = \sigma^{\mu}_{a\dot{a}} \lambda_{\mathbf{i}}^{\mathbf{a}} \widetilde{\lambda}_{\mathbf{i}}^{\dot{\mathbf{a}}}$$

$$\langle ij \rangle = \lambda_{i}^{a} \lambda_{\mathbf{j}}^{\mathbf{b}} \epsilon_{\mathbf{ab}}$$

$$[ij] = \widetilde{\lambda}_{\mathbf{i}}^{\dot{\mathbf{a}}} \widetilde{\lambda}_{\mathbf{j}}^{\dot{\mathbf{b}}} \epsilon_{\dot{\mathbf{a}}\dot{\mathbf{b}}}$$




$$\lambda_{\mathbf{i}}^{\mathbf{a}} = Z_{\mathbf{i}}^{\mathbf{a}}$$

$$\widetilde{\lambda}_{\mathbf{i}}^{\dot{\mathbf{a}}} = \frac{\langle \mathbf{i}-1\, \mathbf{i} \rangle \mathbf{Z}_{\mathbf{i}+1}^{\mathbf{a}+2} + \langle \mathbf{i}+1\, \mathbf{i}-1 \rangle \mathbf{Z}_{\mathbf{i}}^{\mathbf{a}+2} + \langle \mathbf{i}\, \mathbf{i}+1 \rangle \mathbf{Z}_{\mathbf{i}-1}^{\mathbf{a}+2}}{\langle \mathbf{i}-1\, \mathbf{i} \rangle \langle \mathbf{i}\, \mathbf{i}+1 \rangle}$$

$$\langle \mathbf{abcd} \rangle = \mathbf{det}(\mathbf{Z_a}\, \mathbf{Z_b}\, \mathbf{Z_c}\, \mathbf{Z_d})$$

Hodges 0905.1473

# Functions

- **Uniform Transcendentality m=2L @ L-loops**
- **Polylogarithms**

$$Li_m(x) = \int_0^x \frac{dt}{t} Li_{m-1}(t) \qquad Li_1(x) = -\log(1-x)$$

- **Generalized Polylogarithms (MPLs)**

$$G(a_1, \ldots, a_m; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_m; t) \qquad G(; x) = 1$$

- **Elliptic Polylogarithms (eMPLs) n=10 L=2 k=3**

Bourjaily, Broedel, Chaubey, Duhr, Frellesvig, Hidding, Marzucca, McLeod, Spradlin, Tancredi, Vergu, Volk, AV, Hippel, Weinzierl, Wilhelm, Zhang 2203.07088

# Symbol

Goncharov, Spradlin, Vergu, AV 1006.5703

$$S(\log x) = x \qquad S[Li_2(x)] = -(1-x)\otimes x$$

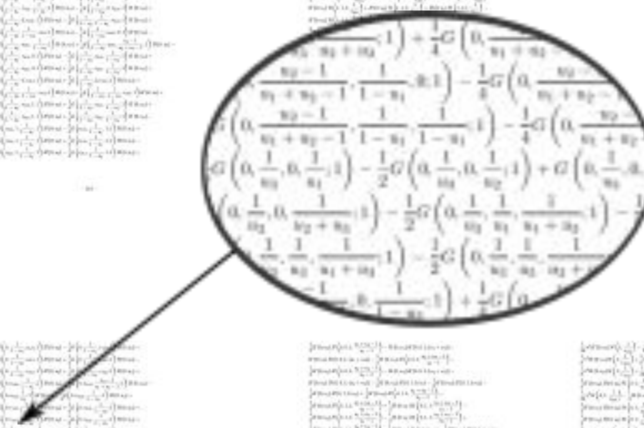
$$dF^{(m)} = \sum_i F_i^{(m-1)}\, d\log x_i$$

$$S(F^{(m)}) := \sum_i S(F_i^{(m-1)}) \otimes x_i$$

The kinematic variables that appear in a symbol are called letters, and collectively they are called the symbol alphabet.

Symbol encodes all the singularities of a function.

# 2-loop 6-point MHV in N=4 Yang-Mills



$$R_6^{(2)} = \sum_{\text{cyclic}} \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$$+\text{products of } \mathrm{Li}_k(-x) \text{functions of lower weight}$$

$$\text{symbol alphabet} = \langle \mathrm{a}\; \mathrm{a{+}1}\; \mathrm{b}\; \mathrm{c}\rangle$$

**15 symbol letters**

# 2-loop 7-point in N=4 Yang-Mills

**Symbol alphabet for 2-loop 7-point amplitudes in N=4 Yang-Mills has been worked out in**

**Goncharov, Spradlin, Vergu, AV 1006.5703**

**Caron-Huot 1105.5606.**

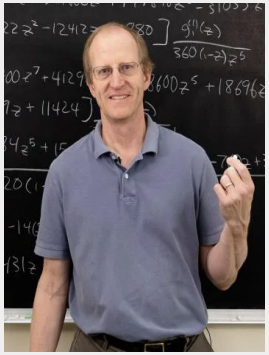
$$\{\langle \mathbf{a\,a{+}1\,b\,c}\rangle, \langle \mathbf{1(23)(45)(67)}\rangle, \langle \mathbf{1(27)(34)(56)}\rangle\}$$

