Tame multi-leg Feynman integrals beyond one loop

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Based on works: L.H. Huang, R.J. Huang, Y.Q. Ma, arXiv: 2412.21053
R.J. Huang, D.S. Jian, Y.Q. Ma, D.M. Mu, W.H. Wu, arXiv: 2412.21054

北京大学

Outline

I. Introduction

II. A new representation

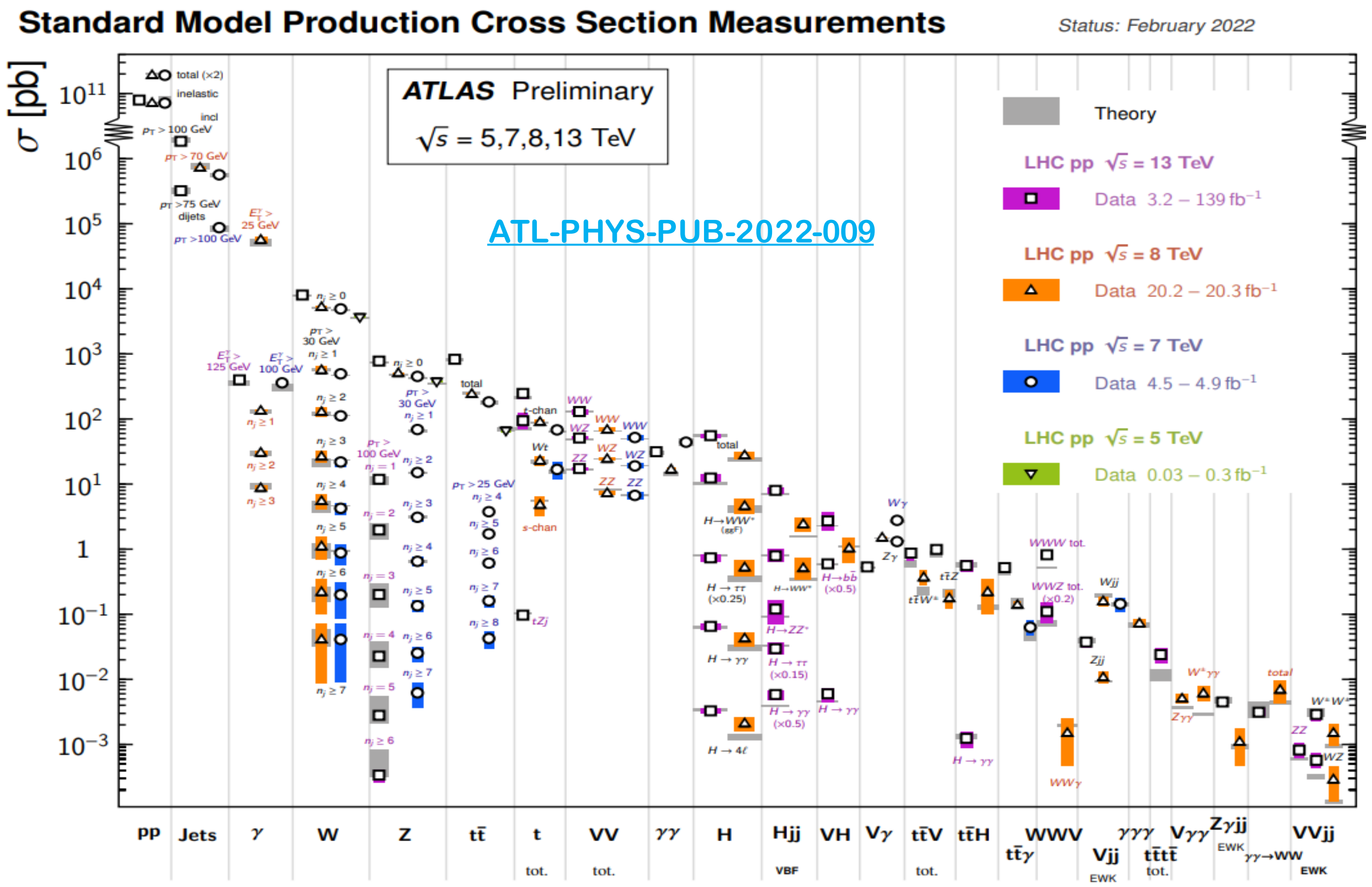
III. Calculate FBIs

IV. Integrate branch variables

V. Summary and outlook

Era of precision physics

- **High-precision data**
  - Many observables probed at precent level precision
  - HL-LHC: 30 times more data



- **QCD cor. requirement: ideally**
