RG-improved low-energy theorem for the effective Higgs-gluon-gluon coupling for simultaneous decoupling of several heavy quarks

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Introduction: idea and hystory of decoupling and Low Energy Energy Theorems

the full QCD with 6 flavoures is inconvenient if charactersitic scale $Q << m_{top}$.

Two main reasons:

- appearance potentially large logs ot type $\ln Q^2/m_{top}^2$
- calculations become unnecessarily complicated

It is much more convenient to use low-energy efficient QCD without the top-quark. The corresponding Lagrangian has the standard QCD form, but with 5 active quarks plus (if necessary) power-like $(1/m_{top})^n$ corrections.

But there is a subtlety: the famous decoupling theorem of T. Appelquist and J. Carazzone (1975) in its literal form does not work for the /most computationally efficient/ MSbar scheme. (For instance the β-function is **mass-independent**)

The situation for minimal subtractions was clarified in the works of S. Weinberg (1980); B. Ovrut, H. Schnitzer (1981): the concept of an effective theory with a corresponding effective Lagrangian (i.e. two different (but one to one connected!) Lagrangians for two different kinematic modes)

More formally: consider the QCD Lagrangian with one heavy quark h and n_ℓ light:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_{i=1-n_\ell} \bar{\psi}_i \left(\frac{i}{2} \overleftrightarrow{\not{D}} - m_i \right) \psi_i + \bar{h} \left(\frac{i}{2} \overleftrightarrow{\not{D}} - m_h \right) h$$

$$+\lambda_0 H \left(m_h \bar{h} h + \sum_{i=1-n_\ell} m_i \bar{\psi}_i \psi_i \right)$$

where we introduced the interaction of the Higgs field H with quarks. This Lagrangian is suitable for calculating the decay of $H \to hadrons$. The effective $\mathcal{L}'$ with n_ℓ light quarks assumes the form

$$\mathcal{L}' = -\frac{1}{4} \left(G^a_{\mu\nu} G^{a\,\mu\nu} \right)' + \sum_{i=1-n_\ell} \bar{\psi}'_i \left(\frac{i}{2} \overleftrightarrow{\not{D}}' - m'_i \right) \psi'_i$$

$$+ + \lambda_0 H \left(C_1 \left(-\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} \right)' + C_2 \sum_{i=1-n_\ell} m_i \bar{\psi}'_i \psi'_i \right)$$

It describes physics on scale below the heavy quark mass m_h (all power suppressed corrections are ignored!)

Connection between $\mathcal{L}$ and $\mathcal{L}'$

$$\mathcal{L}' = -\frac{1}{4}\left(G^a_{\mu\nu}G^{a\,\mu\nu}\right)' + \sum_{i=1-n_\ell} \bar{\psi}'_i \left(\frac{i}{2}\overleftrightarrow{\not{D}}' - m'_i\right)\psi'_i$$

$$+ + \lambda_0 H\left(C_1\left(-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}\right)' + C_2 \sum_{i=1-n_\ell} m'_i\bar{\psi}'_i\psi'_i\right)$$

where all primed variables refer to QCD $\mathcal{L}'$ with n_ℓ quarks related with the unprimed ones by simple formulas

$(a' \equiv {\alpha_s}'/\pi \equiv (g')^2_s(\mu)/(4\pi^2), a \equiv \alpha_s/\pi \equiv (g_s)^(\mu)/(4\pi^2))$:

$$a' = \alpha_s\,\zeta_\alpha\left(a, \ln\frac{\mu^2}{m_h^2(\mu)}\right),\ m' = m\,\zeta_m\left(a, \ln\frac{\mu^2}{m_h^2(\mu)}\right)$$

$+$ similar relations for fields A, ψ'
$+$ dependence of both "decoupling" functions ζ_α, ζ_m as well as C_1 and C_2 **only** on $a(\mu)$ and $\ln\frac{\mu^2}{m_h^2(\mu)}$.

