Recent Developments in Scattering Amplitudes

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Introduction

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My talk will have two short pieces

(1) an appetizer, with some recent mathematical developments on tree amplitudes for gluons;

(2) a long motivational digression on why I think the "curve integral formalism" of Arkani-Hamed, Frost, Plamondon, Salvatori, Thomas intrigues me; this led us to generalize it from scalar ϕ^3 theory to introduce fermions with a Yukawa interaction (based on work with Shounak De, Andrzej Pokraka, Marcos Skowronek and Anastasia Volovich).

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Sometimes, very occasionally, new formalism can help us discover new physics (example: Lagrangian mechanics);

more frequently (but, I confess, not always), it is at best useful (sometimes, extremely useful) for better understanding “known” physics.

A Tree-Level Teaser

Open problem: prove the optimal upper bound on the complexity of computing A_{n_+,n_-}, the (color-ordered) tree-level scattering amplitude of $n = n_+ + n_-$ massless particles with helicities ± 1 (i.e., gluons in Yang-Mills theory).

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Real Motivation: Is there something about the quantum field theory of spin-one particles that we fundamentally do not understand?

A Tree-Level Teaser

Comments: A_{n_+,n_-} is a rational function of the spinor helicity variables that describe the energy and momenta of the n-particle configuration.

In fact, from physics we know where all of the possible poles are, so if we define a rescaled amplitude

$$\tilde{A}_{n_+,n_-} = \left(\prod_{i<j} \langle i\, j \rangle [i\, j] \right) A_{n_+,n_-}$$

it is guaranteed to be a polynomial (c.f. Kosower's talk). So if we want to capture the attention of a mathematician or computer scientist, we can declare that all of our kinematic variables are integers, then the amplitude is also an integer, and the problem of computing this amplitude is the problem of evaluating a certain function

$$\tilde{A}_{n_+,n_-} : \mathbb{Z}^{4n} \mapsto \mathbb{Z}$$

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Currently, the best known algorithm (BCFW recursion) expresses the answer as a sum of

$$\frac{1}{n-3}\binom{n-3}{k-1}\binom{n-3}{k-2} \qquad k = \min(n_+, n_-)$$

terms – each of which is a rational number – even though we know that the sum must be an integer.

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Looks can be deceiving: a famous example of a polynomial with n! terms that can be computed in $\mathcal{O}(n^3)$ time is the determinant.

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This has recently been proven by mathematicians Even-Zohar, Lakrec, Parisi, Sherman-Bennett, Tessler, Williams have recently proven this to be correct, and Galashin has revealed a crazy but beautiful direct connection between this problem and origami folding.

I'm excited by the prospect that this problem has been put on solid enough footing to attract the interest of serious mathematicians.

A Long Motivational Introduction

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What does it mean to "evaluate a loop integral"?

In practice, it often means to express it as a linear combination of "known" special functions, like

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Here I'm putting on an unusually practical hat – imagining I'm someone who wants to get digits of precision in order to compare to some data.

A Long Motivational Introduction

A practically minded person might be happy to consider a loop integral to be evaluated if there is a fast algorithm for either (1) writing down a rapidly convergent series expansion, around some useful points or at least (2) processing it down into an integral formula with as few integrations remaining as possible – then one can define the things that appear as a new class of "special functions" and consider the job done.

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In D spacetime dimensions, L-loop Feynman integrals are expressible as

$$\int d^D\ell_1\, d^D\ell_2 \cdots d^D\ell_L \,[\text{rational function of } p_1, \ldots, p_n \text{ and } \ell_1, \cdots]$$

$$= \pi^{DL-d} \int d^d\vec{\mathbf{x}}\ [\text{rational function of } p_1, \ldots, p_n \text{ and } \vec{\mathbf{x}}]$$

where $p_1, \ldots, p_n$ are the momenta of the particles and $d \leq DL/2$ but there are two big things currently lacking.

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(2) There is no effective algorithm for determining if two expressions of the type shown on the second line are equal to each other.

That's a big problem because one can have a long expression, resulting from a sum over many Feynman diagrams – maybe gigabytes long – that integrates to something very simple, or even to zero!

A Long Motivational Introduction

One way in which problem (2) could, hypothetically, be solved, is: consider the infinite set of different ways of writing a given integral

$$I = \int d^d\vec{\mathbf{x}}\ [\text{rational function of } p_1, \ldots, p_n \text{ and } \vec{\mathbf{x}}]$$

that differ from each other by combinations of arbitrary changes of variables and/or integration by parts.