  - Most processes: N2LO
  - Many processes: N3LO
  - Some processes: N4LO

Current status of perturbative calculation

➢ Accomplished processes

- NLO solved, automatic codes exist: MadGraph, Helac, etc
- Need to push calculation to 1-2 orders in α_s

Legs / Order	2→1	2→2	2→3	2→4	2→5	2→6
NLO	★★★	★★★	★★★	★★★	★★★	★★★
N2LO	★★★	★★	★	?	?	
N3LO	★★	★	?			
N4LO	★	?				
N5LO	?					

Efficient methods for high-order computation are highly demanded!!!

Feynman integrals computation

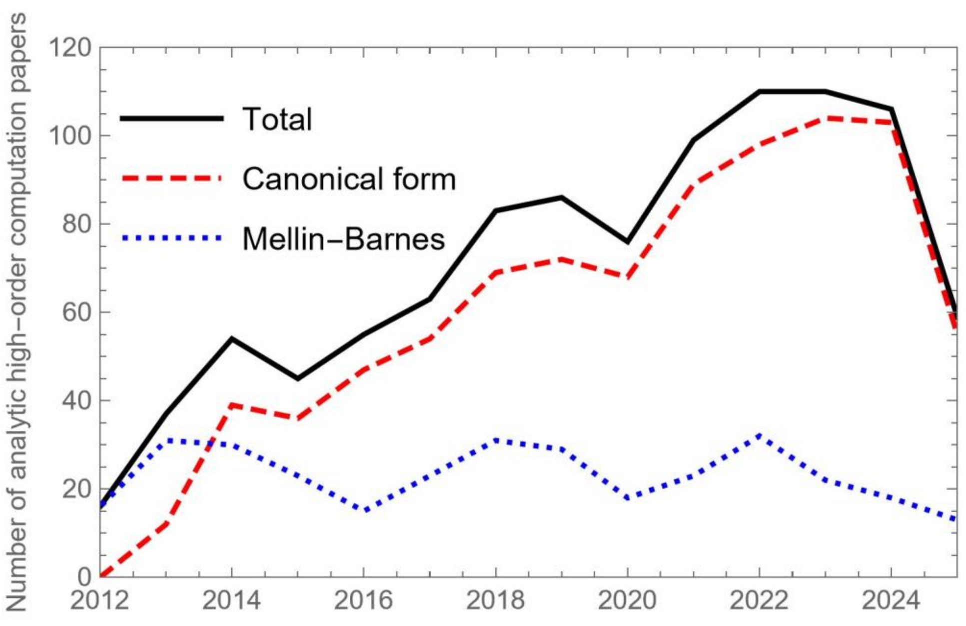
- A key obstacle in high-order computation
- Mainstream method:

