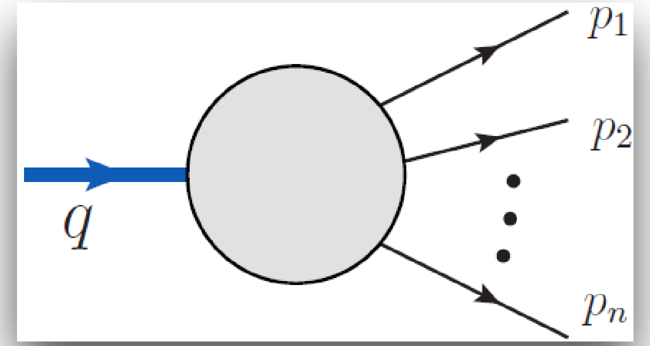


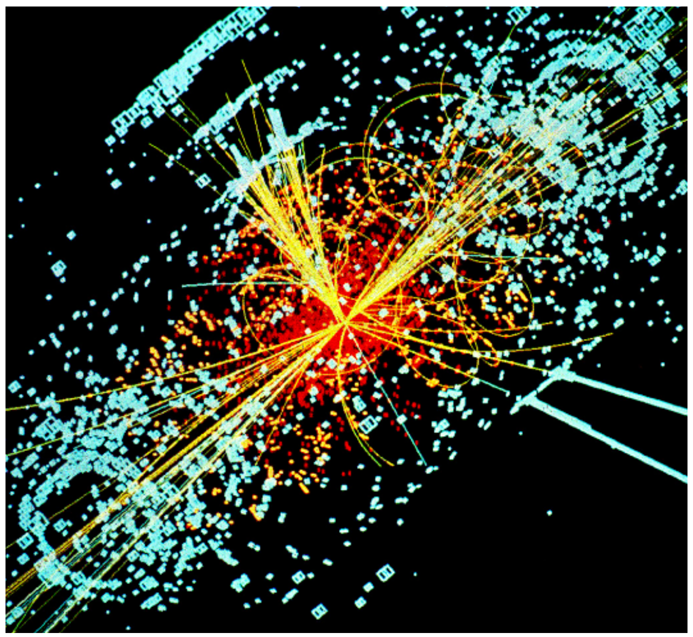
Form factor in N=4 SYM and QCD

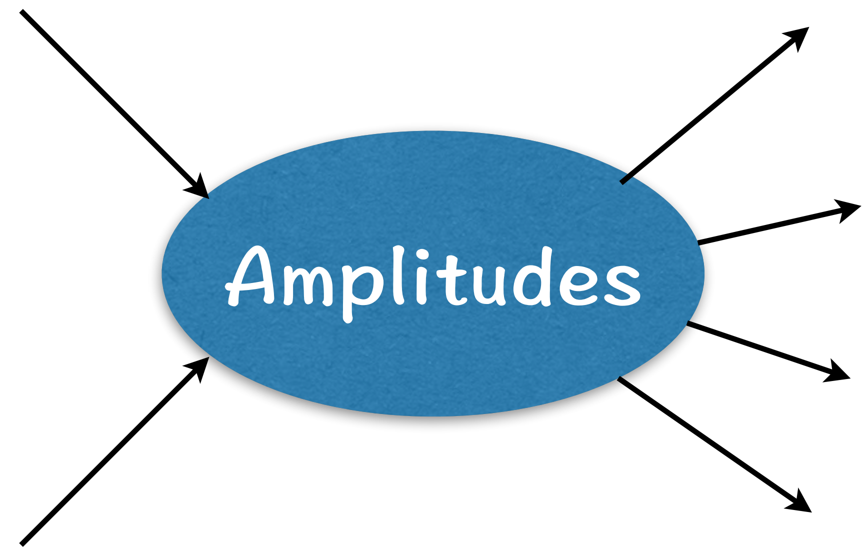
Gang Yang

ITP-CAS

International Workshop on New Opportunities for Particle Physics (NOPP)
July 18-20, 2025, IHEP-CAS

Scattering amplitudes



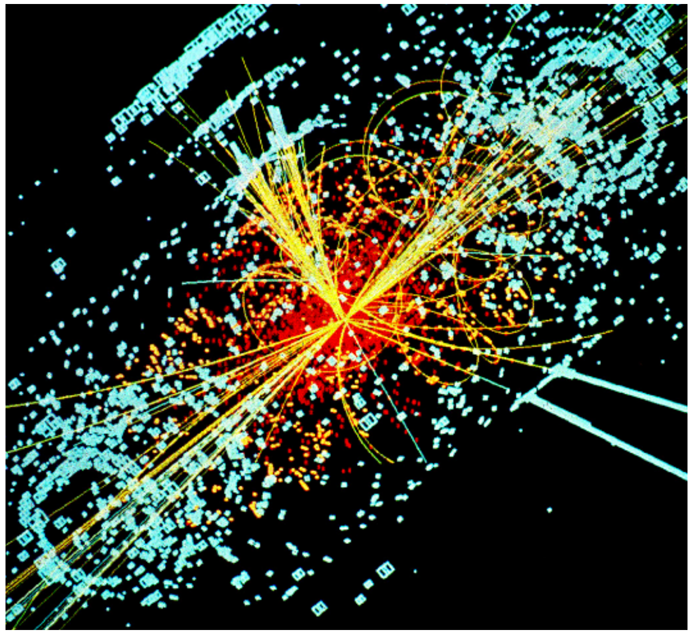


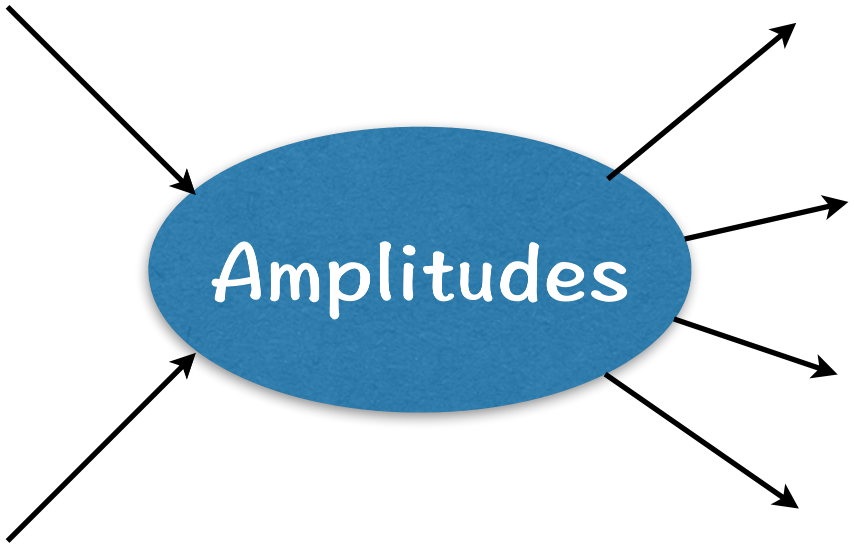
In past over 30 years, significant progress has been made in the studies of scattering amplitudes.

[Parke, Taylor, 1986]

$$A_n^{\text{tree}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

Scattering amplitudes





In past over 30 years, significant progress has been made in the studies of scattering amplitudes.

New structures

New methods

Amplitudes

$\downarrow$

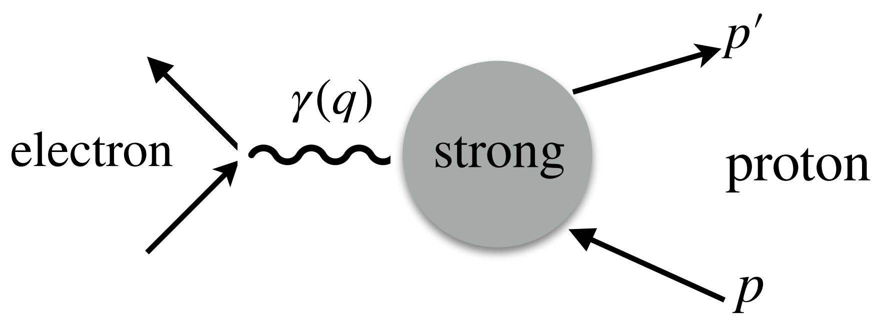
Form Factors

What are form factors?

Some history

1) Nuclear "structure factor"

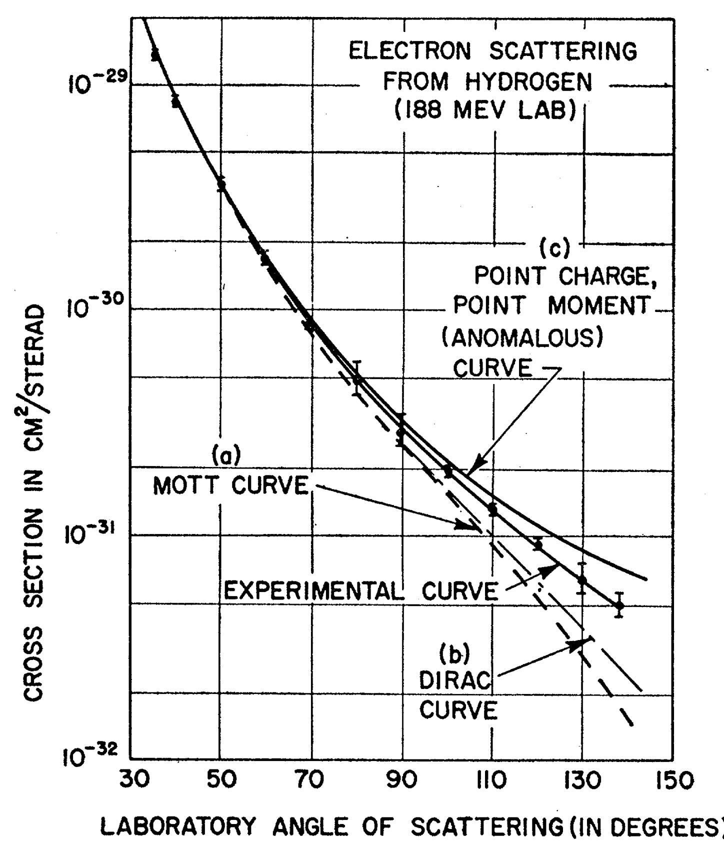
- Form factor characterizes the deviation from the point-particle picture.



$$\sigma_s(\theta) = \left(\frac{Ze^2}{2E}\right)^2 \frac{\cos^2\frac{1}{2}\theta}{\sin^4\frac{1}{2}\theta}\left[\int_0^\infty \rho(r)\frac{\sin qr}{qr}4\pi r^2 dr\right]^2.$$

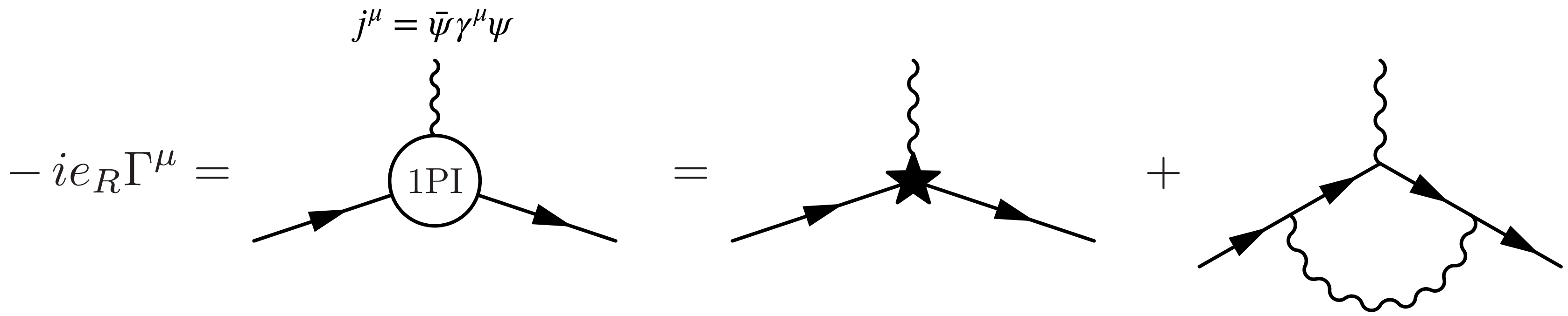
$$F = \frac{4\pi}{q}\int_0^\infty \rho(r)\sin(qr)r\,dr$$

"form factor"



McAllister and Hofstadter, Phys.Rev. (1956)

2) Form factor in text book



$$\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$$

$$\bar{\psi}\gamma^\mu\psi \rightarrow \bar{\psi}\Gamma^\mu\psi$$

Form factors

Leading order: $F_1(p^2) = 1, \quad F_2(p^2) = 0$

One-loop order: $F_2(0) = \frac{\alpha}{2\pi}$ → $g - 2 = 2F_2(0) = \frac{\alpha}{\pi}$

Anomalous magnet moment

3) Sudakov form factor

- Pioneer work by Vladimir Sudakov in 1954

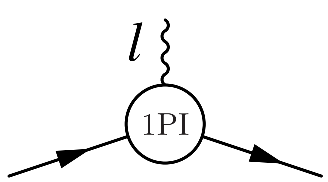
Vertex Parts at Very High Energies in Quantum Electrodynamics

V. V. SUDAKOV

(Submitted to JETP editor Nov. 4, 1954)
J. Exper. Theoret. Phys. USSR 30, 87-95 (January 1956)

A method is developed for calculating Feynman integrals with logarithmic accuracy, working to any order of perturbation theory. The method is applied to calculate the vertex part in quantum electrodynamics for a certain range of values of the momenta. The result is displayed as the sum of a perturbation series.