**49 symbol letters**

# Higher Loops

**Dixon et al have computed n=6 and n=7 amplitudes to higher number of loops (8 and 4, respectively) by starting with the symbol alphabet presented on the last two slides and using a bootstrap approach, based on mathematical and physical inputs.**

| Constraint | $L=1$ | $L=2$ | $L=3$ | $L=4$ | $L=5$ | $L=6$ |
|---|---|---|---|---|---|---|
| 1. $\mathcal{H}_6$ | 6 | 27 | 105 | 372 | 1214 | 3692? |
| 2. Symmetry | (2,4) | (7,16) | (22,56) | (66,190) | (197,602) | (567,1795?) |
| 3. Final-entry | (1,1) | (4,3) | (11,6) | (30,16) | (85,39) | (236,102) |
| 4. Collinear | (0,0) | (0,0) | $(0^*,0^*)$ | $(0^*,2^*)$ | $(1^{*3},5^{*3})$ | $(6^{*2},17^{*2})$ |
| 5. LL MRK | (0,0) | (0,0) | (0,0) | (0,0) | $(0^*,0^*)$ | $(1^{*2},2^{*2})$ |
| 6. NLL MRK | (0,0) | (0,0) | (0,0) | (0,0) | $(0^*,0^*)$ | $(1^*,0^{*2})$ |
| 7. NNLL MRK | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | $(1,0^*)$ |
| 8. $N^3$LL MRK | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 9. Full MRK | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 10. $T^1$ OPE | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 11. $T^2$ OPE | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) |

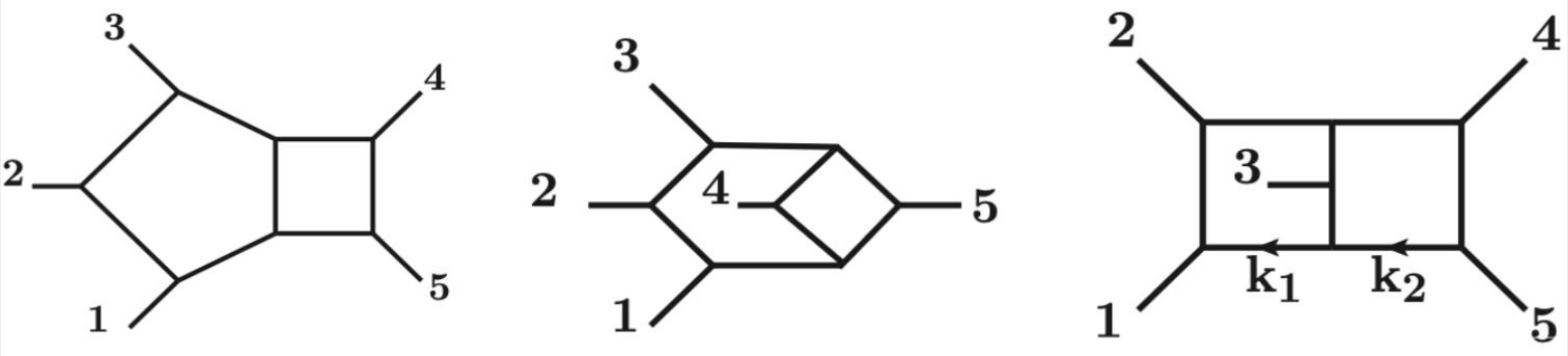


Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
Dixon, Liu 2308.08199

# Higher Points

- 2-loop all-n MHV — Caron-Huot 1105.5606
- 2-loop 8/9-point NMHV — He, Li, Zhang 1911.01290, 2009.11471
- 3-loop 8-point MHV — Li, Zhang 2110.00350
- Some symbol letters involve square roots when written in terms of Pluckers.

# 2-loop 5-point Massless Integrals




**The full symbol alphabet for planar & non-planar 2-loop 5-point massless integrals has been worked out by Germann, Henn, Lo Presti 1511.05409, Chicherin, Henn, Mitev 1712.09610.**

$$\left\{ s_{12}, s_{12}+s_{23}, s_{12}-s_{45}, s_{45}-s_{12}-s_{23}, s_{34}+s_{45}-s_{12}-s_{23}, \frac{a-\sqrt{\Delta}}{a+\sqrt{\Delta}} \right\}$$