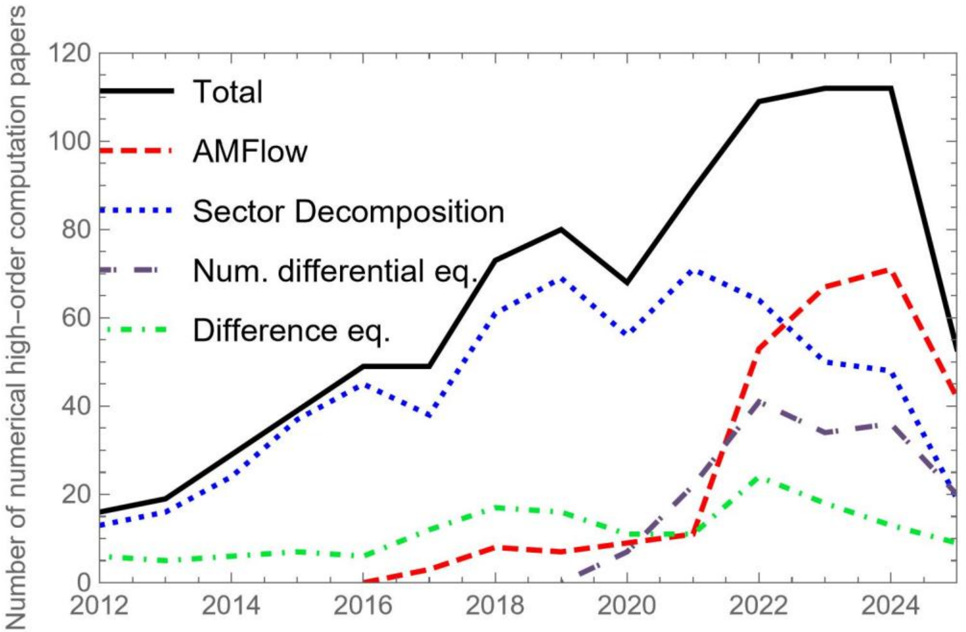
1) Integration-by-parts: Reduce loop integrals to basis (Master Integrals)

$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

2) Compute MIs

Computation of MIs





Canonical form: Henn, PRL2013

AMFlow: Liu, YQM, Wang, PLB2018

See also Yang Zhang's talk

Systematic and efficient for: both massless and massive MIs

But, all depend on reduction!!!

Integration-by-parts reduction: the bottleneck!

- **The state-of-the-art IBP method: very challenging**
  - 4-loop DGLAP kernel cannot be obtained
  - $H+2j$ production: exact two-loop contribution is missing — Chen, et al., JHEP2022
  - $H+t\bar{t}$ production: exact two-loop contribution is missing — Catani, et al., PRL2023
- **Improvements for IBPs**
  - Syzygy equations: trimming IBP system — See also David Kosower's talk
    - Gluza, Kajda, Kosower, PRD2011
    - Böhm, Georgoudis, Larsen, Schulze, Zhang, PRD2018
    - *NeatIBP*: Wu, et al. CPC2024
  - Block-triangular form: search simple IBP system
    - Liu, YQM, PRD2019
    - Guan, Liu, YQM, CPC2020
    - *Blade*: Guan, Liu YQM, Wu, 2405.14621

  } Improve efficiency by a hundredfold $\approx$ half order in α_s

Need to calculate two more orders in α_s! Ways to bypass IBP?

Lessons after many-years study

- Reduction is very hard, no matter using any method
  - IBP
  - Intersection number See also Hjalte Axel Frellesvig's talk
  - Asymptotic expansion
  - Iterative reduction
  - …
- The reason: too many integration variables

Conservation of suffering!

Unless a deeper understanding of FIs?

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A possible simplification?

➢ Feynman parametrization

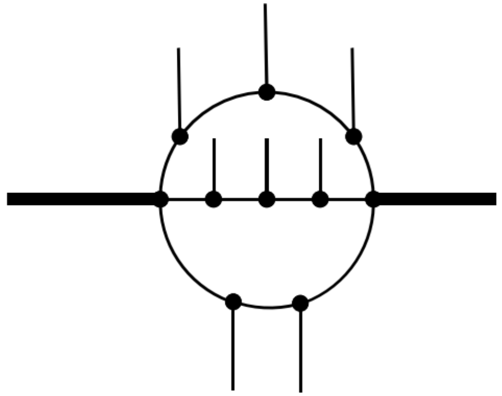
$$J(\vec{\nu}; D) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_K)} \int \prod_{i=1}^{K} (x_i^{\nu_i - 1} \mathrm{d}x_i) \delta(1 - X) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

- U: degree L in the Feynman parameters x_i
- F: degree $L+1$

$$\mathcal{U} = \sum_{T \in T(G)} \prod_{e_i \notin T} x_i$$

➢ Will things be simpler if we fix U unintegrated?