$-ie_R\Gamma^\mu =$



$$\Gamma_\sigma(p, q; l) = \gamma_\sigma \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right)^n = \gamma_\sigma \exp\left\{-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right\}.$$

A closed formula of summing up the leading-logarithm terms.

Generalization to non-leading logarithms in QED of Mueller 1979, Collins 1980, and in QCD Sen 1980

IR divergences

Infrared structure of amplitudes:

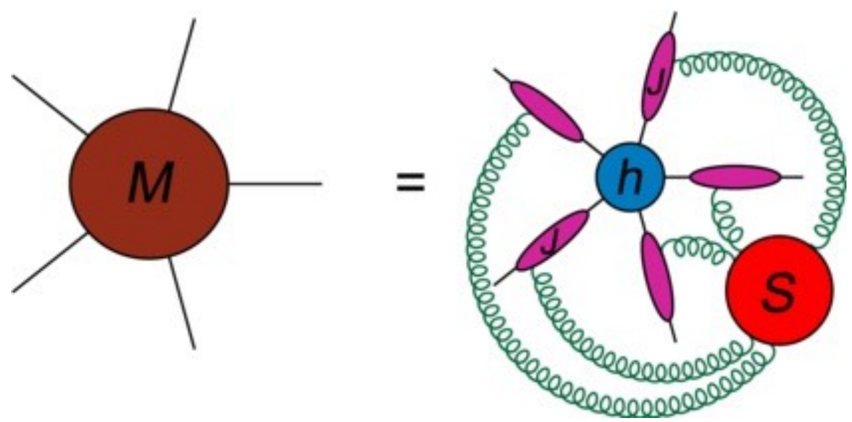


figure from L. Dixon 1105.0771

For modern dim-reg representation, see:
Magnea and Sterman 1990;
Catani 1998,
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

$$\mathcal{M}_n = \prod_{i=1}^{n}\left[\mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon\right)\right]^{1/2} \times h_n\left(k_i, \mu, \alpha_s, \epsilon\right)$$

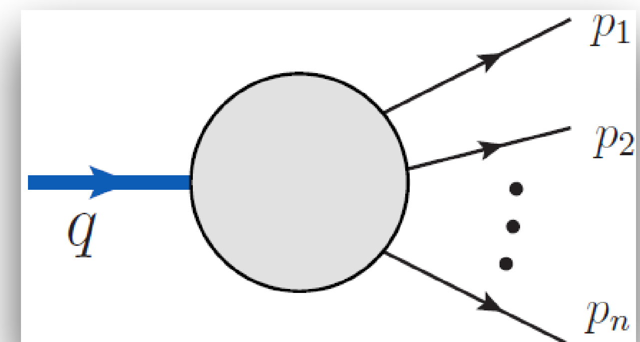
Sudakov form factor $= \exp\left[-\frac{1}{4}\sum_{l=1}^{\infty} a^l \left(\frac{\mu^2}{-Q^2}\right)^{l\epsilon}\left(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon}\right) + \text{finite}\right]$

Leading IR singularity -> Cusp anomalous dimension

4) "Modern" general form factors

Hybrids of on-shell states and off-shell operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\, \langle p_1 \ldots p_n|\mathcal{O}(x)|0\rangle$$
$$= \delta^{(4)}(\sum_{i=1}^{n} p_i - q)\, \langle p_1 \ldots p_n|\mathcal{O}(0)|0\rangle$$



$$q = \sum_i p_i, \quad q^2 \neq 0$$

form factors

$\langle p_1 p_2 \ldots p_n|0\rangle$

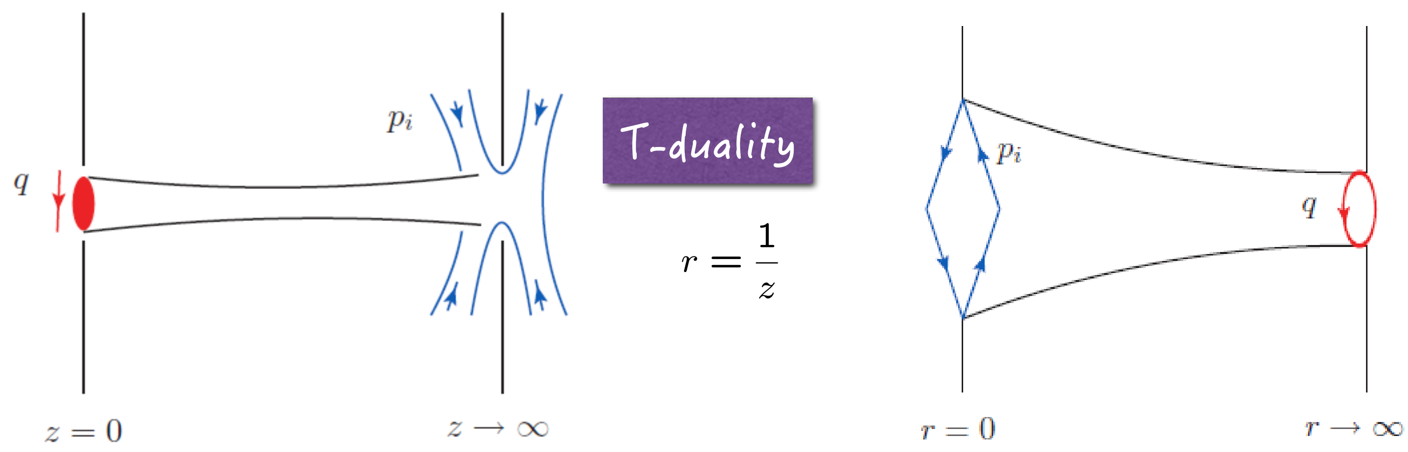
Scattering amplitude

$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n\rangle$

Correlation functions

4) “Modern” general form factors

- **Maldacena** and **Zhiboedov** (2010) considered high-point form factors at strong coupling using AdS/CFT duality.



- **Brandhuber, Spence, Travaglini, GY** (2010) and **Bork, Kazakov, Vartanov** (2010) studied high-point form factors at weak coupling.

MHV structure of form factors: Brandhuber, Spence, Travaglini, GY 2010

$$F_n^{\mathrm{MHV}}(1^+, .., i_\phi, .., j_\phi, .., n^+; \mathrm{tr}(\phi^2)) = \delta^4\left(\sum_{i=1}^{n} p_i - q\right) \frac{\langle ij\rangle^2}{\langle 12\rangle \cdots \langle n1\rangle}$$

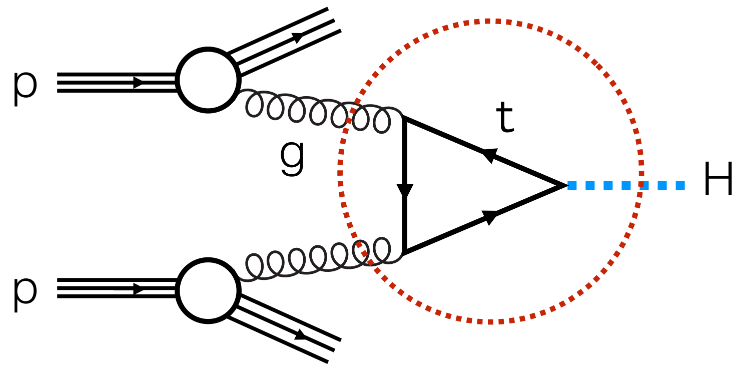
$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

Applications of form factors

- Operator classification and spectrum
- EFT amplitudes
- IR divergences (Sudakov FF)
- Correlation functions (EEC, etc..)
- New hidden structures beyond amplitudes

$$\text{Loop form factor} = \underbrace{(\text{Universal IR div.})}_{\text{Sudakov form factor}} + \underbrace{(\text{UV div.})}_{\text{Renormalization}} + \underbrace{(\text{Finite part})}_{\text{EFT amplitudes}}$$

"Higgs" EFT



$m_t \to \infty$

Effective gluon-Higgs vertex:

$$\mathcal{L}_{\text{eff}} = C_0 O_0 + \frac{1}{m_{\text{t}}^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{t}}^4}\right)$$

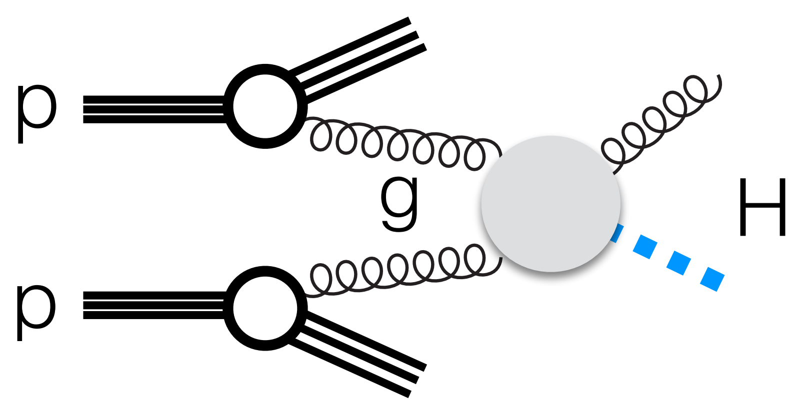
Dimension-5 operator

$$O_0 = H\text{tr}(F_{\mu\nu}F^{\mu\nu})$$

Dimension-7 operators

$$O_1 = H\,\text{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu)\,,$$
$$O_2 = H\,\text{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu})\,,$$
$$O_3 = H\,\text{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu})\,,$$
$$O_4 = H\,\text{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu})\,.$$

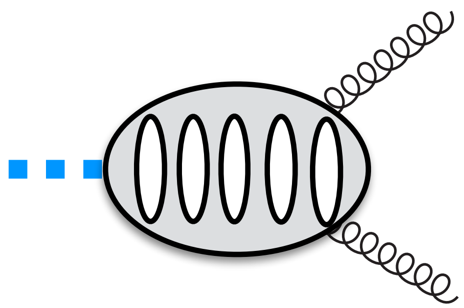
Higgs plus jet production



$$A(q^H, 1^g, 2^g, \ldots, n^g) = F_{\mathcal{O}=\text{tr}(F^2)}(1^g, 2^g, \ldots, n^g)$$

Progress in N=4 SYM

$$\mathscr{F}_n = \int d^4x\, e^{-iq\cdot x} \langle p_1, \ldots, p_n \,|\, \mathrm{tr}(F^2)(x) \,|\, 0 \rangle$$



Full-color integrand up to 5 loops

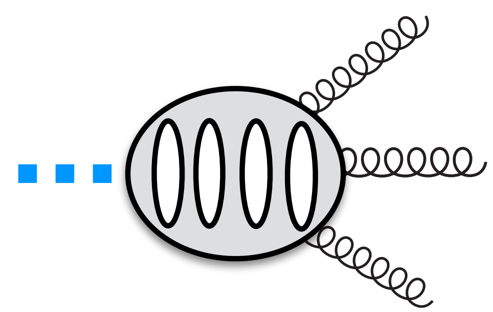
Boels, Kniehl, Tarasov, GY 2012

GY, 2016

Integrated results at 4 loops

Boels, Huber, GY 2017

Huber, von Manteuffel, Panzer, Schabinger, GY 2020



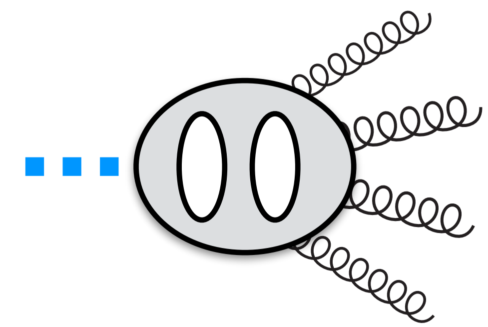
Full-color integrand up to 4 loops

Lin, GY, Zhang, 2021

Integrated results at 3 loops

Lin, GY, Zhang, 2021

Guan, Lin, Liu, Ma, GY 2023



Integrated results at 2 loops

Guo, Wang, GY, 2022

Guo, Wang, GY, Yin 2024

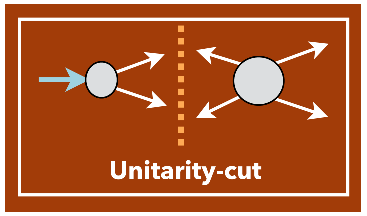
See also: Dixon, Gurdogan, Liu, McLeod, Wilhelm 2021, 2022; Dixon, Xin 2024