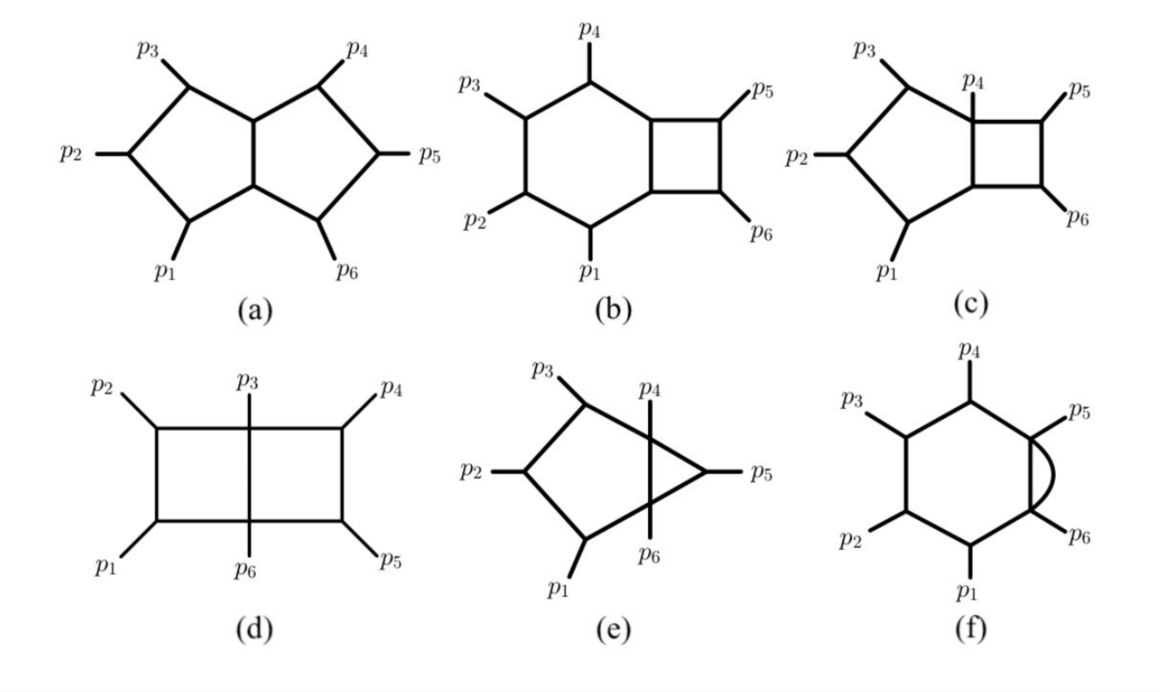
$$a = s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} - s_{51}s_{12}$$

$$\sqrt{\Delta} = \langle 12\rangle\langle 45\rangle[24][15] - \langle 24\rangle\langle 15\rangle[12][45]$$

**+ 5 cyclic**

**Planar: 25+1 letters & Non-planar: +5 letters**

# 2-loop 6-point Massless Integrals




**The full symbol alphabet for planar 2-loop 6-point massless integrals has been worked out by**

**Abreu, Monni, Page, Usovitsch 2412.19884;**

**Henn, Matijasic, Miczajka, Peraro, Xu, Zhang 2403.19742, 2501.01847.**

# 2-loop 6-point Symbol Alphabet

**Choose a basis given by 9 variables:**

$$v_1 = (p_1 + p_2)^2 \,, \qquad v_{i\in[2,6]} = T^{i-1} v_1$$

$$v_7 = (p_1 + p_2 + p_3)^2 \,, \quad v_{i\in[8,9]} = T^{i-7} v_7$$

$$T p_i = p_{i+1 \bmod 6}$$

**In addition we will use the notation:**

$$\epsilon_{ijkl} = [ij]\langle jk\rangle[kl]\langle li\rangle - \langle ij\rangle[jk]\langle kl\rangle[li] \,,$$

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - [12]\langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle$$

# 2-loop 6-point Symbol Alphabet

$$W_{i\in[1,289]}$$

$$W_1 = v_1$$
$$W_{49} = v_7 v_9 - v_1 v_4$$
$$W_{100} = v_2 v_5 (v_9 - v_3) + v_1 v_3 (v_8 - v_6) + v_7 v_8 (v_3 - v_9) + v_6 v_7 (v_9 - v_3)$$
$$W_{118} = \sqrt{v_1^2 + v_3^2 + v_5^2 - 2v_1 v_3 - 2v_1 v_5 - 2v_3 v_5} \equiv r_1$$
$$W_{123} = \epsilon_{1234}$$
$$W_{138} = \Delta_6$$
$$W_{139} = v_7 \epsilon_{5612} - v_1 \epsilon_{4561}$$
$$W_{157} = \frac{-r_1 + v_1 + v_3 - v_5}{r_1 + v_1 + v_3 - v_5}$$
$$W_{182} = \frac{-\epsilon_{1234} + v_1 v_2 - v_3 v_2 + v_5 v_2 + v_3 v_7 - v_1 v_8 - v_7 v_8}{\epsilon_{1234} + v_1 v_2 - v_3 v_2 + v_5 v_2 + v_3 v_7 - v_1 v_8 - v_7 v_8}$$
$$W_{275} = \frac{-r_1 \epsilon_{1234} + w_{275}}{r_1 \epsilon_{1234} + w_{275}}$$