If there were an algorithm for picking, from among this infinite dimensional equivalence class, a certain preferred representative – let's suggestively call it a canonical form – then we could easily compare whether two things are equal, or if some long combination of objects sums to zero.

A Long Motivational Introduction

Aside: A period is a number that is the volume of some region in $\mathbb{R}^n$ carved out by polynomial inequalities with coefficients in $\mathbb{Q}$.
Example:

$$\pi = \int_{\mathbb{R}^2 : x^2 + y^2 < 1} dx\, dy$$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset$ algebraic numbers $\subset$ periods $\subset$ transcendental numbers

Indeed, a giant open conjecture – far, far beyond our current understanding of number theory – is that if two periods are equal to each other, then there is an explanation for it: i.e., the two defining integrals can be mapped into each other by some combination of changes of variables and integration by parts (that involve only algebraic numbers). (Kontsevich, Zagier) Problem (2) mentioned above is a reflection of this...

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These are the so-called generalized polylogarithm functions, and the tool for unlocking their structure is called the symbol. (Goncharov, Spradlin, Vergu, Volovich).

Curve Integrals

The curve integral formalism is applicable to any colored theory; that means all particles are in the adjoint representation of some group (let's say U(N)), but they can be massless or massive, and the spacetime dimension is arbitrary (so we can use dim reg if we need to regulate IR or UV divergences).

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It provides a combinatorial algorithm for writing down an expression of the form

$$\int d^{n+3(L-1)}\vec{\mathbf{t}}\ [\text{~~rational~~ function of } p_1, \ldots, p_n \text{ and } \vec{\mathbf{t}}]$$

that represents the sum of all L-loop Feynman diagrams to the n-particle amplitude. (One can restrict to planar, or any order in the $1/N$ expansion.)

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that represents the sum of all L-loop Feynman diagrams to the n-particle amplitude. (One can restrict to planar, or any order in the $1/N$ expansion.)

The complexity of computing the integrand is $\mathcal{O}(n^2)$ rather than $\mathcal{O}(4^n)$ as it would be for summing tri-valent Feynman graphs.

Curve Integrals

The complexity is $\mathcal{O}(n^2)$ because the key ingredients that enter are not individual Feynman diagrams, but curves that can be drawn on a surface. For example, consider all non-planar Feynman diagrams that contribute to the three-point amplitude shown here:

Individual Feynman diagrams correspond to triangulations of this surface, but the curve integral (representing the sum over all Feynman diagrams) can be written down by computing certain quantities associated to compatible curves on the surface; the curve from 1 to 3 is shown in purple.

Examplitude

The 2-point 1-loop bubble diagram:

$$A(p^2) = \int_{t_1+t_2\leq 0} dt_1\, dt_2\; exp\Big(\frac{p^2}{t_1+t_2}(\max(0,t_2) - t_2 - \max(0,t_1))^2$$
$$+ m^2(2\max(0,t_2)+t_1) - \frac{D}{2}\log(t_1+t_2)\Big)$$

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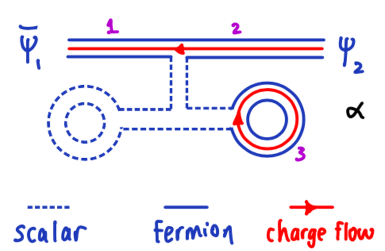
(2) however, a big drawback of the fact that D is just a parameter is that none of the very special technology (spinor helicity, momentum twistors) specially tuned to $D = 4$ can help out here.

(3) The non-analytic integrand may frighten you, but there are recently-developed techniques (tropical sampling; M. Borinsky) for numerically evaluting integrals precisely of this type extremely fast (see G. Salvatori's talk at Amplitudes 2025).

Colored Yukawa Theory

In order to take this formalism one step closer to the real world, in arXiv:2406.04411 we should how to incorporate (adjoint) fermions with Yukawa interactions: we gave an explicit formula for the curve integral (sum of all Feynman diagrams, at any loop order and any order in $1/N$) involving certain determinants.



Each curve can be assigned to be either bosonic or fermionic; for an L loop amplitude we must sum over the 2^L possible assignments (bosonic or fermionic) for each puncture.

Summary and Conclusion

Dramatic progress has been made in recent decades, but many fundamental questions about the structure of quantum field theory remain unsolved.

Seeking new formalisms may help to shed light on some of these questions and perhaps some may even ultimately be useful!