Computational tools

- On-shell unitarity method

Bern, Dixon, Durban, Kosower 1994;
Britto, Cachazo, Feng, 2003

Simple tree blocks -> Higher loop results



- Color-kinematics duality

Bern, Carrasco, Johansson 2008

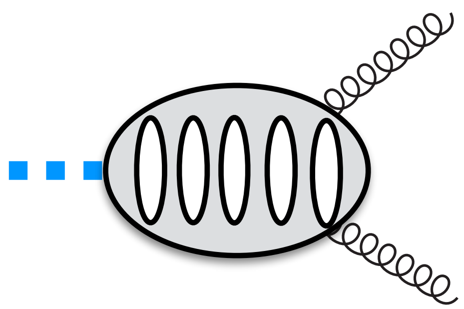
Large number of diagrams -> Very few “master” diagrams

- Master-integral bootstrap

Guo, Wang, GY 2021

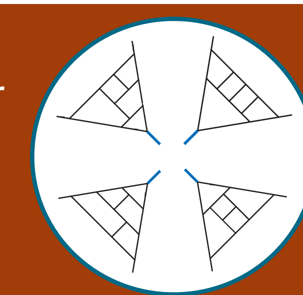
Construct final results directly using physical constraints

Full color form factors



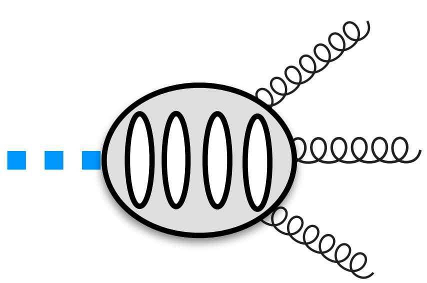
L-loop	L=1	L=2	L=3	L=4	L=5
# of topologies	1	2	6	34	306
# of masters	1	1	1	2	4

Four master graphs @ 5-loop:



Boels, Kniehl, Tarasov, GY 2012

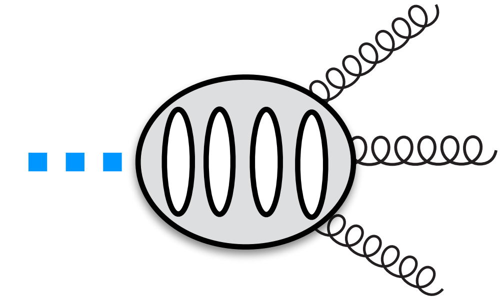
GY, 2016



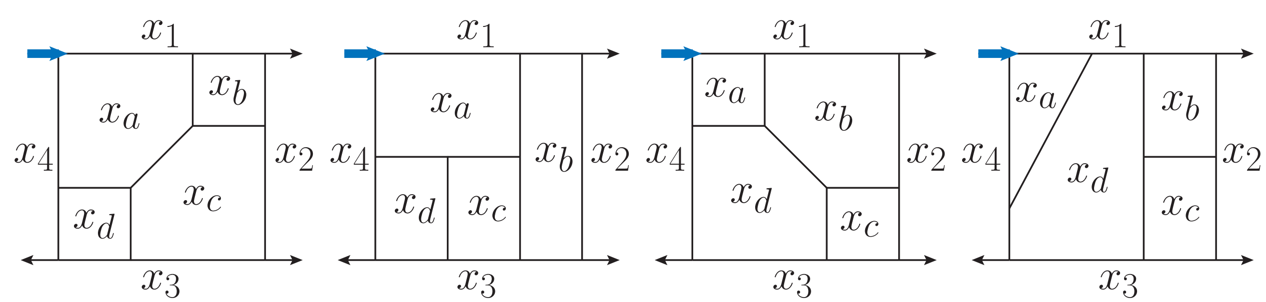
L loops	L=1	L=2	L=3	L=4
# of cubic graphs	2	6	29	229
# of planar masters	1	2	2	4
# of free parameters	1	4	24	133

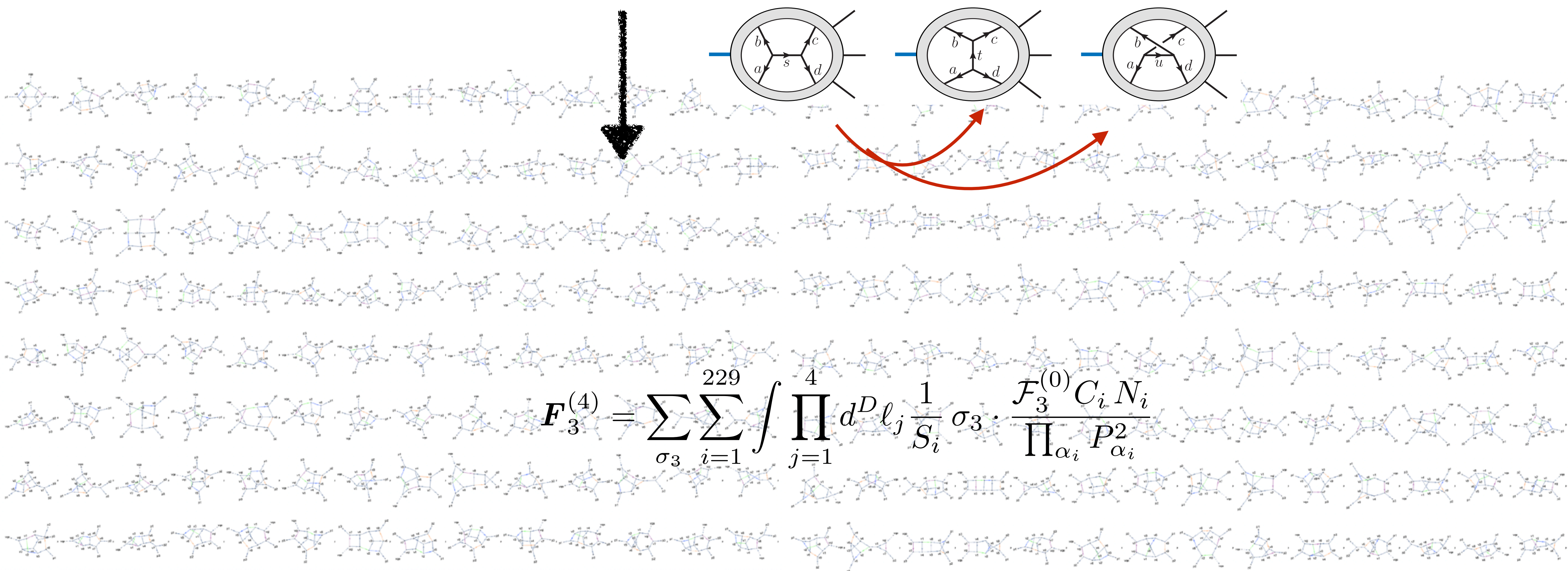
Lin, GY, Zhang, 2021

Three-point form factors



Master graphs

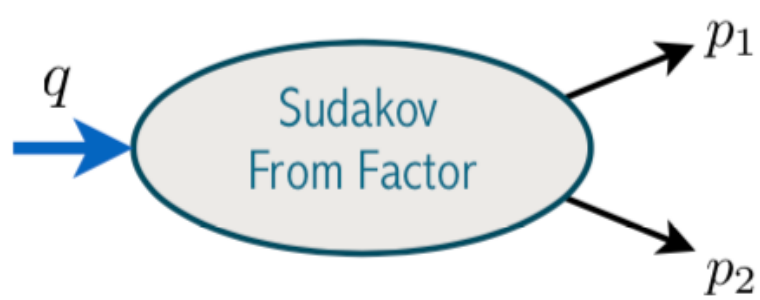




$$\boldsymbol{F}_3^{(4)} = \sum_{\sigma_3} \sum_{i=1}^{229} \int \prod_{j=1}^{4} d^D \ell_j \, \frac{1}{S_i} \, \sigma_3 \cdot \frac{\mathcal{F}_3^{(0)} C_i \, N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

Sudakov form factor
and Casimir scaling conjecture

Sudakov form factor



$p_1^2 = p_2^2 = 0$

$q^2 = (p_1 + p_2)^2$

Logarithm behavior is well-understood:

For dim-reg representation, see:
Magnea and Sterman 1990;
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

$$\log F_2(1,2) \simeq -\sum_{l=1}^{\infty} g^{2l}\left(\frac{\gamma_{\rm cusp}^{(l)}}{\epsilon^2} + \frac{\mathcal{G}_{\rm coll}^{(l)}}{\epsilon}\right)(-q^2)^{-l\epsilon} + \mathcal{O}(\epsilon^0)$$

Leading IR singularity -> Cusp anomalous dimension

Color structure

Up to three loops, only quadratic Casimir appears:

L-loop	L=1	L=2	L=3	L=4
Color Factor	C_A	C_A^2	C_A^3	C_A^4, d_{44}

For $SU(N)$:

$C_A = N$

$$d_{44} = \frac{N^2(N^2+36)}{24}$$

At four-loop, there is a new quartic Casimir which contains non-planar part

Casimir scaling conjecture

In "On the Structure of Infrared Singularities of Gauge- Theory Amplitudes", JHEP 0906, 081 (2009)

Thomas Becher and Matthias Neubert conjectured that:

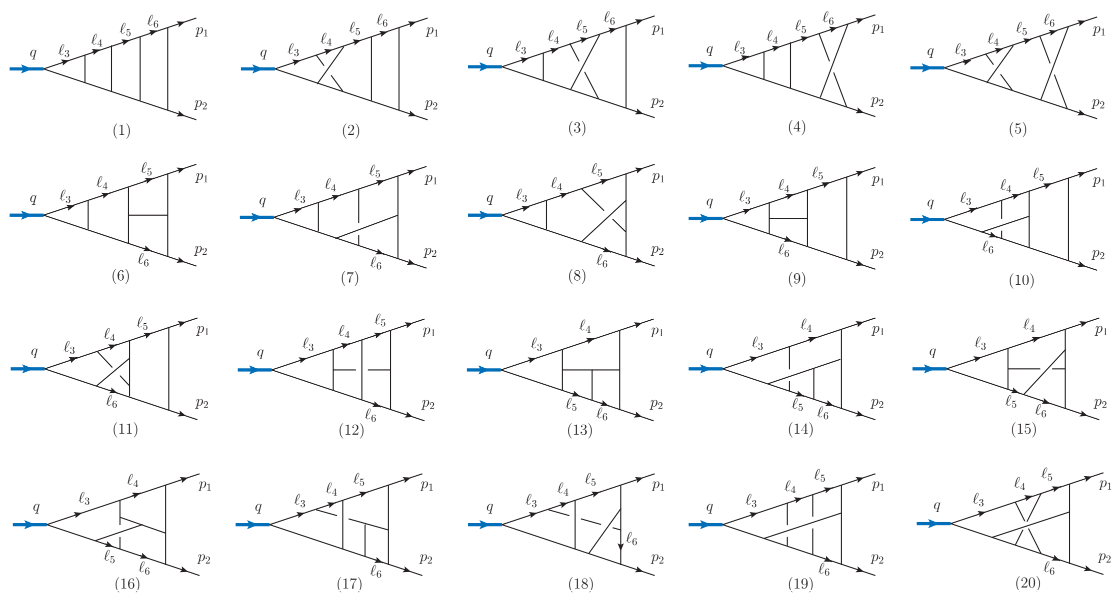
"Our formula predicts Casimir scaling of the cusp anomalous dimension to all orders in perturbation theory, and we explicitly check that the constraints exclude the appearance of higher Casimir invariants at four loops."

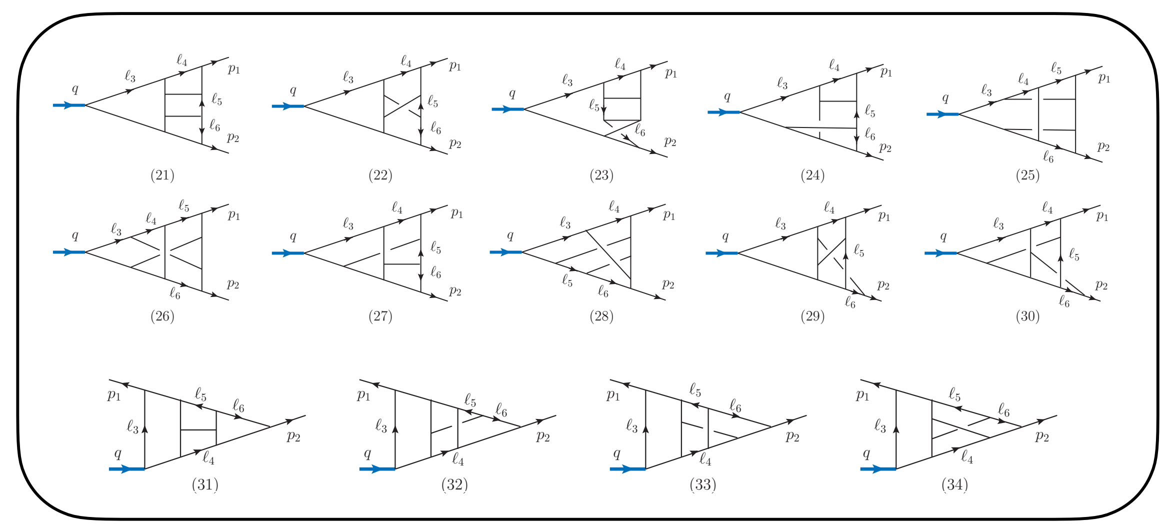
An explicit four-loop computation is needed.