$$w_{275} = (v_2 - v_8) v_1^2 + \ldots$$

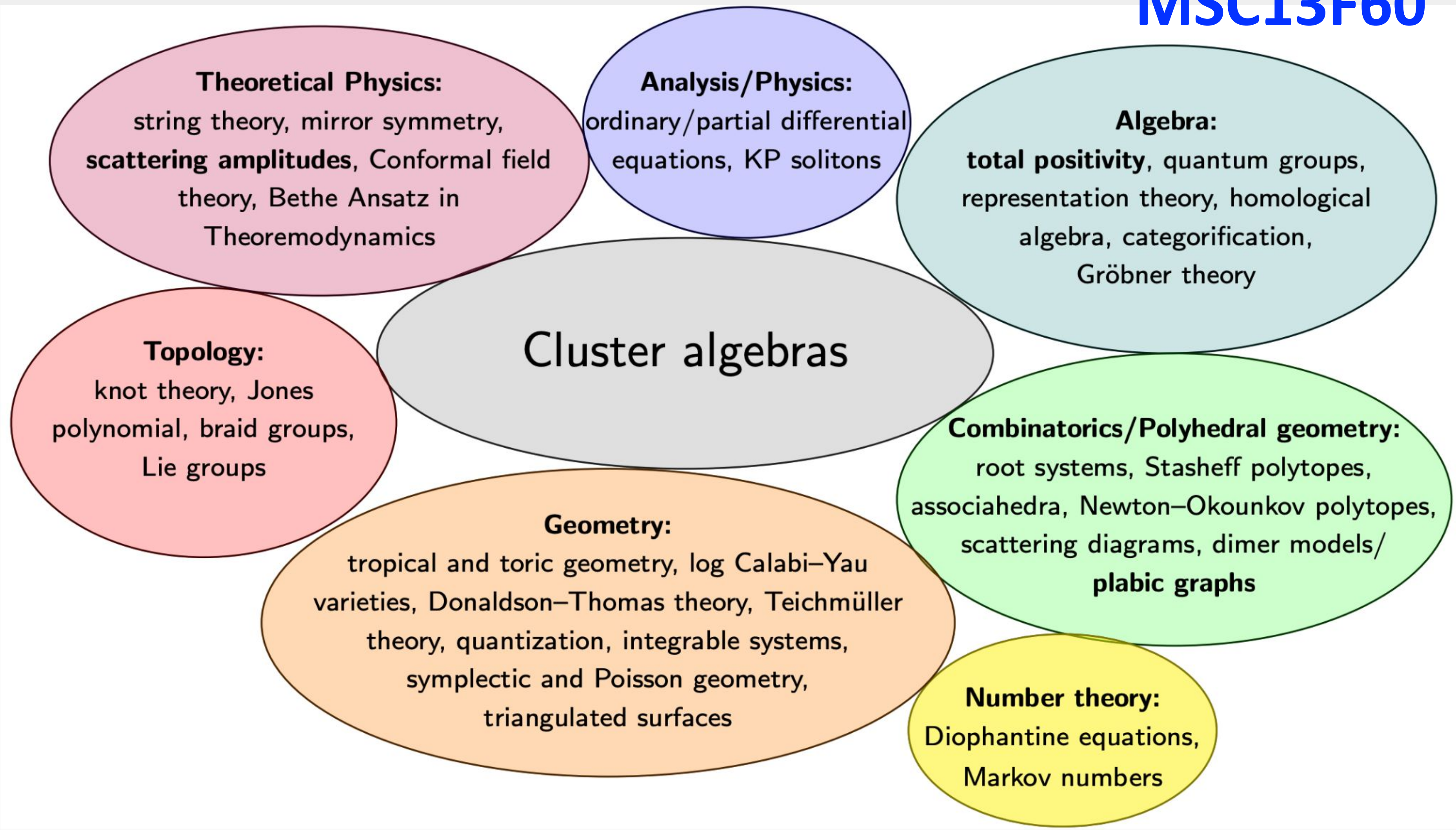
## 245 independent letters, we will omit:

$$W_{i\in[191,193]\cup[212,217]\cup[221,223]\cup[248,274]\cup[279,280]\cup[287,289]}$$

**Is there a mathematical description of symbol alphabets?**

# Cluster Algebras in Mathematics

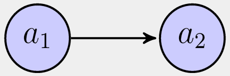
**MSC13F60**




**Slide from Bossinger, Harvard CMSA Summer School (2025)**

# $A_2$ Cluster Algebra

**Initial Seed**




**Mutation Rule**

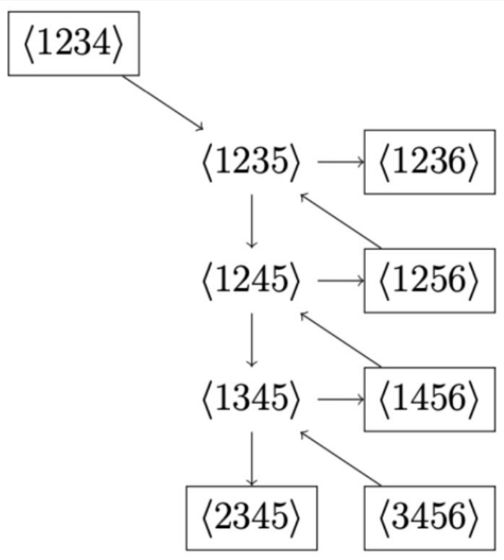
$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