Convenient notations:

in the full theory with n_f quarks:

$$O_1 = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} \equiv O_1^{(n_f)}, \quad O_2 = \sum_{i=1-n_f} m_i \bar{\psi}_i \psi \equiv O_2^{(n_f)},$$

in the effective theory with $n_\ell = n_f - 1$ quarks

$$O_1' = -\frac{1}{4} \left(G^a_{\mu\nu} G^{a\,\mu\nu} \right)' \equiv O_1^{(n_\ell)}, \quad O_2' = \sum_{i=1-n_f} \left(m_i \bar{\psi}_i \psi \right)' \equiv O_2^{(n_\ell)},$$

Similar notations will be used for CF's C_1, C_2 and α:

$$C_1 \equiv C_1^{(n_f)}, \quad C_2 \equiv C_2^{(n_f)} \quad \alpha_s \equiv {\alpha_s}^{(n_f)}, \quad {\alpha_s}' \equiv {\alpha_s}^{(n_\ell)}$$

History: evaluation of decoupling constants $\zeta_\alpha, \ \zeta_m$

- 2 loops: W. Bernreuther, W. Wetzel (1982); Erratum (1998)
  S. A. Larin, T. van Ritbergen, J. A. M. Vermaseren (1995)
- 3 loops: K. Ch., B. A. Kniehl, M. Steinhauser (1998)
- 4 loops: K. Ch., J. H. Kühn, C. Sturm (2006)
  Y. Schröder, M. Steinhauser (2006)

first 2-loops results were obtained with (over)complicated calculations dealing with massive diagrams depending on an external momentum (no smth like "method" of regions was available then)

3- and 4-loop results were made possible by:

advances in theory:

the projector method /Gorishny, S. Larin, F. Tkachev (1983,1988)/ (deals with massive (that is one scale) vacuum diagrams (tadpoles)

Integration By Parts (IBP)

advances in computer algebra based approaches:

the FORM language /J. Vermaseren (1990 . . .)
and (FORM) program MATAD /M. Steinhauser (1996 . . . /3-loops/)
Laporta method /Laporta (1996 . . . /4-loops/)

Effective couplings Higgs with gluons and light quarks: LET

Thus, after heavy quark decoupling h the effective Lagrangian for Higgs reads:

$$\lambda_0 H \left(C_1 O_1' + C_2 O_2' \right)$$

Many years ago there were derived two pretty Low energy Theorems (LET, /K.Ch., Kniehl, Steinhauser (1998)/) which is valid in (all orders of PT!) in α_s /$\overline{\text{MS}}$ assumed!/

$$C_1 = m_h \frac{\partial}{\partial m_h} \ln \zeta_\alpha \quad \textbf{and} \quad C_2 = m_h \frac{\partial}{\partial m_h} \ln \zeta_m \equiv m_h \frac{\partial}{\partial m_h} \ln m'$$

Remarkable feature of the both decoupling constants ζ_α and ζ_m: their dependence only on the ratio μ^2/m_h^2 (due to trivial dimensional considerations), that is $\zeta_\alpha \equiv \zeta_\alpha(a_s, \frac{\mu^2}{m_h^2})$ Thus, neither C_1 nor C_2 depend on constant parts of ζ_α and ζ_m respectively! This opens a way for RG-improvement (as logs of μ^2/m_h^2 could be found and restored via the RG-equation). Already in 1998 the original 3-loop results for ζ_α and ζ_m led to 4-loop ones for C_1 and C_2 (with the use of the 4-loop QCD β-function and the 4-loop quark anomalous dimension γ_m /just computed in 1997 by J. Vermaseren, S. Larin, T. van Ritbergen and K.Ch. respectively/)

Decoupling for several heavy quarks and LET's

Having in mind various extensions of the SM containing either additional quarks heavier than the top one or Higgs-like scalar particles with mass of order a few GeV or even less[★] let us consider a generic case with the field H not necessarily being the one from the SM[★★]. Our only assumptions are: (i) the field H couples with quarks via a top-like (that is proportional to the corresponding quark masses) Yukawa couplings and (ii) its mass M_H is larger than masses of light quarks and less than masses of heavy quarks:

$$m_i \gg M_H \gg m_j \quad \text{with} \quad (n_\ell + 1) \leq i \leq n_f \quad \text{and } 1 \leq j \leq n_\ell.$$

In the framework of the SM we naturally have $n_h = 1$ and $n_\ell = 5$.