Four-loop compact integrand

[Boels, Kniehl, Tarasov, GY 2012]

Four-loop form factor integrand was obtained by:
color-kinematics duality and unitarity:





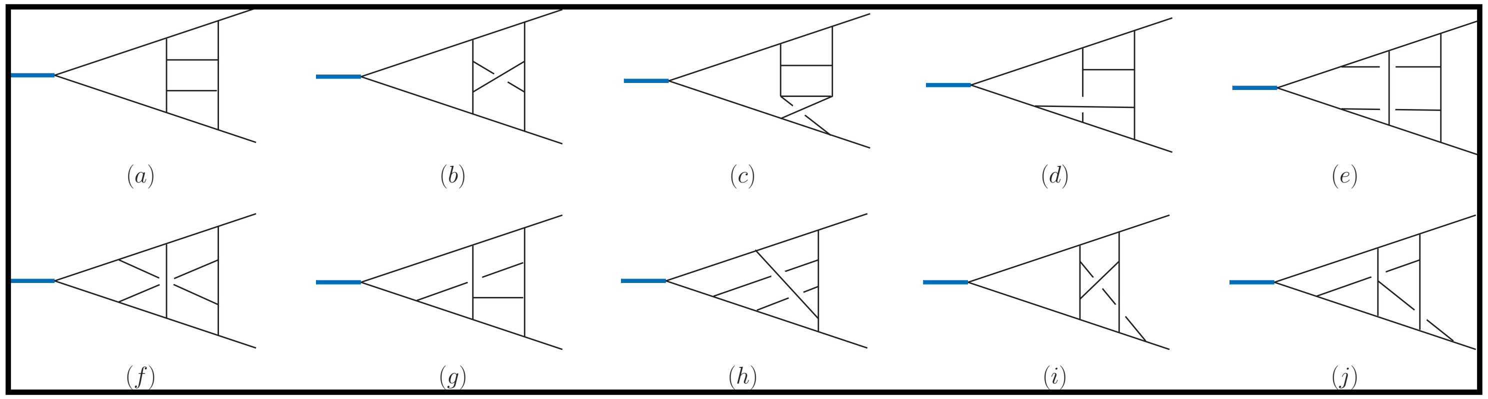
non-planar

compact form and with only quadratic loop momenta in the numerator.

$$
\begin{aligned}
N_{21} = \; & -(\ell_3 \cdot p_1)^2 - (\ell_3 \cdot p_2)^2 - 6(\ell_3 \cdot p_1)(\ell_3 \cdot p_2) \\
& +(p_1 \cdot p_2)\big[2(\ell_3 \cdot \ell_3) + 4(\ell_3 \cdot p_1) + p_1 \cdot p_2\big] \\
& +(\alpha_1 + 1)\big[(\ell_3 \cdot p_{12} - p_1 \cdot p_2)^2 \\
& -\tfrac{2}{7}(\ell_3 \cdot (\ell_3 - p_{12}) + p_1 \cdot p_2)(p_1 \cdot p_2)\big]
\end{aligned}
$$

Four-loop Sudakov form factor

- Integrand: unitarity + color-kinematics duality — Boels, Kniehl, Tarasov, GY 2012



- Numerical integration: — Boels, Huber, GY 2017

$$\gamma^{(4)}_{\text{cusp, NP}} = -3072 \times (1.60 \pm 0.19) \frac{1}{N_c^2}$$

Finding Uniform Transcendental (UT) basis is the key

- Analytic integration: — Huber, von Manteuffel, Panzer, Schabinger, GY 2020
  (See also: Henn, Korchemsky, Mistlberger 2020)

$$\gamma^{(4)}_{\text{cusp,NP}} = -3072 \times (\frac{3}{8}\zeta_3^2 + \frac{31}{140}\zeta_2^3)\frac{1}{N_c^2} = -3072 \times 1.52 \frac{1}{N_c^2}$$

Casimir scaling conjecture is incorrect.

High-dimensional operators in QCD

- 1804.04653, 1904.07260, 1910.09384, with Qingjun Jin
- 2011.02494 with Qingjun Jin, Ke Ren;
- 2202.08285, 2208.08976, 2301.01786 with Qingjun Jin, Ke Ren; Rui Yu

High dimensional YM operators

Gauge invariant operators:

$$\mathcal{O} \sim \quad c(a_1, ..., a_n)\big(D_{\mu_{11}}...D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1} \cdots \big(D_{\mu_{n1}}...D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n}.$$

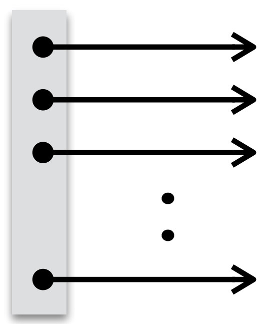
$$D_\mu \star = \partial_\mu + ig[A_\mu, \star]\,, \qquad [D_\mu, D_\nu] \star = ig[F_{\mu\nu}, \star] \qquad\qquad F_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad [T^a, T^b] = if^{abc}T^c$$

D-dimensional on-shell methods using form factor formalism:

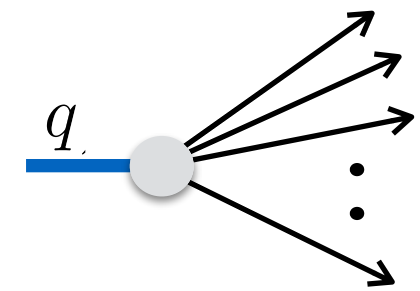
$$\mathcal{F}_n = \int d^4x\, e^{-iq\cdot x}\langle p_1, \ldots, p_n | \mathcal{O}(x) | 0\rangle$$

- Operator basis
- Two-loop renormalization and spectrum
- Two-loop EFT amplitudes

Minimal tree form factors



$$\mathcal{O}_L \;\Leftrightarrow\; \mathcal{F}_{\mathcal{O}_L,L}(1,\ldots,L)$$



Dictionary for YM operators:

operator	$D_{\dot{\alpha}\alpha}$	$f_{\alpha\beta}$	$\bar{f}_{\dot{\alpha}\dot{\beta}}$
spinor	$\tilde{\lambda}_{\dot{\alpha}}\lambda_{\alpha}$	$\lambda_{\alpha}\lambda_{\beta}$	$-\tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}$

4-dim

$$F_{\mu\nu} \to F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}$$

Used in N=4 SYM: Zwiebel 2011, Wilhelm 2014

operator	D_{μ}	$F_{\mu\nu}$
kinematics	p_{μ}	$p_{\mu}\varepsilon_{\nu} - p_{\nu}\varepsilon_{\mu}$

D-dim

Important for capturing
“Evanescent operators”

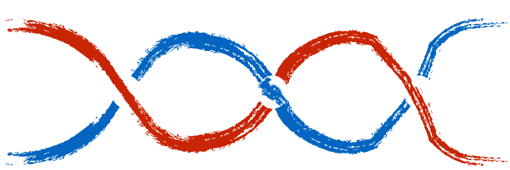
One can translate any local operator into “on-shell” kinematics.

Unitarity-IBP strategy

D-dimensional unitarity-cut:

$$\mathcal{F}^{(l)}\Big|_{\rm cut} = \prod(\text{tree blocks}) = \text{cut integrand} = \sum_i c_i \, M_i\big|_{\rm cut}$$

On-shell unitarity



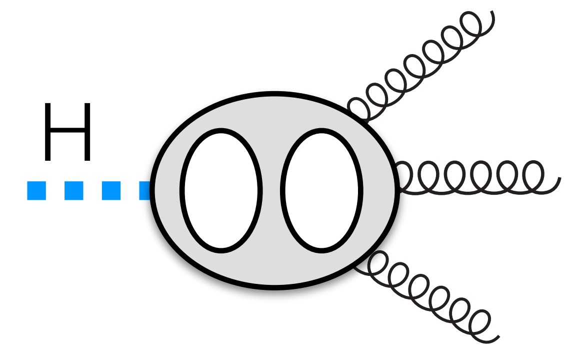
(cut) IBP reduction

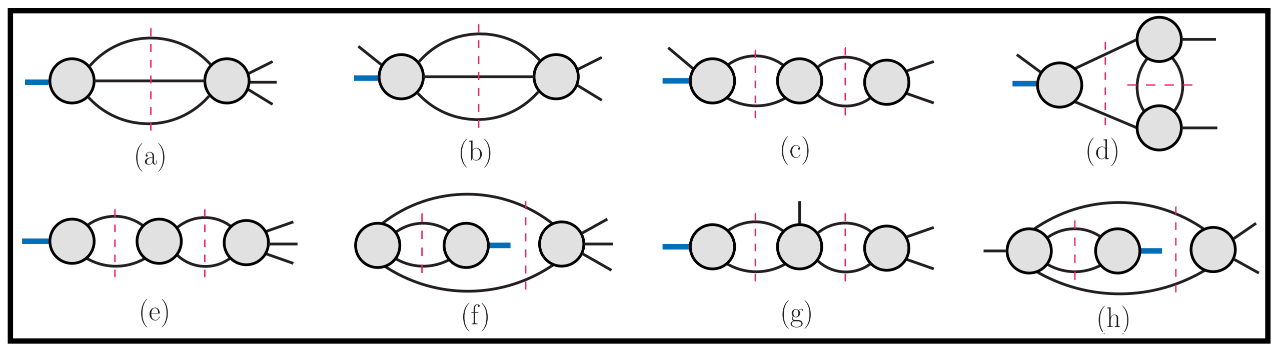
Jin, GY 2018 Boels, Jin, Luo 2018

Numerical unitarity: Abreu, Cordero, Ita, Jaquier, Page, Zeng 2017

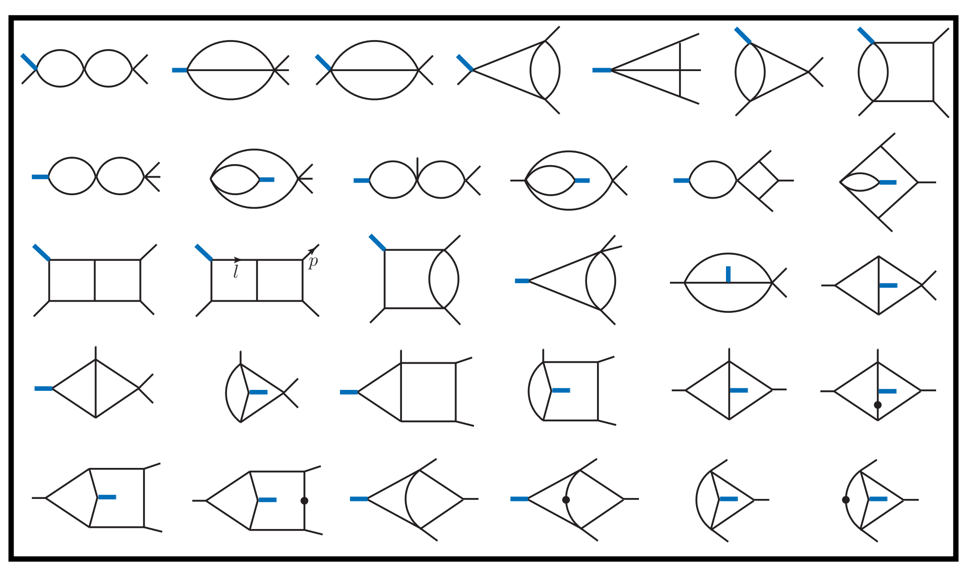
Unitarity cuts and master integrals

All cuts that are needed:





Master integrals are known in terms of 2d Harmonic polylogarithms.