## Cluster Variables

$$a_1,\ a_2,\ a_3 = \frac{1+a_2}{a_1},\ a_4 = \frac{1+a_1+a_2}{a_1 a_2},\ a_5 = \frac{1+a_1}{a_2}$$

Fomin, Williams, Zelevinsky "Introduction to Cluster Algebras"

# Gr(4,6) Cluster Algebra & 2-loop 6-point N=4 YM Symbol Alphabet




Golden, Goncharov
Spradlin, Vergu, AV
1305.1617

**Start with the initial seed**

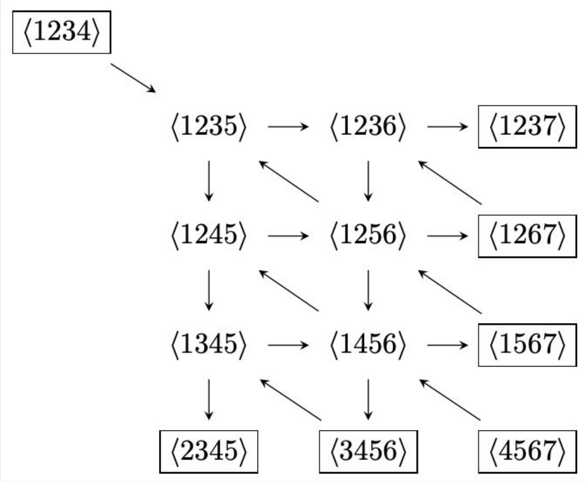
**Perform all mutations**

**Obtain 15 cluster variables**

$$\{\langle \mathbf{a}\,\mathbf{a{+}1}\,\mathbf{b}\,\mathbf{c}\rangle\}$$

**Match 2-loop 6-point symbol alphabet!**

# Gr(4,7) Cluster Algebra & 2-loop 7-point N=4 YM Symbol Alphabet




Golden, Goncharov
Spradlin, Vergu, AV
1305.1617

**Start with the initial seed**

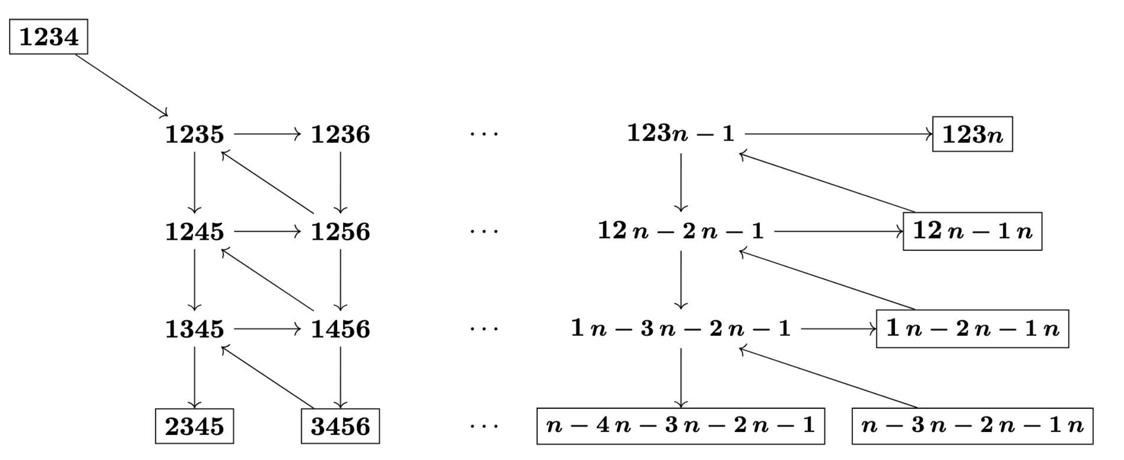
**Perform all mutations**

**Obtain 35+7+7 cluster variables**

$$\{\langle \mathbf{a\,a{+}1\,b\,c} \rangle, \langle \mathbf{1(23)(45)(67)} \rangle, \langle \mathbf{1(27)(34)(56)} \rangle\}$$

**Match 2-loop 7-point symbol alphabet!**

# Grassmannian Cluster Algebra Gr(4,n)




Fomin, Zelevinsky 0104151, Scott 0311148, Gekhtman, Shapiro, Vainshtein 0208033, Williams 1212.6262, Fomin, Williams, Zelevinsky "Introduction to Cluster Algebras"