End of 2023



$$J(\vec{\nu}; D) = \int [\mathrm{d}\mathbf{X}] \prod_{a=1}^{B} X_a^{\nu_a - 1} \mathcal{U}^{\nu - \frac{(L+1)D}{2}} I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X})$$

X_a: the summation of Feynman parameter for the a-th branch

A surprising observation!

L.H. Huang, R.J. Huang, YQM, 2412.21053

➢ The integrands are as simple as one-loop FIs!

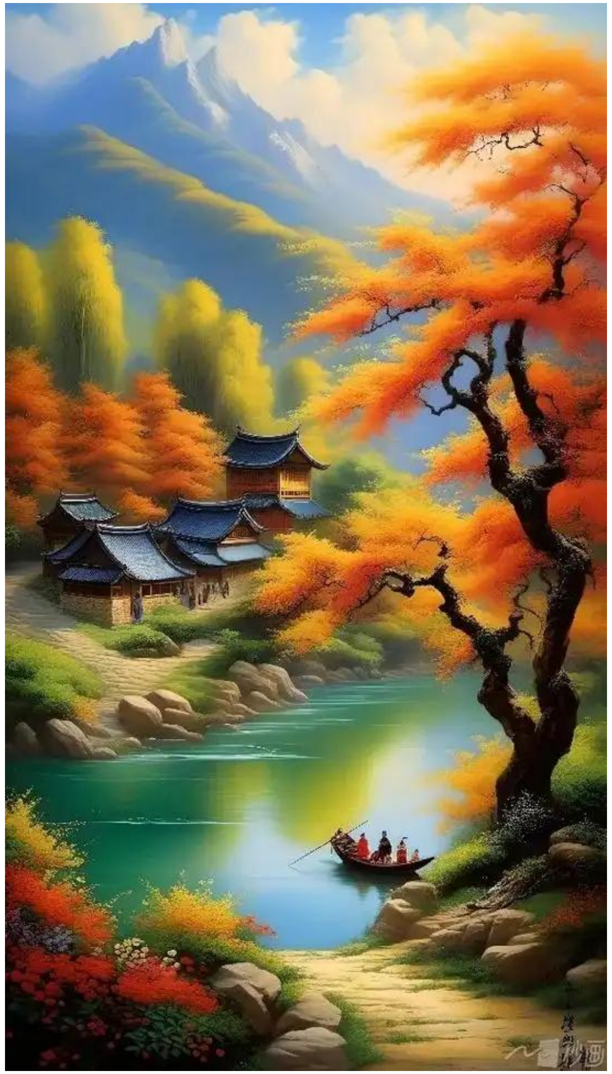
$$J\left(\vec{\nu};D\right)=\int[\mathrm{d}\mathbf{X}]\prod_{a=1}^{B}X_a^{\nu_a-1}\mathcal{U}^{\nu-\frac{(L+1)D}{2}}I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X})$$

A new representation

- Because F is then degree 2 (explain later)
- Integrand can be computed easily

➢ Much less unintegrated parameters!