[Gehrmann, Remiddi 2001]

Mixing matrix and spectrum

Jin, Ren, GY 2020

Dim-16 length-3 operators at 2-loop:

$$
Z^{(2)}_{\mathcal{O}_{16,f}}\Big|_{\frac{1}{\epsilon}-\text{part.}} = \frac{N_c^2}{\epsilon}
\left(\begin{array}{c|cccccccccccccc}
-\frac{34}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-\frac{1}{3} & \frac{269}{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & 0 & 0 & 0 \\
-\frac{209}{900} & -\frac{5579}{18000} & \frac{712}{125} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1493}{1200} & \frac{5}{36} & 0 & 0 & 0 \\
-\frac{31}{180} & \frac{53}{3600} & -\frac{36227}{28800} & \frac{3575983}{432000} & \frac{9793}{21600} & 0 & 0 & 0 & 0 & 0 & \frac{13}{16} & \frac{16877}{14400} & -\frac{7319}{14400} & 0 & 0 \\
-\frac{181}{900} & -\frac{60979}{36000} & \frac{78487}{72000} & -\frac{2177}{2000} & \frac{704167}{72000} & 0 & 0 & 0 & 0 & 0 & \frac{1229}{1200} & \frac{115501}{43200} & -\frac{9803}{43200} & 0 & 0 \\
-\frac{523}{3920} & -\frac{2201287}{29635200} & \frac{605939}{1975680} & -\frac{64128769}{24696000} & \frac{3303367}{9878400} & \frac{332422343}{29635200} & \frac{6699071}{14817600} & 0 & 0 & 0 & \frac{37547}{78400} & \frac{75071}{39200} & -\frac{497}{576} & \frac{103}{1440} & 0 \\
-\frac{809}{5600} & -\frac{12166789}{21168000} & \frac{11202299}{7056000} & -\frac{73487}{36750} & -\frac{9182209}{7056000} & \frac{37249}{156800} & \frac{26302879}{2116800} & 0 & 0 & 0 & \frac{1613}{3360} & \frac{17401}{6720} & \frac{19}{225} & \frac{1187}{2880} & 0 \\
-\frac{269}{2520} & \frac{125599}{10584000} & \frac{50369}{1323000} & -\frac{98317}{1176000} & \frac{73489}{392000} & -\frac{8625329}{3528000} & -\frac{97913}{756000} & \frac{90760559}{7408800} & \frac{25354501}{21168000} & \frac{40519}{56448} & \frac{184259}{1058400} & \frac{65297}{23520} & -\frac{420373}{211680} & \frac{248791}{235200} & -\frac{2747}{9408} \\
-\frac{19717}{176400} & \frac{3374557}{7408800} & -\frac{102465523}{74088000} & \frac{5260289}{1764000} & -\frac{6201763}{4939200} & -\frac{115070197}{24696000} & \frac{10687837}{9261000} & \frac{6498287}{9261000} & \frac{1025255701}{74088000} & -\frac{25511}{493920} & \frac{347437}{1764000} & \frac{863371}{302400} & -\frac{230747}{105840} & \frac{938797}{705600} & -\frac{78243}{196000} \\
-\frac{19717}{176400} & -\frac{2733089}{9261000} & \frac{88146899}{74088000} & -\frac{5678651}{3528000} & -\frac{1966229}{12348000} & \frac{17842339}{18522000} & -\frac{6878309}{4630500} & -\frac{58976629}{37044000} & \frac{8569667}{9261000} & \frac{179275483}{12348000} & \frac{28489}{661500} & \frac{54403}{14700} & -\frac{228689}{88200} & \frac{687461}{264600} & -\frac{485507}{5292000} \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{12} & 0 & 0 & 0 & 0 \\
-\frac{19}{36} & \frac{139}{2400} & \frac{499}{800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{143}{288} & \frac{2195}{288} & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{4}{15} & \frac{121}{400} & \frac{637}{800} & -\frac{211}{800} & 0 & 0 & 0 & 0 & 0 & \frac{119}{120} & -\frac{15643}{7200} & \frac{79313}{7200} & 0 & 0 \\
-\frac{209}{900} & \frac{6299}{21168} & \frac{6767}{35280} & \frac{71063}{88200} & -\frac{34723}{176400} & \frac{25841}{58800} & -\frac{36091}{264600} & 0 & 0 & 0 & \frac{22723}{21600} & -\frac{35}{48} & -\frac{2861}{5400} & \frac{443801}{36000} & 0 \\
-\frac{31}{180} & \frac{13843}{105840} & \frac{8317}{15120} & -\frac{797}{35280} & \frac{5477}{35280} & \frac{2417}{3528} & \frac{611}{105840} & \frac{13975}{14112} & -\frac{5377}{10584} & -\frac{3581}{10080} & \frac{114221}{151200} & \frac{6017}{15120} & \frac{121}{216} & -\frac{3661627}{1411200} & \frac{63879443}{4233600}
\end{array}\right)
$$

$$
Z^{(2)}_{\mathcal{O}_{16,d}}\Big|_{\frac{1}{\epsilon}-\text{part.}} = \frac{N_c^2}{\epsilon}
\begin{pmatrix}
\frac{575}{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{23347}{14400} & \frac{46517}{5760} & 0 & 0 & 0 & 0 & \frac{487}{1800} & 0 \\
\frac{3883}{4032} & -\frac{171823}{37800} & \frac{36597791}{3024000} & -\frac{29581}{16800} & 0 & 0 & -\frac{1789}{4800} & 0 \\
-\frac{9271}{11200} & -\frac{35239}{50400} & \frac{74209}{168000} & \frac{188599}{18900} & 0 & 0 & \frac{2101}{4800} & 0 \\
\frac{3287}{84000} & -\frac{2048479}{1176000} & \frac{422283}{392000} & -\frac{2501309}{1764000} & \frac{49211483}{3528000} & \frac{293221}{392000} & \frac{2764807}{2116800} & -\frac{61}{20160} \\
\frac{947587}{1058400} & -\frac{1555357}{705600} & \frac{16831}{29400} & -\frac{239641}{75600} & -\frac{381527}{2116800} & \frac{5839021}{423360} & -\frac{5807}{201600} & \frac{118933}{1411200} \\
\frac{3349}{7200} & -\frac{2591}{2400} & 0 & 0 & 0 & 0 & \frac{150391}{14400} & 0 \\
-\frac{45083}{44100} & \frac{16564}{11025} & \frac{5447}{117600} & \frac{380791}{176400} & \frac{1063}{29400} & -\frac{545189}{352800} & \frac{1176541}{1058400} & \frac{174229}{12600}
\end{pmatrix}
$$

Mixing matrices and spectrum

Two-loop anomalous dimensions for length-3 operators up to dimension 16:

Jin, Ren, GY 2020

dim	4	6	8	10	12	14	16
$\gamma^{(1)}_{f,\alpha}$	$-\frac{22}{3}$	/	$\frac{7}{3}$	$\frac{71}{15}$	$\frac{241}{30}, \frac{101}{15}$	$\frac{61}{6}, \frac{172}{21}$	$\frac{331}{35}, \frac{1212\pm\sqrt{3865}}{105}$
$\gamma^{(2)}_{f,\alpha}$	$-\frac{136}{3}$	/	$\frac{269}{18}$	$\frac{2848}{125}$	$\frac{49901119}{1404000}, \frac{8585281}{234000}$	$\frac{4392073141}{87847200}, \frac{685262197}{15373260}$	$\frac{231568398949}{4253886000}, \frac{355106171452034\pm95588158951\sqrt{3865}}{6576507756000}$
$\gamma^{(1)}_{f,\beta}$	$-\frac{22}{3}$	1	/	$\frac{17}{3}$	9	$\frac{43}{5}$	$\frac{67}{6}$
$\gamma^{(2)}_{f,\beta}$	$-\frac{136}{3}$	$\frac{25}{3}$	/	$\frac{2195}{72}$	$\frac{79313}{1800}$	$\frac{443801}{9000}$	$\frac{63879443}{1058400}$
$\gamma^{(1)}_{d,\alpha}$	/	/	/	$\frac{13}{3}$	$\frac{41}{6}$	$\frac{551\pm3\sqrt{609}}{60}$	$\frac{321\pm\sqrt{1561}}{30}$
$\gamma^{(2)}_{d,\alpha}$	/	/	/	$\frac{575}{36}$	$\frac{46517}{1440}$	$\frac{5809305897\pm19635401\sqrt{609}}{131544000}$	$\frac{229162584707\pm225658792\sqrt{1561}}{4130406000}$
$\gamma^{(1)}_{d,\beta}$	/	/	/	/	9	/	$\frac{67}{6}$
$\gamma^{(2)}_{d,\beta}$	/	/	/	/	$\frac{150391}{3600}$	/	$\frac{174229}{3150}$

Two-loop renormalization for higher length operators. Jin, Ren, GY, Yu 2022

Finite remainder

The transcendentality degree-4 part is universal:

$$\boxed{\begin{aligned} & -\frac{3}{2}\mathrm{Li}_4(u) + \frac{3}{4}\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{4}\log(w)\left[\mathrm{Li}_3\left(-\frac{u}{v}\right) + \mathrm{Li}_3\left(-\frac{v}{u}\right)\right] \\ & + \frac{\log^2(u)}{32}\left[\log^2(u) + \log^2(v) + \log^2(w) - 4\log(v)\log(w)\right] \\ & + \frac{\zeta_2}{8}\left[5\log^2(u) - 2\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 + \mathrm{perms}(u,v,w)\,, \end{aligned}}$$

It also appears as a universal function for length-3 operators in N=4 SYM

[Brandhuber, Kostacinska, Penante, Travaglini, Wen, Young 2014, 2016]

[Loebbert, Nandan, Sieg, Wilhelm, GY 2015, 2016]

“Maximal transcendentality principle” [Kotikov, Lipatov, Onishchenko, Velizhanin 2004]

Evanescent operators

Evanescent operator ("倏逝算符"):
Vanishing in 4 dimension but non-zero in $d = 4 - 2\epsilon$

$$\boxed{\mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, n\geq L}\Big|_{\text{4-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, L}\Big|_{d\text{-dim}} \neq 0\,.}$$

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Four-fermion dimension-6 operators:

$$\mathcal{O}^{(n)}_{\text{4-ferm}} = \bar{\psi}\gamma^{[\mu_1}\ldots\gamma^{\mu_n]}\psi\bar{\psi}\gamma_{[\mu_1}\ldots\gamma_{\mu_n]}\psi\,, \qquad n \geq 5\,.$$

Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich and U. Nierste 1994

Evanescent operators

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$$\mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, n\geq L}\Big|_{4\text{-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, L}\Big|_{d\text{-dim}} \neq 0\,.$$

Gluonic evanescent operators (start to appear at dimension 10):

$$\mathcal{O}_{\rm e} = \frac{1}{16}\delta^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{\nu_1\nu_2\nu_3\nu_4\nu_5}\mathrm{tr}(D_{\nu_5}F_{\mu_1\mu_2}F_{\mu_3\mu_4}D_{\mu_5}F_{\nu_1\nu_2}F_{\nu_3\nu_4})$$

$$\delta^{\mu_1\cdot\cdot\mu_n}_{\nu_1\ldots\nu_n} = \det(\delta^{\mu}_{\nu}) = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_1}_{\nu_n} \\ \vdots & & \vdots \\ \delta^{\mu_n}_{\nu_1} & \ldots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}$$

Length-4 basis counting

Δ_0	$N^{\rm p}_+ = N^{\rm p}$	$N^{\rm e}_+$	$N^{\rm e}_-$
8	4	0	0
10	20	4	0
12	82	24	1
14	232	88	4
16	550	246	13

Jin, Ren, GY, Yu, 2202.08285, 2208.08976

Evanescent operators

Systematic classification and renormalization at two-loop order.

Jin, Ren, GY, Yu, 2022

Evanescent operators are important for renormalization beyond one-loop order.

$$\begin{pmatrix} Z_{\rm pp}^{(1)} & Z_{\rm pe}^{(1)} \\ 0 & Z_{\rm ee}^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} Z_{\rm pp}^{(l)} & Z_{\rm pe}^{(l)} \\ Z_{\rm ep}^{(l)} & Z_{\rm ee}^{(l)} \end{pmatrix}, \quad l \geq 2$$

One can use finite renormalization scheme such that

$$\begin{pmatrix} \hat{\mathcal{D}}_{\rm pp}^{(l)} & \hat{\mathcal{D}}_{\rm pe}^{(l)} \\ 0 & \hat{\mathcal{D}}_{\rm ee}^{(l)} \end{pmatrix}$$

but the lower-loop evanescent operator result are needed.

For example, $\hat{\mathcal{D}}_{\rm pp}^{(2)}$ contains $(-2\epsilon \hat{Z}_{\rm pe}^{(1)} \hat{Z}_{\rm ep}^{(1)})$

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

The answer is NO.