**Symbol alphabet for n-point amplitude in N=4 YM is described by Grassmannian cluster algebra Gr(4,n).**

Golden, Goncharov, Spradlin, Vergu, AV 1305.1617

# Symbol Alphabet for Higher Points

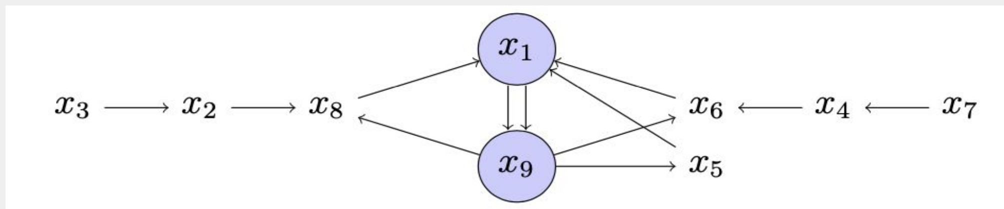
- 2-loop all-n MHV
- 2-loop 8/9-point NMHV
- 3-loop 8-point MHV
- Some symbol letters involve square roots when written in terms of Pluckers

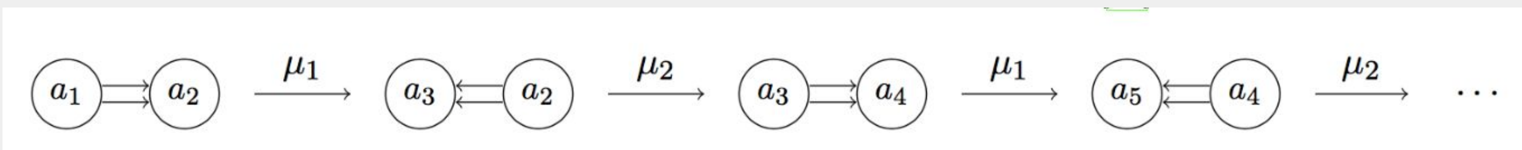
Li, Zhang 2110.00350

He, Li, Zhang 1911.01290
2009.11471

# Square Roots from Infinite Paths

**The infinite cluster variables of Gr(4,8) arise from quivers that have a double arrow like**



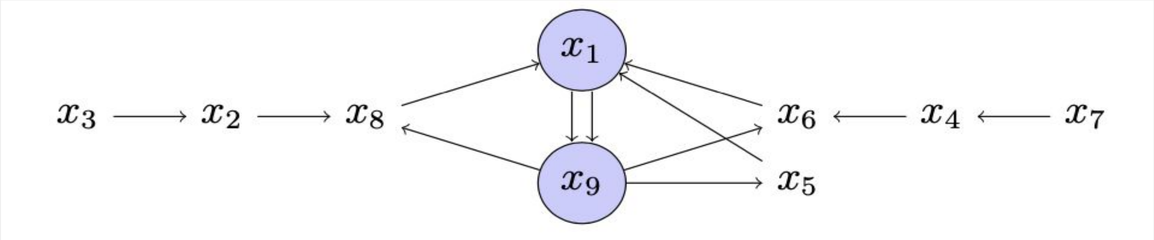

**Square roots appear along infinite mutation sequence**

$$\lim_{i\to\infty} \frac{a_i}{a_{i-1}} = \frac{a_2}{2a_1}\left(1 + x_1 + x_1x_2 + \sqrt{(1 + x_1 + x_1x_2)^2 - 4x_1x_2}\right)$$

$x_1 = 1/a_2^2,\ x_1 = a_1^2.$ Canakci, Schiffler 1608.06568

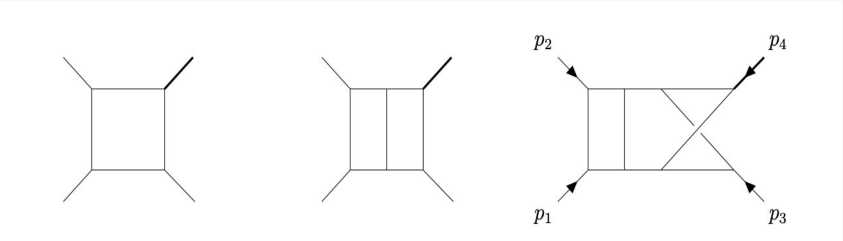
# Square Roots from Infinite Paths

**All square roots symbol letters known to appear in 8 and 9-point amplitudes are known to be associated to such infinite sequences inside Gr(4,8) and Gr(4,9).**




Drummond, Foster, Gürdoğan, Kalousios 1912.08217

# Cluster Algebras for Feynman Integrals




Chicherin, Henn, Papathanasiou 2012.12285,
Aliaj, Papathanasiou 2408.14544 found cluster algebras structure in certain Feynman integrals relevant to QCD.