- 2 loops: $B-1=2$
- 3 loops: $B-1=5$



Definition

- **An L-loop amplitude**

$$\mathcal{M} \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D l_i}{\mathrm{i}\pi^{D/2}} \frac{P(l)}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_N^{\nu_N}},$$

- With $$\mathcal{D}_\alpha = \sum_{i,j=1}^{L} \hat{\mathcal{A}}_{ij}^{\alpha}\, l_i \cdot l_j + 2\sum_{i=1}^{L} \hat{\mathcal{B}}_i^{\alpha} \cdot l_i + \hat{\mathcal{C}}^{\alpha}$$
- Two propagators are in the same branches if they have identical: $\hat{\mathcal{A}}_{i,j}^{\alpha}$ and $\hat{\mathcal{A}}_{i,j}^{\beta}$
- B: number of branches
- $n_1, \cdots, n_b, \cdots, n_B$: number of propagators in each branch
- Corresponding between α and (b, i)

Feynman parametrization

- First combine denominators in each branch, then combine them

$$\frac{1}{\mathcal{D}_1^{\nu_1}\cdots\mathcal{D}_N^{\nu_N}} \equiv \prod_{b=1}^{B}\prod_{i=1}^{n_b}\frac{1}{\mathcal{D}_{(b,i)}^{\nu_{(b,i)}}} = \frac{\Gamma(\nu)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)}\int_0^\infty [\mathrm{d}\mathbf{X}]\,[\mathrm{d}\mathbf{y}]\,\frac{\prod_{b=1}^{B}X_b^{\nu_b-1}\prod_{\alpha=1}^{N}y_\alpha^{\nu_\alpha-1}}{\left(\sum_{b=1}^{B}\sum_{i=1}^{n_b}X_b y_{(b,i)}\mathcal{D}_{(b,i)}\right)^{\nu}}$$

- With: $\nu_b = \sum_{i=1}^{n_b}\nu_{(b,i)},\ \nu = \sum_{\alpha=1}^{N}\nu_\alpha$

$$[\mathrm{d}\mathbf{X}] = \prod_{b=1}^{B}\mathrm{d}X_b\,\delta\left(1-\sum_{b=1}^{B}X_b\right),\quad [\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N}\mathrm{d}y_\alpha\prod_{b=1}^{B}\delta\left(1-\sum_{i=1}^{n_b}y_{(b,i)}\right)$$

Feynman parametrization(cont.)

$$[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$$

➢ The denominator

$$\sum_{b=1}^{B} \sum_{i=1}^{n_b} X_b y_{(b,i)} \mathcal{D}_{(b,i)} = \sum_{i,j=1}^{L} \mathcal{A}_{ij}\, l_i \cdot l_j + 2 \sum_{i=1}^{L} \mathcal{B}_i \cdot l_i + \mathcal{C}$$

- A is independent of $\boldsymbol{y}$! B and C are linear in $\boldsymbol{y}$
- Define:

$$\mathcal{U} = \det\left(\mathcal{A}\right), \quad \text{independent of } y$$

$$\mathcal{F} = \left(\mathcal{B}_\mu\right)^T \mathcal{A}^{adj} \mathcal{B}^\mu - \mathcal{C} \det\left(\mathcal{A}\right) = \frac{1}{2} \sum_{\alpha,\beta=1}^{N} R_{\alpha\beta}\, y_\alpha y_\beta = \frac{1}{2} \mathbf{y}^T \cdot R \cdot \mathbf{y}$$

⇧

$$y_{(b,i)} \to y_{(b,i)} \times 1 = y_{(b,i)} \sum\nolimits_j y_{(b,j)}$$

A new representation

➢ **Formula after straightforwardly integrated out loop momenta**

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}(\mathbf{X}) \qquad \hat{\mathcal{M}}(\mathbf{X}) = \mathcal{U}^{-\frac{(L+1)D}{2}} \sum_{\Delta,\vec{\nu}'} K^{\Delta}_{\vec{\nu}'}(\mathbf{X})\, I^{\Delta}_{\vec{\nu}'}(\mathbf{X})$$