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

The answer is NO.

$$
\begin{aligned}
&\partial_\nu \partial_\rho \left[\delta^{12456\nu}_{3789\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right],\\
&\partial_\nu \partial_\rho \left[\delta^{1}_{4}\delta^{2356\nu}_{789\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]\\
&\partial_\nu \partial_\rho \left[\delta^{1}_{4}\delta^{2356\nu}_{789\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]\\
&\partial_\nu \partial_\rho \left[\delta^{1}_{4}\delta^{2357\nu}_{689\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]\\
&\partial_\nu \partial_\rho \left[\delta^{1}_{4}\delta^{2367\nu}_{589\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]\\
&\partial_\nu \partial_\rho \left[\delta^{1}_{4}\delta^{2378\nu}_{569\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]\\
&\partial_\nu \partial_\rho \left[\delta^{1}_{5}\delta^{2347\nu}_{689\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]\\
&\partial_\nu \partial_\rho \left[\delta^{2}_{4}\delta^{1567\nu}_{389\mu\rho}\Big(\mathrm{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \mathrm{Rev.}\Big)\right]
\end{aligned}
$$

Dim-12 evanescent operators

$$
\begin{pmatrix}
-\frac{38}{3\epsilon} & \frac{2}{\epsilon} & -\frac{13}{12\epsilon} & 0 & \frac{14}{3\epsilon} & 0 & \frac{14}{3\epsilon} & \frac{28}{3\epsilon} \\
-\frac{1}{2\epsilon} & -\frac{85}{6\epsilon} & \frac{2}{\epsilon} & \frac{5}{6\epsilon} & -\frac{2}{3\epsilon} & -\frac{5}{12\epsilon} & -\frac{7}{3\epsilon} & -\frac{16}{3\epsilon} \\
0 & -\frac{4}{\epsilon} & -\frac{22}{3\epsilon} & \frac{16}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & \frac{16}{3\epsilon} \\
0 & -\frac{4}{3\epsilon} & \frac{7}{3\epsilon} & -\frac{34}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & 0 \\
\frac{1}{12\epsilon} & -\frac{1}{12\epsilon} & -\frac{3}{8\epsilon} & \frac{1}{12\epsilon} & -\frac{44}{3\epsilon} & \frac{5}{8\epsilon} & \frac{1}{2\epsilon} & \frac{2}{\epsilon} \\
0 & \frac{4}{3\epsilon} & \frac{2}{3\epsilon} & 0 & 0 & -\frac{18}{\epsilon} & 0 & -\frac{16}{3\epsilon} \\
\frac{1}{6\epsilon} & \frac{3}{2\epsilon} & \frac{9}{16\epsilon} & -\frac{1}{2\epsilon} & \frac{29}{6\epsilon} & -\frac{5}{12\epsilon} & -\frac{49}{6\epsilon} & \frac{13}{3\epsilon} \\
-\frac{5}{6\epsilon} & -\frac{1}{3\epsilon} & \frac{13}{32\epsilon} & -\frac{5}{6\epsilon} & \frac{3}{4\epsilon} & \frac{1}{4\epsilon} & \frac{5}{12\epsilon} & -\frac{91}{6\epsilon}
\end{pmatrix}
$$

One-loop mixing matrix

A pair of complex eigenvalues:

$$1.90386 \pm 0.181142\,\mathrm{i}\,.$$

Jin, Ren, GY, Yu, 2023

Maximally transcendental principle

N=4 SYM ⟷ QCD

Maximal Transcendentality Principle

Conjecture for certain quantities

N=4 SYM ⟷ **QCD**

The maximally transcendental parts are equal in two theories.

- Such a relation was first observed for anomalous dimension of twist-2 operators

$$\boxed{\gamma^{\mathcal{N}=4}(j) = \gamma^{\mathrm{QCD}}(j)|_{\text{max. trans}}}$$

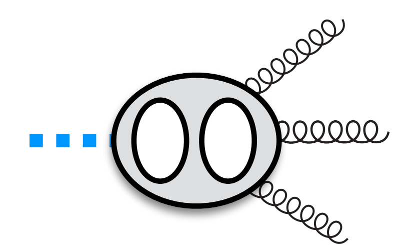
Kotikov, Lipatov 2001; Kotikov, Lipatov, Onishchenko, Velizhanin 2004

Finite remainder of two-loop form factor

$$
\begin{aligned}
&-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
&+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
&-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
&-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
&-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
&+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
&-G(0,-v,1-v,u)H(\ldots 1-v,0,1-v,u)H(0,v)\\
&-G(1-v,1,0,u)H(0, \ldots)H(0,v)+H(1,0,0,1,v)\\
&-G(0,0,1-v,u)H(1, \ldots -v,0,u)H(1,v)\\
&+2G(0,-v,1-v,u)H \ldots -G(1-v,0,-v,1-v,u)\\
&+G(1-v,-v,0,u)H(\ldots v,u)H(1,v)\\
&-4G(-v,-v,1-v,u)H(1,v)+4G(-v,-v,-v,u)H(1,v)+G(0,0,u)H(0,0,v)+G(0,1-v,u)H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
&-G(0,0,u)H(0,1,v)+G(0,-v,u)H(0,1,v)-G(1,0,u)H(0,1,v)+2G(1-v,0,u)H(0,1,v)+2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
&-G(-v,0,u)H(0,1,v)-2G(-v,1-v,u)H(0,1,v)+4G(-v,-v,u)H(0,1,v)-G(0,0,u)H(1,0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
&+2G(1-v,0,u)H(1,0,v)-2G(1-v,1-v,u)H(1,0,v)+G(1-v,-v,u)H(1,0,v)-G(-v,0,u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
&-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
&+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
&-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
&+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
&+4G(-v,1-v,-v,1-v,u)
\end{aligned}
$$

Maximal transcendental part of QCD Higgs-3-gluon amplitude

Gehrmann, Jaquier, Glover, Koukoutsakis 2011



The same !

N=4 SYM three-point form factor

$$
-2\left[J_4\left(-\frac{uv}{w}\right)+J_4\left(-\frac{vw}{u}\right)+J_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]
$$

$$
-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4
$$

Brandhuber, Travaglini, GY 2012

Master-integral bootstrap

A bootstrap strategy to compute amplitudes or form factors: Guo, Wang, GY 2021

Ansatz → **Physical constraints** → **Solution**

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR, collinear limits, spurious pole cancellation, unitarity-cut, etc

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

Proof of MTP for form factors

A bootstrap strategy to compute amplitudes or form factors: Guo, Wang, GY 2021

Ansatz → **Physical constraints** → **Solution**

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR, collinear limits, spurious pole cancellation, unitarity-cut, etc

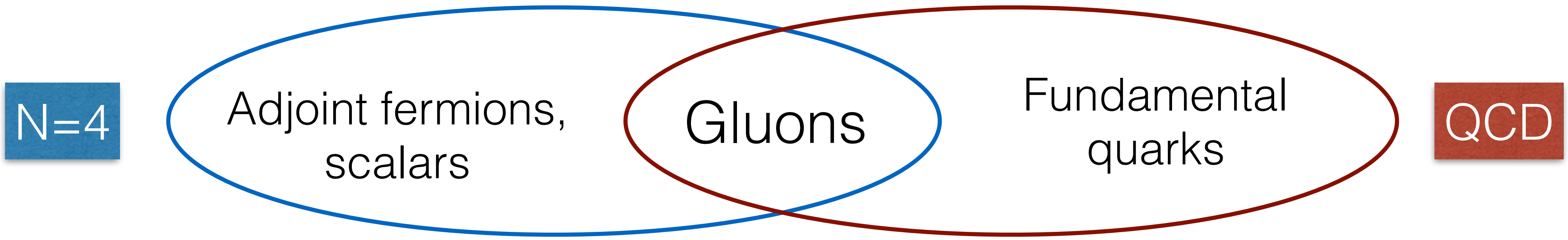
$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

Theory-independent constraints → **Theory-independent results**

IR and collinear are **universal** at MT level, and some unitarity cuts are also universal.

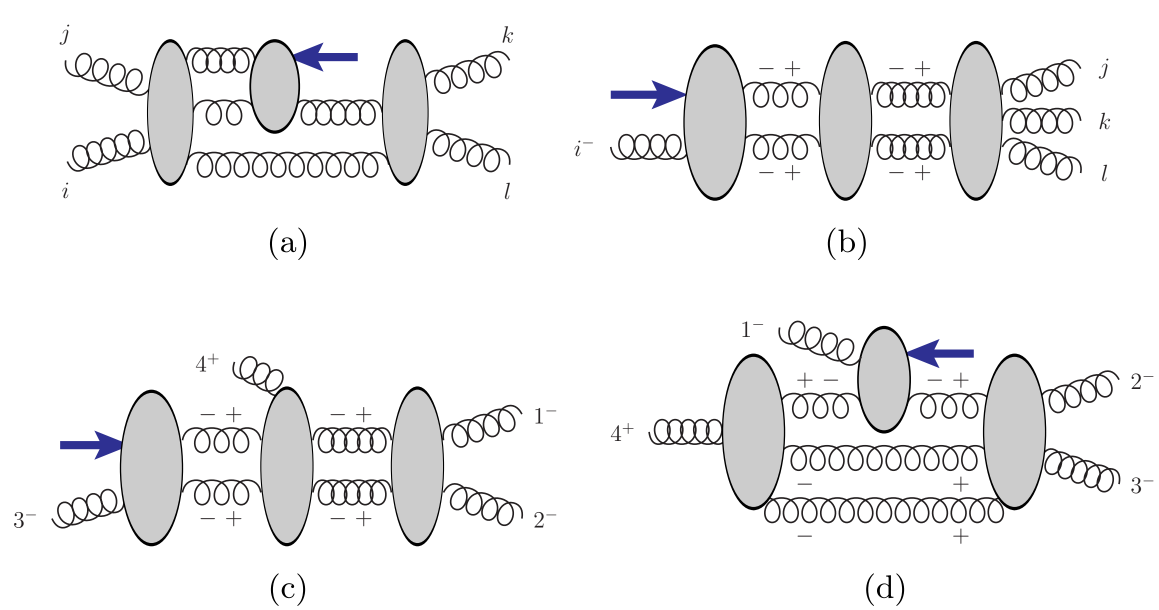
Guo, Jin, Wang, GY 2022

Universal gluon cuts

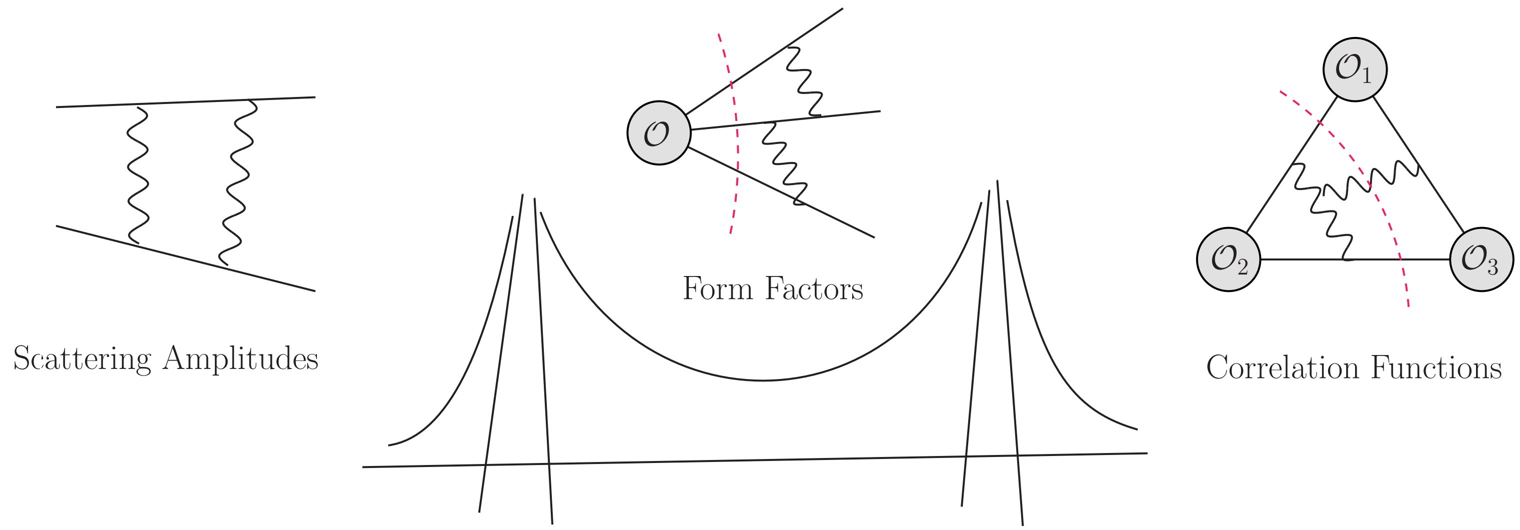


Certain cut channels involving only gluons and therefore are same in both N=4 and QCD, e.g.