$A_2, C_2, G_2$
for
L=1,2,3

$$\{a_1, a_2\}$$

$$a_{m+1} = \begin{cases} \frac{1+a_m}{a_{m-1}} & m \text{ odd} \\ \frac{1+a_m^L}{a_{m-1}} & m \text{ even} \end{cases}$$

5,6,8 cluster variables match symbol alphabet after the change of variables

$$a_1 = \frac{s - p_4^2}{t} \qquad a_2 = \frac{s}{t}\frac{p_4^2 - s - t}{p_4^2}$$

# 2-loop Integrals & Flag Cluster Algebra

Bossinger, Li 2408.14956 presented

an embedding of

the partial flag variety F(2,n-2;n) cluster algebra

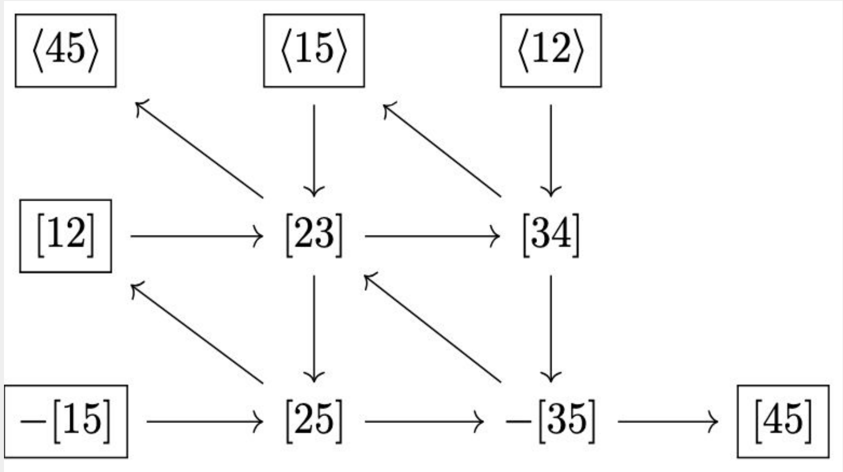
into

the Grassmannians Gr(n-2,2n-4) cluster algebra.

n=5: F(2,3;5) $\longrightarrow$ Gr(3,6)

n=6: F(2,4;6) $\longrightarrow$ Gr(4,8)

# F(2,3;5) Flag Cluster Algebra




$$\{\langle \mathbf{i\,j}\rangle, [\mathbf{i\,j}], \langle \mathbf{2\,3}\rangle[\mathbf{2\,3}] - \langle \mathbf{4\,5}\rangle[\mathbf{4\,5}], \langle \mathbf{1\,2}\rangle[\mathbf{1\,2}\rangle - \langle \mathbf{3\,4}\rangle[\mathbf{3\,4}]\}$$