- $\Delta = \frac{\mathrm{LD}}{2}$, K's are rational in $\boldsymbol{X}$
- Fixed-Branch Integrals (FBIs) defined as

$$I^{\Delta}_{\vec{\nu}}(\mathbf{X}) = \frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)} \int [\mathrm{d}\mathbf{y}]\, \frac{\prod_{\alpha=1}^{N} y_\alpha^{\nu_\alpha-1}}{\left(\frac{1}{2}\mathbf{y}^T\!\cdot\! R\!\cdot\!\mathbf{y} - \mathrm{i}0^+\right)^{\nu-\Delta}}$$

- The same as one-loop integrals, except for more delta functions $[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$

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Compute FBIs: from matrix *R* to matrix *S*

- **Add a line for each branch; number of 1's equals to n_b**
  - E.g., if $B = 3$ and $(n_1, n_2, n_3) = (2, 1, 1)$

$$S = \begin{pmatrix} & & & 1 & 1 & 0 & 0 \\ & 0_{3\times 3} & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & R & & \\ 0 & 0 & 1 & & & & \end{pmatrix}$$

Generalized Gram matrix

Reduction relations for FBIs

➢ Recursion relation

$$S\cdot(t_1,\cdots,t_B,\nu_1 I^{\Delta}_{\vec{\nu}+\vec{e}_1},\cdots,\nu_N I^{\Delta}_{\vec{\nu}+\vec{e}_N})^T=(-I^{\Delta-1}_{\vec{\nu}},\cdots,-I^{\Delta-1}_{\vec{\nu}},I^{\Delta-1}_{\vec{\nu}-\vec{e}_1},\cdots,I^{\Delta-1}_{\vec{\nu}-\vec{e}_N})^T$$

- With t_b determined by the equation itself

➢ Dimension-shift relation

$$CI^{\Delta-1}_{\vec{\nu}}=(2\Delta-\nu-B)\,z_0 I^{\Delta}_{\vec{\nu}}+\sum_{\alpha=1}^{N} z_\alpha I^{\Delta-1}_{\vec{\nu}-\vec{e}_\alpha}$$

- With $z_0=0$ or 1 depending on generalized Gram determinant $\det S=0$ or not
- Other parameters determined by

$$S\cdot(C_1,\cdots,C_B,z_1,\cdots,z_N)^T=(z_0,\cdots,z_0,0,\cdots,0)^T$$

- Choose $C=\sum_{b=1}^{B} C_b$ as nonzero as possible

Reduction: 4 different cases

- **FBIs have at most one master integral in each sector**

1. $\det(S) \neq 0$ and $C \neq 0$: using recursion relation, leaving one master integral

2. $\det(S) \neq 0$ and $C = 0$: $(2\Delta - \nu - B)\, I_{\vec{\nu}}^{\Delta} = -\sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$

3. $\det(S) = 0$ and $C \neq 0$: $C I_{\vec{\nu}}^{\Delta-1} = \sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$

4. $\det(S) = 0$ and $C = 0$: $I_{\vec{\nu}}^{\Delta} = -\sum_{\alpha\neq\beta} \frac{z_\alpha}{z_\beta} I_{\vec{\nu}+\vec{e}_\beta-\vec{e}_\alpha}^{\Delta}$

2-4: no master integral

Compute master integrals of FBIs - numerical

➢ Using auxiliary mass flow method:

$$\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta) = \frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)} \int [\mathrm{d}\mathbf{y}] \frac{\prod_{\alpha=1}^{N} y_\alpha^{\nu_\alpha-1}}{\left(\frac{1}{2}\mathbf{y}^T\cdot R\cdot\mathbf{y}+\eta\right)^{\nu-\Delta}}$$

- Equivalent to $R_{\alpha\beta} \to R_{\alpha\beta} + 2\eta/B^2$, thus have

$$(2z_0\eta - C)\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta) = (2\Delta-\nu-B)\, z_0\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta) + \sum_{\alpha=1}^{N} z_\alpha \mathcal{I}_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}(\eta)$$