Guo, Jin, Wang, GY 2022

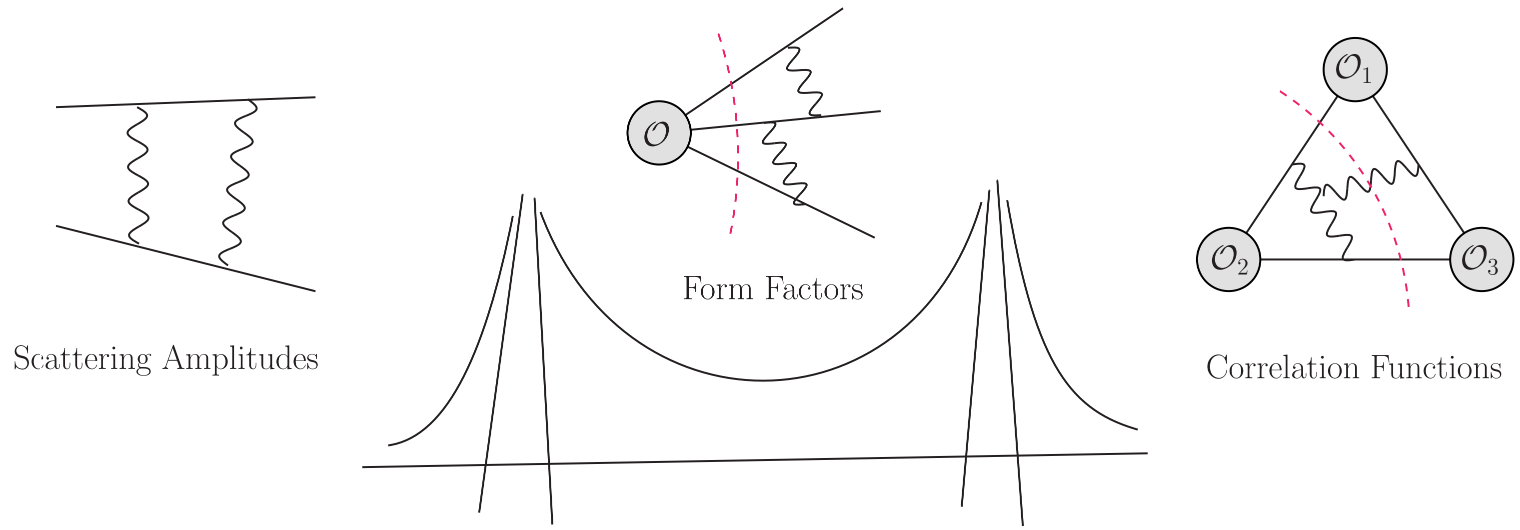


Summary



- Form factors provide a framework to study many interesting physical quantities using powerful on-shell amplitude methods:
  - IR divergences
  - UV renormalization
  - Finite remainder
- New hidden structure of form factor
  - CK-duality, MTP
  - double-copy of form factors, DDCI, FFOPE, etc.

Summary



- Form factors provide a framework to study many interesting physical quantities using powerful on-shell amplitude methods:
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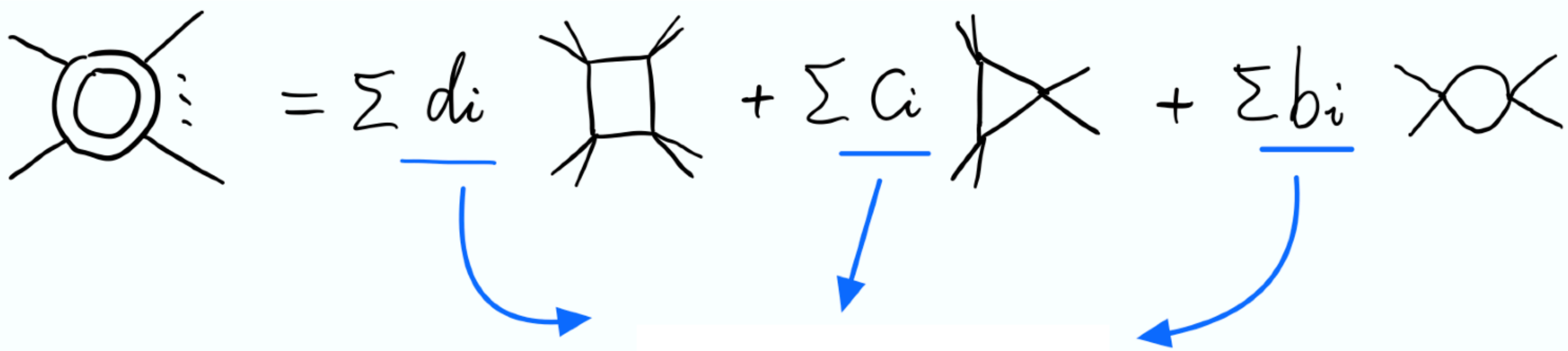
Thank you for your attention!

Backup slides

Thank you for your attention!

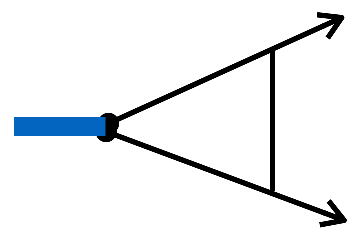
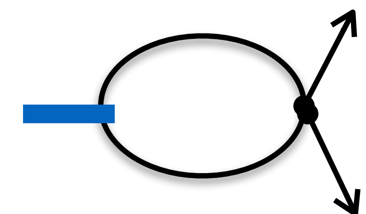
Unitarity computation

Consider one-loop amplitudes:

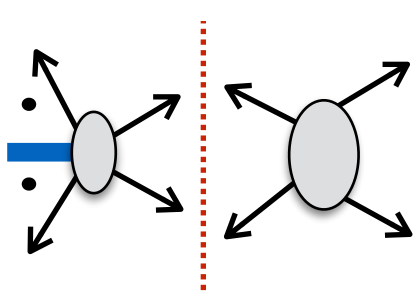
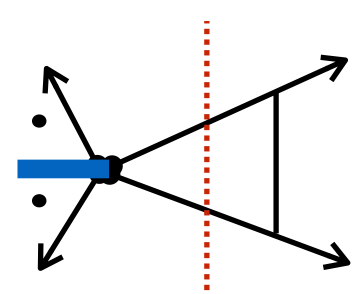
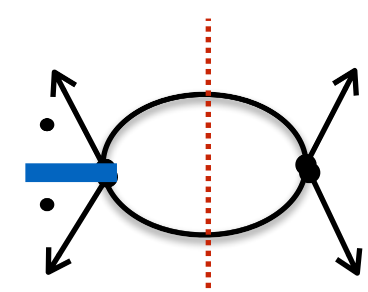


What we really want

Unitarity computation

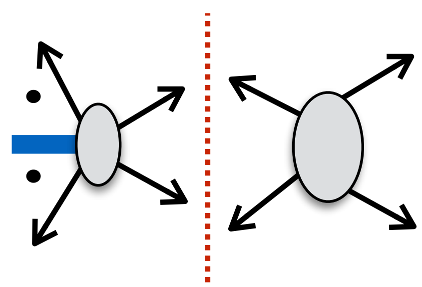
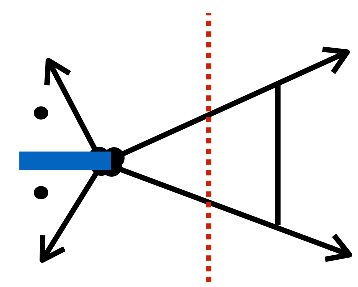
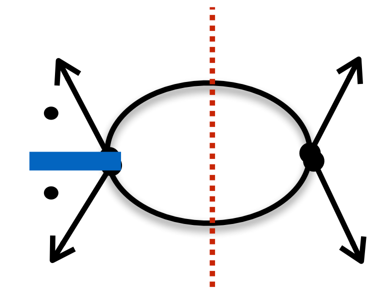
$F_2^{(1)} =$ Ctri  + Cbub 

The basis coefficient can be computed by cuts:

 → Ctri  + Cbub 

$$\mathscr{F}_2^{(1)}(1,2)\Big|_{s_{12}\text{-cut}} = \int d\text{PS}_2\, \mathscr{F}_2^{(0)}(-l_1, -l_2)\mathscr{A}_4^{(0)}(1,2,l_2,l_1)$$

Unitarity computation

 $\longrightarrow$ $C_{\rm tri}$  $+$ $C_{\rm bub}$ 

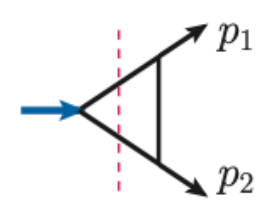
$$\mathcal{F}_2^{(1)}(1,2)\big|_{s_{12}\text{-cut}} = = \int d\mathrm{PS}_2 \sum_{\text{helicity of } l_i} \mathcal{F}_2^{(0)}(-l_1,-l_2) \times \mathcal{A}_4^{(0)}(1,2,l_2,l_1)$$

$$= \mathcal{F}_2^{(0)}(1,2)\, i \int d\mathrm{PS}_2\, 1 \times \frac{\langle l_1 l_2\rangle\langle 12\rangle}{\langle l_1 p_1\rangle\langle l_2 2\rangle}$$

$$\mathcal{O} = \mathrm{tr}(\phi_{12}^2)$$

$$= \mathcal{F}_2^{(0)}(1,2)\, i \int d\mathrm{PS}_2\, \frac{-s_{12}}{(l_1+p_1)^2}$$

$$= \mathcal{F}_2^{(0)}(1,2)(-s_{12})$$

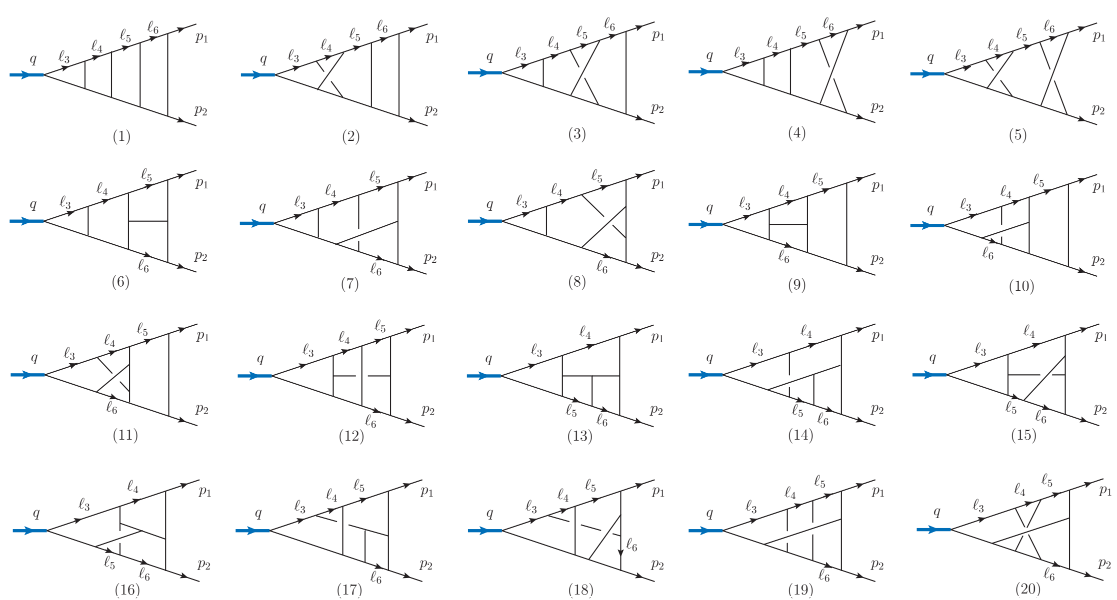


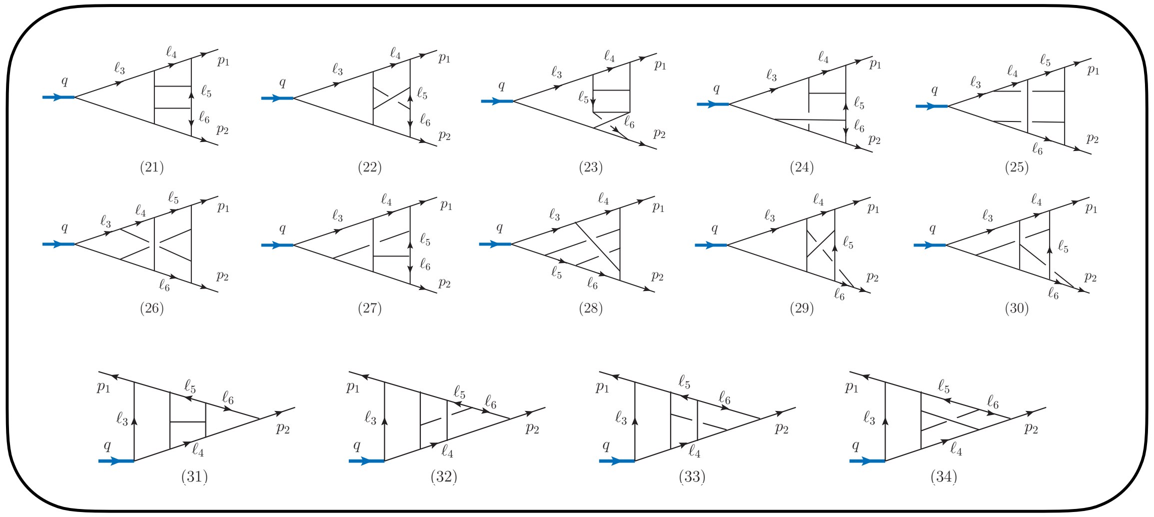
$$\longrightarrow \qquad C_{\rm tri} = -\, s_{12}\,, \qquad C_{\rm bub} = 0$$

Four-loop integrand with NO Feynman diagrams

[Boels, Kniehl, Tarasov, GY 2012]

Four-loop form factor integrand was obtained by:
color-kinematics duality and unitarity:





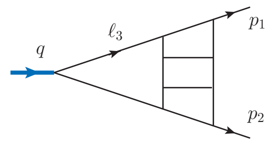
non-planar

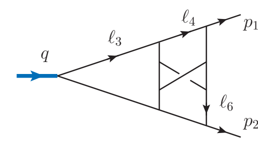
compact form and with only quadratic loop momenta in the numerator.