[★] see, e.g. D. Gorbunov, E. Kriukova and O. Teryaev, "Scalar decay into pions via Higgs portal," [arXiv:2303.12847]

[★★] For simplicity we continue to refer to the field H as the the Higggs one.

$$\mathcal{L}^{n_f} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_{i=1-n_\ell} \overline{\psi}_i \left(\frac{i}{2} \overleftrightarrow{\not{D}} - m_i \right) \psi_i + \sum_{i=(n_{ell}+1)-n_h} \overline{\psi}_i \left(\frac{i}{2} \overleftrightarrow{\not{D}} - m_i \right) \psi_i$$

$$+\lambda_0 H \left(\sum_{i=1-n_\ell} m_i \overline{\psi}_i \psi_i + \sum_{i=(n_\ell+1)-n_f} m_i \overline{\psi}_i \psi_i \right).$$

$$\mathcal{L}^{n_l} = -\frac{1}{4} \left(G^a_{\mu\nu} G^{a\,\mu\nu} \right)' + \sum_{i=1-n_\ell} \overline{\psi}'_i \left(\frac{i}{2} \overleftrightarrow{\not{D}} - m_i \right) \psi'_i + \lambda_0 H \left(C_1 \, O'_1 + C_2 \, O'_2 \right)$$

$$O'_1 \equiv -\frac{1}{4} \left(G^a_{\mu\nu} G^{a\,\mu\nu} \right)', O'_2 = \sum_{i=1-n_\ell} m_i \overline{\psi'}_i \psi'_i$$

Here all primed quantities refer to QCD c n_ℓ active quark flavours; the effective quark-gluon coupling constant g' and effective light quark masses m'_i are connected to the original ones via the corresponding *decoupling constants* (we assume that one and the same $\overline{\text{MS}}$ normalization parameter μ is employed in full and effective theories)

$$g'(\mu) = g(\mu) \, \zeta_g(\mu, a_s(\mu), \underline{m}_h), \quad m'_i(\mu) = m_i \, \zeta_m \, (\mu, a_s(\mu), \underline{m}_h),$$

where $\underline{m}_h = m_{(n_l+1)}, \ldots m_{n_f}$ stands for heavy quarks masses. Similar relations connect the fields in the effective Lagrangian with the corresponding ones in the full one.

One can easily check that all-order (RG-not-improved!) LET's for C_1 and C_2 are easily generalized on the generic case of many heavy quarks and read:

$$(^\star) \quad C_1 = \sum_h m_h \frac{\partial}{\partial m_h} \ln \zeta_\alpha \quad \text{and} \quad C_2 = 1 + \sum_h m_h \frac{\partial}{\partial m_h} \ln \zeta_m$$

where the index h runs over all heavy quarks (that is $h \in \{n_\ell + 1, \ldots n_f\}$)

- At 1- and 2 loops there is no computational difference between $n_h = 1$ and $n_h > 1$ cases because all diagrams contain only one quark flavour at a time)
- At first glance RG-improvement should stop to work for $n_h > 1$ case due to possible appearance **constant** terms like m_h/m_h' in ζ_α and ζ_m
- The only work dealing with simultaneous decoupling of many heavy quarks is:

  A.G. Grozin, M. Hoeschele, J. Hoff, M. Steinhauser, M. Hoschele, J. Hoff et al., Simultaneous decoupling of bottom and charm quarks, JHEP 09 (2011) 066 [1107.5970]

  The authors have had to deal (for the first time) with 3-loop vacuum diagrams contibuting to the decoupling constant and dependig on *two* different quark masses. The results for ζ_α and ζ_m are **complicated** /dilogs, etc./ functions of heavy masses. Nevertheless, the functions C_1 and C_2 obtained via LET's (*) happen to be very simple. The authors literally write: "It is remarkable that although ζ_α contains di- and tri-logarithms there are only linear logaithms (of heavy masses) present in C_1" . . .
- This simplicity is a direct consequence of the RG-improved versions of LET's *, which is our **new result** and which we start to discuss now

RG-improved LET's in QCD with $n_h > 1$

We start from standard RG nomenclature for main QCD RG-functions ($a \equiv \alpha_s/\pi$)

$$\mathrm{d}_{\mu^2} \ln a = \beta(a) = \sum_{i>1} \beta_i a^i, \quad \mathrm{d}_{\mu^2} \ln m_i = \gamma_m(a) = \sum_{i>1} (\gamma_m)_i a^i,$$

Here $\mathrm{d}_{\mu^2} = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}$, it is also covenient to define $\partial_{\mu^2} = \mu^2 \frac{\partial}{\partial \mu^2}$ and $\partial_h = \sum_h m_h \frac{\partial}{\partial m_h}$. For the case of the effective QCD with n_l quarks we will use the same notations with added prime, that is $\beta'(a')$ is just a shortcut for $\beta^{(n_l)}(a^{(n_l)})$ and so on.