**10+10+1+1 cluster variables**

# F(2,3;5) Cluster Algebra & 2-loop 5-point Symbol Alphabet

**This set is not closed under the natural cyclic group $i \to i+1$ of the 5-particles**

**but if we take the union under cyclic permutation (or under all permutations)**

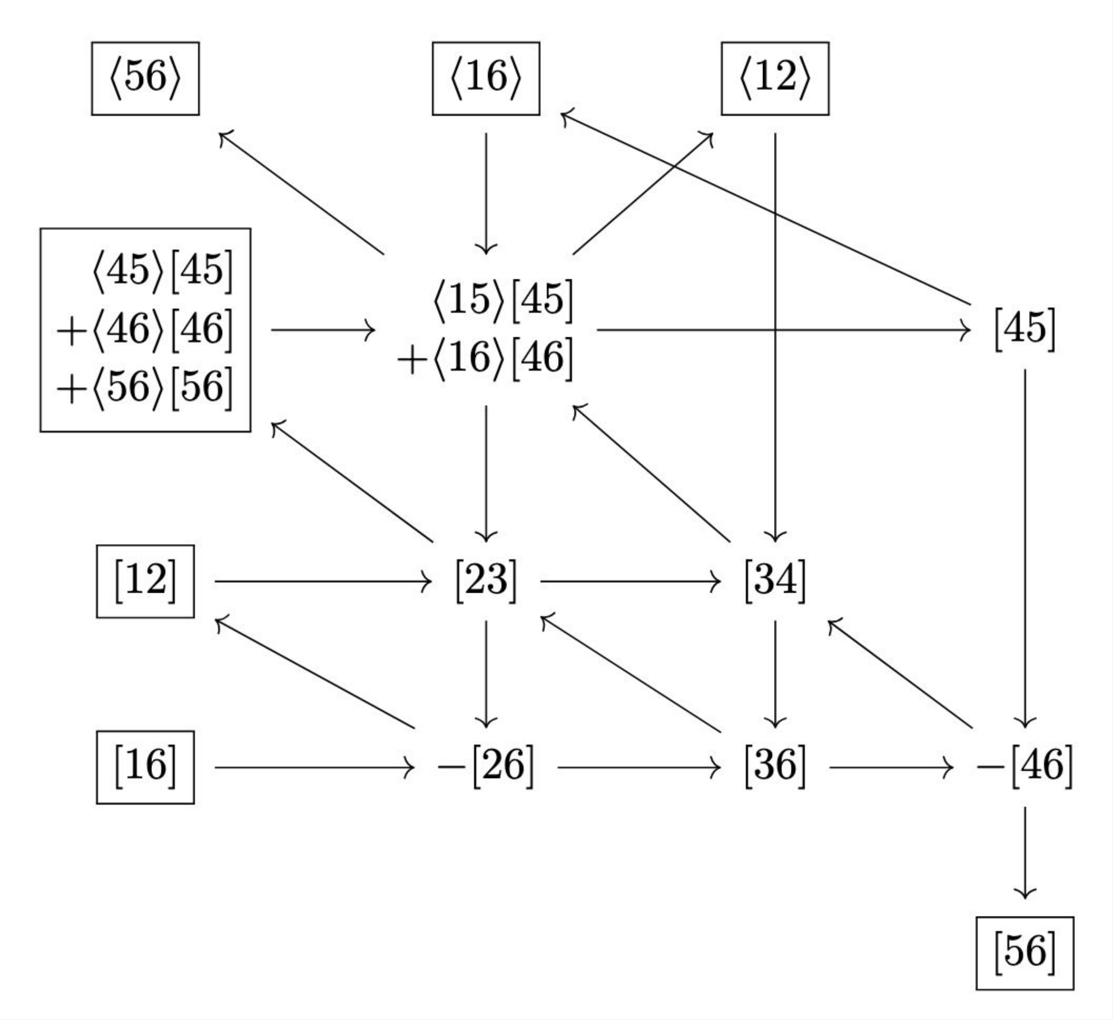
**it produces exactly**

**25 planar (and 5 non-planar) letters!**

**(except for one letter $\sqrt{\Delta}$ that drops out of all known amplitudes in four dimensions).**

Pokraka, Spradlin, AV, Weng 2506.11895

# F(2,4;6) Flag Cluster Algebra



# F(2,4;6) Cluster Algebra & 2-loop 6-point Symbol Alphabet

**245 symbol letters through all orders in dim reg**

- **7 are analogs of $\sqrt{\Delta}$ in the 5-point case**
- **135 are F(2,4;6) cluster variables (including permutations)**
- **40 are algebraic and we reproduce all of them from infinite mutation sequences**
- **27 of them are rational, are NOT cluster variables, but only appear at O(epsilon) so are not relevant in four dimensions**
- **36 remain somewhat mysterious…**

**(24 of 36 appear in Carrolo, Chicherin, Henn, Yang, Zhang 2505.01245)**

**Pokraka, Spradlin, AV, Weng 2506.11895**

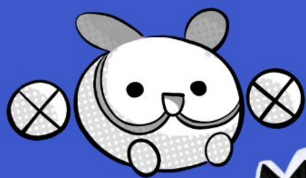
# Conclusion

ABSTRACT: The full 245-letter symbol alphabet for all planar massless two-loop six-point Feynman integrals was recently determined in arXiv:2412.19884 and arXiv:2501.01847. In a parallel mathematical development, it was shown in arXiv:2408.14956 that there is an embedding of the cluster algebra associated to the partial flag variety $\mathcal{F}\ell_{2,n-2;n}$, which describes the kinematics of $n$ massless particles, into that of the Grassmannian $\mathrm{Gr}(n-2, 2n-4)$. In this paper we connect these developments by showing that most of the rational symbol letters can be expressed in terms of flag cluster variables, and that all of the algebraic symbol letters arise from infinite mutation sequences.

- **Missing Letters**
- **Cluster Adjacency** — Bossinger, Drummond, Glew, Gürdoğan, Li 2507.01015
- **Cluster Functions** — Rudenko, Matveiakin 2208.01564
- **Amplituhedron Tilings** — Even-Zohar, Lakrec, Parisi, Sherman-Bennett, Tesser, Williams 2310.17727 Galashin 2410.09547
- **More data**

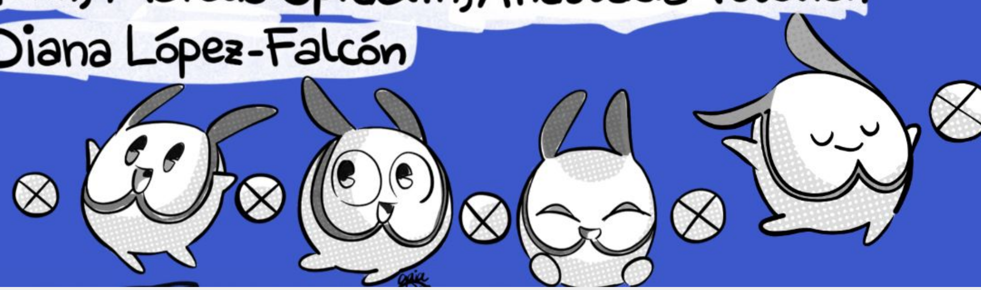
# Symbology@15

## 15-18 December 2025



## Max Planck Institute for Physics, Munich

Organizers: Johannes Henn, Marcus Spradlin, Anastasia Volovich
Scientific Coordinator: Diana López-Falcón
Design: Gaia Fontana



**Thank you!**