- Solve it with $\eta \to \infty$ as boundary condition
- Using Dimension-Change Transformation to obtain desired FBIs Huang, Jian, YQM, Mu, Wu,PRD2025

$$I_{\vec{\nu}}^{\Delta+\delta} = \frac{1}{\Gamma(\delta)} \int_{-\mathrm{i}0^+}^{-\mathrm{i}\infty} \mathrm{d}\eta\, \eta^{\delta-1}\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)$$

Compute master integrals of FBIs - analytical

- Canonical form are obtained for all cases, e.g., Chen, Feng, Zhang, JHEP2025

$$d\mathcal{I}_{2m} = c_{2m\to 2m}\mathcal{I}_{2m} + \sum_i c_{2m\to 2m-1;i}\mathcal{I}^{(i)}_{2m-1} + \sum_{i\neq j} c_{2m\to 2m-2;ij}\mathcal{I}^{(ij)}_{2m-2}$$

$$c_{2m\to 2m} = -2\epsilon d\log\mathcal{D}$$

$$c_{2m\to 2m-2;ij} = \frac{\epsilon N}{2} d\log\left(\frac{\sqrt{(\mathcal{D}_{\hat{i}}-\mathcal{D})\mathcal{D}_{\widehat{i,j}}} - \sqrt{(\mathcal{D}_{\hat{i}}-\mathcal{D}_{\widehat{i,j}})\mathcal{D}}}{\sqrt{(\mathcal{D}_{\hat{i}}-\mathcal{D})\mathcal{D}_{\widehat{i,j}}} + \sqrt{(\mathcal{D}_{\hat{i}}-\mathcal{D}_{\widehat{i,j}})\mathcal{D}}}\right) + (i\leftrightarrow j)($$

- Enabling the analytical computation of FBIs, like one-loop cases

Comparison

- One-loop FIs: a special case of FBIs, with $B = 1$

$$[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$$

- B is an unimportant parameter in the computation of FBIs

$$C I_{\vec{\nu}}^{\Delta-1} = (2\Delta - \nu - B)\, z_0 I_{\vec{\nu}}^{\Delta} + \sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$$

- FBIs are as simple as one-loop FIs, thus a solved problem

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Numerical method: contour deformation

- Integral with known integrand

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}\,(\mathbf{X})$$

Avoid IBP reduction!

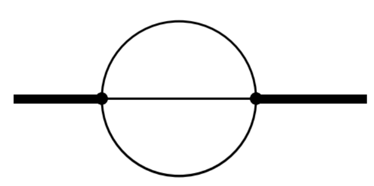
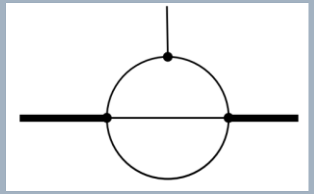
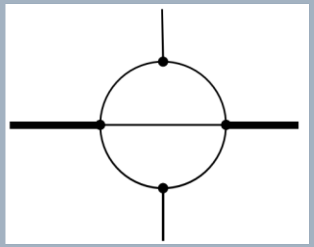
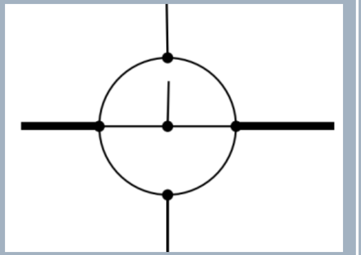
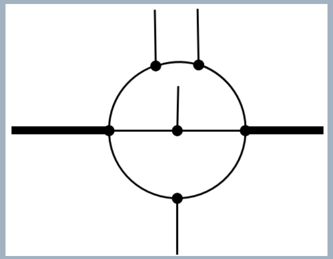
- Contour deformation to avoid divergences

$$\tilde{X}_b = X_b + \mathrm{i} X_b(1 - X_b) G_b(\mathbf{X})$$

$$G_b(\mathbf{X}) = \kappa \sum_j \lambda k_j \frac{\partial_{X_b} P_j}{P_j^2 + (\partial_{X_b} P_j)^2} \exp(-\frac{P_j^2}{\lambda^2 k_j^2})$$