RG-evolution of the decoupling fuinction ζ_α: we apply d_{μ^2} to obvious relation

$$\ln a' = \ln(\zeta_\alpha) + \ln a \ (\text{since } a' = \zeta_\alpha\, a)$$

with the result

$$\mathrm{d}_{\mu^2} \ln(\zeta_\alpha) = \beta'(a') - \beta(a).$$

or, equivalently,

$$\left(\partial_{\mu^2} + \gamma_m(a)\partial_h + \beta(a) a \frac{\partial}{\partial a}\right) \ln \zeta_\alpha = \beta'(a') - \beta(a). \qquad (1)$$

Since elementary dimensional analysis implies

$$\rho \frac{\partial}{\partial \rho} \zeta_\alpha(\alpha, \rho^2 \mu^2, \rho\, \underline{m}_h) \equiv 0 \equiv (\partial_{\mu^2} + \frac{1}{2}\, \partial_h)\ \ln \zeta_\alpha \qquad (2)$$

we combine (1), (2) and arrive to our final RG-improved LETs for C_1 and C_2

Final RG-improved LET's for QCD with many heavy quarks

$$C_1 = -\frac{2}{1 - 2\,\gamma_m(a)} \left(\beta'(a') - \beta(a) - \beta(a)\, a \frac{\partial \ln \zeta_\alpha}{\partial a} \right)$$

$$C_2 = 1 - \frac{2}{1 - 2\,\gamma_m(a)} \left(\gamma'(a') - \gamma_m(a) - \beta(a)\, a \frac{\partial \ln \zeta_m}{\partial a} \right).$$

Important to note:

- RG-improved LETs are essentially identical for the cases $n_h = 1$ and $n_h > 1$.

  This is exlusively due to (assumed) proportionality of Yukawa couplings of the Higgs field to the corresponding quark masses)

- The main advantage of RG-improved LETs is that the factor $\beta(a)$ in the third terms in round brackets **decrease by one** the required loop order of the decoupling constants ζ_α and ζ_m

Example: C_1 at three loops in QCD with many heavy quarks

Below is esentially the very old★ 2-loop result for ζ_α ($L_{\mu h} \equiv \sum_h \ln \frac{\mu^2}{m_h^2}$)

$$\zeta_\alpha = 1 - a(\mu)\,\frac{T_F}{3} L_{\mu h}$$
$$+\, a(\mu)^2 \left(\frac{2}{9} C_A T_f n_h - \frac{13}{48} C_F T_f n_h + (-\frac{-5}{12} C_A F + \frac{C_F}{4})\, T_f\, L_{\mu h} + \frac{T_F^2}{9}\, L_{\mu h}^2 \right)$$

Now, a direct use the RG-improved LET for C_1 leads to a general result for the CF C_1 at the 3-loop level ($a = a^{(nf)}(\mu)$)

$$C_1(a_s) = a\, n_h \frac{2T_F}{3} + a^2 \left((\frac{5}{6} C_A F - C_F/2)\, T_f n_h - \frac{2}{9} T_F^2 n_h\, L_{\mu h} \right) + C_{1,3}\, a^3 + C_{1,4}\, a^4$$

$$\begin{aligned} C_{1,3} \quad &= C_F^2 T_F \frac{9}{16} n_h - C_F C_A T_F \left[\frac{25}{18} n_h + \frac{11}{24} L_{\mu h} \right] \\ &- C_F T_F^2 \left[\frac{5}{24} n_h n_l + \frac{17}{72} n_h^2 - L_{\mu h} \left(n_h \frac{1}{2} + \frac{1}{3} n_l \right) \right] - C_A^2 T_F \left[\frac{1063}{864} n_h + \frac{7}{24} L_{\mu h} \right] \\ &- C_A T_F^2 \left[\frac{47}{216} n_l - \frac{49}{432} n_h + \frac{5}{6} L_{\mu h} \right] n_h + \frac{2}{27} T_F^3 L_{\mu h}^2 n_h \end{aligned}$$