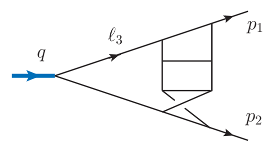
$$
\begin{aligned}
N_{21} = \; & -(\ell_3 \cdot p_1)^2 - (\ell_3 \cdot p_2)^2 - 6(\ell_3 \cdot p_1)(\ell_3 \cdot p_2) \\
& + (p_1 \cdot p_2)\big[2(\ell_3 \cdot \ell_3) + 4(\ell_3 \cdot p_1) + p_1 \cdot p_2\big] \\
& + (\alpha_1 + 1)\big[(\ell_3 \cdot p_{12} - p_1 \cdot p_2)^2 \\
& - \tfrac{2}{7}(\ell_3 \cdot (\ell_3 - p_{12}) + p_1 \cdot p_2)(p_1 \cdot p_2)\big]
\end{aligned}
$$

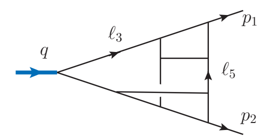
Four-loop non-planar cusp AD

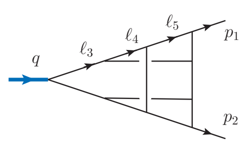
[Boels, Huber, GY 2017]

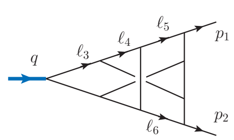
$I_1^{(21)} =$  $\times\ [(\ell_3 - p_1)^2]^2$

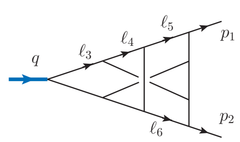
$I_2^{(22)} =$  $\times\ (\ell_3 - p_1)^2\,[\ell_4^2 + \ell_6^2 - \ell_3^2 + (\ell_3 - \ell_4 + p_1)^2 + (\ell_3 - \ell_6 - p_1)^2]$

$I_3^{(23)} =$  $\times\ [(\ell_3 - p_1)^2]^2$

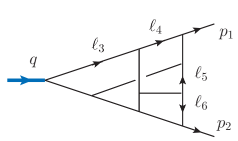
$I_4^{(24)} =$  $\times\ (\ell_3 - p_1)^2\,[(q - \ell_3 - \ell_5)^2 + (\ell_5 + p_2)^2]$

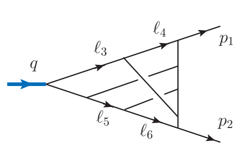
$I_5^{(25)} =$  $\times\ \left\{ \left[(p_1 - \ell_5)^2 + 2(\ell_4 - \ell_5)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - \ell_5)^2 - (p_1 - \ell_4)^2\right]^2 - 4\,(\ell_4 - \ell_5)^2\,(p_1 - \ell_3 + \ell_4 - \ell_5)^2 \right\}$

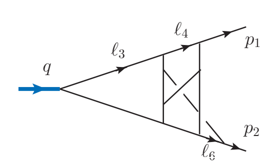
$I_6^{(26)} =$  $\times\ \left\{[(\ell_3 - \ell_4 - \ell_5)^2 - (\ell_3 - \ell_4 - p_1)^2 - (\ell_6 - p_2)^2 - \ell_5^2]\,[\ell_5^2 - \ell_4^2 - \ell_6^2 + (\ell_4 - \ell_6)^2] + 4\,\ell_5^2\,(\ell_6 - p_2)^2 + (\ell_4 - \ell_5)^2\,(\ell_3 - \ell_4 + \ell_6 - p_2)^2\right\}$

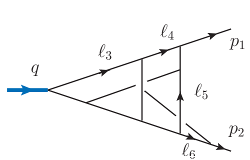
$I_7^{(26)} =$  $\times\ \Big\{4\,[(\ell_4 - \ell_5)(\ell_3 - \ell_4 + \ell_5 - p_1)]\,[(\ell_4 - \ell_6)(\ell_3 - \ell_4 + \ell_6 - p_2)]\ - \ell_5^2\,(\ell_6 - p_2)^2 - 4\,(\ell_4 - \ell_5)^2\,(\ell_3 - \ell_4 + \ell_6 - p_2)^2 - \ell_6^2\,(\ell_5 - p_1)^2$

$- (\ell_3 - \ell_4)^2\,(\ell_5 + \ell_6 - \ell_4)^2 - \ell_4^2\,(\ell_3 - \ell_4 + \ell_5 + \ell_6 - p_1 - p_2)^2\Big\}$

$I_8^{(27)} =$  $\times\ \frac{1}{2}\,[\ell_3^2 - \ell_4^2 - (\ell_4 - \ell_3 - p_1)^2]\ [(\ell_3 - \ell_4 - \ell_5)^2 + (\ell_5 + p_2)^2]$

$I_9^{(28)} =$  $\times\ (\ell_3 - \ell_4 - p_2)^2\ [(\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2]$

$I_{10}^{(29)} =$  $\times\ \frac{1}{2}\,[\ell_3^2 - \ell_4^2 - (\ell_4 - \ell_3 - p_1)^2]\ [\ell_6 \cdot (\ell_6 - \ell_4 + \ell_3 - p_2)]$

$I_{11}^{(30)} =$  $\times\ (\ell_3 - \ell_4 - p_2)^2\,[(p_1 - \ell_4)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2]$

Plus 12 simpler 11- and 10-line integrals

Results

The full form factor result is:

ϵ order	-8	-7	-6	-5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	-4	-3	-2	-1
result	$+3.5 \times 10^{-6}$	$+ 0.0007$	$+1.60$	-17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	± 0.0186	± 0.19	± 3.25

- Four-loop non-planar cusp AD:

Boels, Huber, GY 2017

$$\gamma^{(4)}_{\text{cusp, NP}} = -3072 \times (1.60 \pm 0.19) \frac{1}{N_c^2}$$

- Analytic result in 2019:

Huber, von Manteuffel, Panzer, Schabinger, GY 2019;
Henn, Korchemsky, Mistlberger 2019

$$\gamma^{(4)}_{\text{cusp,NP}} = -3072 \times \left(\frac{3}{8}\zeta_3^2 + \frac{31}{140}\zeta_2^3\right)\frac{1}{N_c^2} = -3072 \times 1.52 \frac{1}{N_c^2}$$

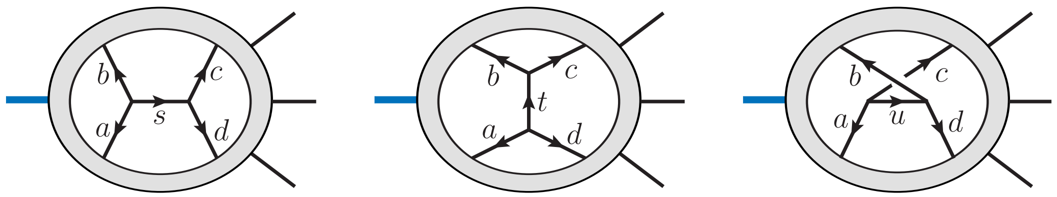
Strategy of loop computation

CK-duality

$$\mathscr{F}^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D}$$

Conjecture !

Compact ansatz of the loop integrand



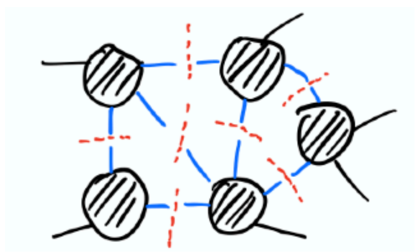
$$C_s = C_t + C_u \longrightarrow N_s = N_t + N_u$$

Strategy of loop computation

CK-duality

Conjecture !

Compact ansatz of the loop integrand



Unitarity cuts

$$\text{Loop-ansatz}\,|_{\text{cut}} = \prod \text{tree-blocks}$$

Solving linear equations

Main challenge: it is a priori not known whether the solution exists

Other 4-point form factors

Ansatz → **Physical constraints** → Solution

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR, collinear limits,
unitarity-cut, etc

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i {\color{red}C_i} I_i^{(l)}$$

The strategy has been used to compute four-point form factors of length-2 and length-3 operators:

$$F^{(2)}_{\text{tr}(\phi^3)}(1^\phi,2^\phi,3^\phi,4^g) \qquad F^{(2)}_{\text{tr}(F^3)}(1^g,2^g,3^g,4^g) \qquad F^{(2)}_{\text{tr}(F^2)}(1^g,2^g,3^g,4^g)$$

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Double copy of form factor

Gauge x Gauge —Double-copy→ Gravity

- An surprising new mechanism for form factors:

Spurious poles (Gauge theory) —Double-copy→ Real propagators (Gravity theory)

- Hidden “factorization” relations of gauge form factors

$$\vec{v} \cdot \vec{\mathcal{F}}_n \Big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

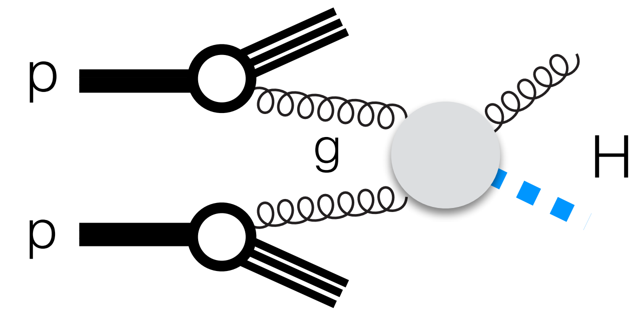
Guanda Lin, GY 2111.12719, 2211.01386, 2306.04672

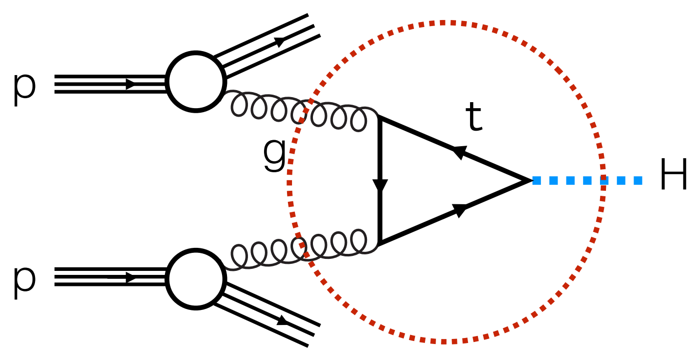
Higgs+gluons scattering

Higgs plus jet production

$$A(q^H, 1^g, 2^g, \ldots, n^g) = F_{\mathcal{O}=\mathrm{tr}(F^2)}(1^g, 2^g, \ldots, n^g)$$

Boughezal, Caola, Melnikov, Petriello, Schulze 2013; Chen, Gehrmann, Glover, Jaquier 2014; Boughezal, Focke, Giele, Liu, Petriello 2015; Harlander, Liebler, Mantler 2016; Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger 2016; Lindert, Kudashkin, Melnikov, Wever 2018; Jones, Kerner, Luisoni 2018; Neumann 2018; …





$m_t \to \infty$

Wilczek, 1977; Shifman et.al., 1979, ….

$$\mathcal{L}_{\mathrm{eff}} = C_0 O_0 + \frac{1}{m_{\mathrm{t}}^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_{\mathrm{t}}^4}\right)$$

Dimension-5 operator

$$O_0 = H\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

2-loop: Gehrmann, Jaquier, Glover, Koukoutsakis 2011

Dimension-7 operators

$$O_1 = H\mathrm{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu)\,,$$
$$O_2 = H\mathrm{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu})\,,$$
$$O_3 = H\mathrm{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu})\,,$$
$$O_4 = H\mathrm{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu})\,.$$

1-loop: Dawson, Lewis, Zeng 2014
2-loop: Jin, GY 2019

Lower degree parts

Degree-3 part and degree-2 part are consist of universal building blocks {T3, T2}, plus simple log functions:

$$
\begin{aligned}
T_3(u,v,w) :=\Big[& -\mathrm{Li}_3\left(-\frac{u}{w}\right)+\log(u)\mathrm{Li}_2\left(\frac{v}{1-u}\right)-\frac{1}{2}\log(u)\log(1-u)\log\left(\frac{w^2}{1-u}\right)\\
& +\frac{1}{2}\mathrm{Li}_3\left(-\frac{uv}{w}\right)+\frac{1}{2}\log(u)\log(v)\log(w)+\frac{1}{12}\log^3(w)+(u\leftrightarrow v)\Big]\\
& +\mathrm{Li}_3(1-v)-\mathrm{Li}_3(u)+\frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right)-\zeta_2\log\left(\frac{uv}{w}\right)\,.
\end{aligned}
$$

$$
T_2(u,v) :=\mathrm{Li}_2(1-u)+\mathrm{Li}_2(1-v)+\log(u)\log(v)-\zeta_2\,.
$$