- Adjust parameters
- Subtract out divergences
- Then use existed techniques to perform integration

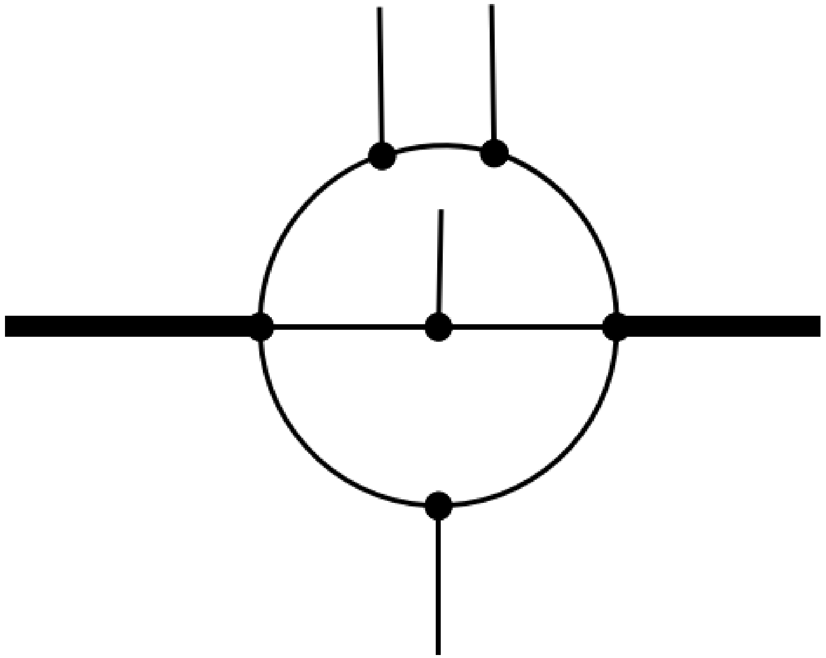
Numerical method: contour deformation

					
DCT/pt (ms)	0.14	0.19	0.34	0.76	2.91
#points	726 =121*6	726	726	726	12826 =121*106

- **To obtain 6-digit precision using Adaptive Gausian-Kronrod Rule with degree 5 (11*11=121 points)**

Numerical method: contour deformation

Method	precision	time (hour)
pySecDec	3	3
	5	108
AMFlow	20	4
New method	6	0.01



- **Computing to $O(\epsilon)$**
- **AMFlow computes all MIs, the other two methods only compute the corner integral**
- **Much faster than previous methods**
- **Note: by combining DCT, we can in fact avoid contour deformation** ⇐ **To appear soon!**

Analytical method: Reduction

- **Combining with intersection theory**
  - Only 3 layers at two loop order
  - More efficient
- **Combining with 1/D expansion**
  - 2 loops: simplifying ~$O(n^{10})$ to ~$O(n^2)$
  - 3 loops: simplifying ~$O(n^{10})$ to ~$O(n^5)$
  - $n{\sim}O(100)$ is the terms to be obtained, Power: the number of integration parameters
  - More efficient

Improve reduction!

To appear soon!

Summary and outlook

- Reveal a deep structure of FIs: simple integrand followed by integration over a few variables:

  2 for two-loop, and 5 for three-loop: independent of number of external legs!

- The integrand (FBIs) can be fully solved, similar to one-loop FIs

- All previous FIs techniques can be applied to resolve the remained integration

  Either fully numerically, or via reduction + computing MIs

- Optimistic to overcome multi-leg FIs computation beyond one-loop, and to meet the requirement of high-precision LHC data

Thank you !

Stay tuned!