★W. Bernreuther and W. Wetzel, (1982); S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren (1995)

$$
\begin{aligned}
C_{1,3} \quad & = C_F^2 T_F \frac{9}{16} n_h - C_F C_A T_F \left[\frac{25}{18} n_h + \frac{11}{24} L_{\mu h} \right] \\
& - C_F T_F^2 \left[\frac{5}{24} n_h n_l + \frac{17}{72} n_h^2 - L_{\mu h} \left(n_h \frac{1}{2} + \frac{1}{3} n_l \right) \right] - C_A^2 T_F \left[\frac{1063}{864} n_h + \frac{7}{24} L_{\mu h} \right] \\
& - C_A T_F^2 \left[\frac{47}{216} n_l - \frac{49}{432} n_h + \frac{5}{6} L_{\mu h} \right] n_h + \frac{2}{27} T_F^3 L_{\mu h}^2 n_h
\end{aligned}
$$

The above result for 3-loop CF C_1 for QCD with $n_h > 1$ was first obtained via a direct (and quite complicated) calculation of a large number of 3-loop diagrams contributing to C_1 including the ones depending on *two* different quark mases[★] It was bit later confirmed in [★★]. where the decoupling were constants computed the 3-loop level. The result includes complicated functions (di-logs, etc.) of ratios $m_h/m_{h'}$. However, the result of RG-non-improved LET for C_1 happend to be extremly simple. . .

Our derivation (with a use of the RG-improved LET is not only extremely straightforward but also reveals the reason behind this remarkable simplicity: at the three loop level the CF C_1 is contibited by the 2-loop decoupling function **only** (not counting mass-independent β and γ_m)

But armed with the RG-impoved LET's we can do more. Indeed, as the four-loop β and γ_m are known since long we could upgrade the result for C_1 to one more loop (that is on the 4-loop level!)

[★] C. Anastasiou, R. Boughezal and E. Furlan, (2010) [1003.4677]

[★★] A.G. Grozin, M. Hoeschele, J. Hoff, M. Steinhauser (2011) [1107.5970].

New result: C_1 at four loops in QCD with many heavy quarks

We start from some notations. The available result of for ζ_α is convenient to present as follows:

$$\zeta_\alpha = d_1\, a + d_2\, a^2 + d_3\, a^3 + d_4\, a^4 + \ldots$$

0 Here the first two coefficients have already been displayed, The coefficient d_3 is a complicated function of μ, $a(\mu)$ and quark mass $m_{n_l+1}, \ldots m_{n_f}$.

Finally, a simple and direct use of RG-improved LET directly leads for the following result for C_1

$$C_1 = -\frac{2}{1 - 2\,\gamma_m(a)} \left(\tilde{C}_{1,1}\, a + \tilde{C}_{1,2}\, a^2 + \tilde{C}_{1,3}\, a^3 + \tilde{C}_{1,4}\, a^4 \right)$$

where

$$\tilde{C}_{1,1} = -\,\beta_1 + \beta_1',$$

$$\tilde{C}_{1,2} = -\,\beta_2 + \beta_2' + (-\beta_1 + \beta_1')\, d_1,$$

$$\tilde{C}_{1,3} = -\,\beta_3 + \beta_3' + (-\beta_2 + 2\,\beta_2')\, d_1 + \beta_1\, d_1^2 + (-2\,\beta_1 + \beta_1')\, d_2,$$

$$\tilde{C}_{1,4} = -\,\beta_4 + \beta_4' + (\beta_2 + \beta_2')\, d_1^2 - \beta_1\, d_1^3 + (-2\,\beta_2 + 2\,\beta_2')\, d_2 + d_1\,(-\beta_3 + 3\,\beta_3' + 3\,\beta_1\, d_2) + (-3\,\beta_1 + \beta_1')\,\mathbf{d_3}$$

One can see that no d_4 appears in $\tilde{C}_{1,4}$ as it should be: RG-improved-LET does work!

Conclusions

- We have demonstrated that RG-improved LET's do work in the case many heavy quarks (asuuming their Higgs-like couplings with the scalar particle)

- As it was shown in (A.G. Grozin, M. Hoeschele, J. Hoff, M. Steinhauser (2011) [1107.5970]) power suppressed corrections should be amended by summing higher oder logs like $(\ln \mu^2/m_h^2)^n$. In the case of more than one heavy quark it could be done only with sequential decoupling.

- and here we encounter a new problem: on the second step of sequential decoupling one should know the transition of the very gluonic operator

$$O_1 = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu}$$

  from the full QCD to the effective one (when decouipling the "second" heavy quark)

- To the best of my knowledge it is not yet solved. We are working on it ...