D-commuting SYK Model:

building quantum chaos from integrable blocks

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Based on **arXiv:2411.12806**,

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Quantum Gravity in the Lab?

In the quest to understand the quantum nature of spacetime and gravity, a key difficulty is the **lack of contact with experiment**.

Holographic application: **experiments on non-gravitational but holographic systems may probe theories of quantum gravity indirectly**.

One such system that could be realized in experiments is the **SYK model**.

SYK Model

Sachdev-Ye-Kitaev (SYK) model is a quantum mechanical model of 2N **Majorana fermions**, with **random interactions** involving 2p of these fermions at a time:

$$\tilde{H}_{SYK} = i^p \sum_{i_1 < \cdots < i_{2p}} J_{i_1 \cdots i_{2p}} \psi_{i_1} \cdots \psi_{i_{2p}} \qquad \{\psi_i, \psi_j\} = 2\delta_{ij}$$

The coupling coefficients $J_{i_1 \cdots i_{2p}}$ have a Gaussian distribution:

$$\mathbb{E}[J_{i_1 \cdots i_{2p}}] = 0, \quad \mathbb{E}[(J_{i_1 \cdots i_{2p}})^2] = J^2 / \binom{2N}{2p}$$

This model exhibits many interesting features : it is an example of $\boldsymbol{NAdS_2/NCFT_1}$ **duality**. It is maximally **quantum chaotic.** It is relatively simple to have **potential experimental realizations**.

A Simulated "Tranversable Wormhole"

D. Jafferis et al. ,
Nature 612, 51 (2022).

A proposal in 2022 claims that one can realize a special SYK-like model on a quantum processor. The Hamiltonian is given by:

$$H_{L,R} = -0.36\psi^1\psi^2\psi^4\psi^5 + 0.19\psi^1\psi^3\psi^4\psi^7 - 0.71\psi^1\psi^3\psi^5\psi^6 + 0.22\psi^2\psi^3\psi^4\psi^6 + 0.49\psi^2\psi^3\psi^5\psi^7,$$

However, **the terms in this Hamiltonian all mutually commute**, therefore the theory is **integrable rather than chaotic.**
Such models are further analyzed under the name of "**commuting SYK model**".

Commuting SYK Model

P. Gao, JHEP 01 (2024) 149

The Hamiltonian of **commuting SYK** (cSYK) model is made up of p Hermitian Majorana bilinears $\mathcal{X}_i$ under Gaussian random couplings:

$$\tilde{H}_{cSYK} = \sum_{i_1<\ldots<i_p} J_{i_1\cdots i_p}\mathcal{X}_{i_1}\cdots\mathcal{X}_{i_p},\ \mathcal{X}_i \equiv \mathrm{i}\psi_{2i-1}\psi_{2i}$$

Since these bilinears all commute with each other, terms in the Hamiltonian commute, and the **bilinears are conserved**. **The theory is integrable, not chaotic**.

Even the Hamiltonian of the SYK model itself can be decomposed into mutually commuting groups. The emergence of quantum chaos must be the consequence of interplay between groups.

D-commuting SYK Model

We model the interplay by **explicitly construct d copies of commuting SYK Hamiltonian,** with d being a tunable parameter. The resulting family of models will be referred to as the "**d-commuting SYK**"(dcSYK) model. The Hamiltonian is:

$$\tilde{H} = \frac{1}{\sqrt{d}} \sum_{a=1}^{d} \tilde{H}_a, \ \tilde{H}_a = \sum_{i_1 < \cdots < i_p} J^a_{i_1 \cdots i_p} \mathcal{X}^a_{i_1} \cdots \mathcal{X}^a_{i_p} \ \equiv \sum_I J^a_I X^a_I$$

We have **different recipes for Majorana bilinears for different groups**, in order to break the integrability of the system:

$$\mathcal{X}^a_j \equiv i\psi_{2j-1}\psi_{[(2j-4+2a) \mod (2N)]+2}, \ j = 1, \cdots, N$$

The random couplings are still picked from a Gaussian ensemble

$$\mathbb{E}[J^a_{i_1 \cdots i_p}] = 0, \quad \mathbb{E}[(J^a_{i_1 \cdots i_p})^2] = \sigma^2 \equiv J^2 / \binom{N}{p}$$

Averaged Spectrum

We focus on the **spectrum of the Hamiltonian** under ensemble average.

$$\rho(E) = \mathbb{E}\left[\sum_{\tilde{\lambda}} \delta(E - \tilde{\lambda})\right]$$

As d increases, we expect the spectrum to interpolate between the integrable commuting SYK and the chaotic SYK.

The spectrum of SYK is of compact support, while the spectrum of cSYK is Gaussian and contains an infinite tail.

We will show that **the spectrum of dcSYK is similar to SYK for even d=2, but non-compact unless $d \to \infty$.**

There exists chaotic-integrable transition under different temperature!

Chord Diagrams

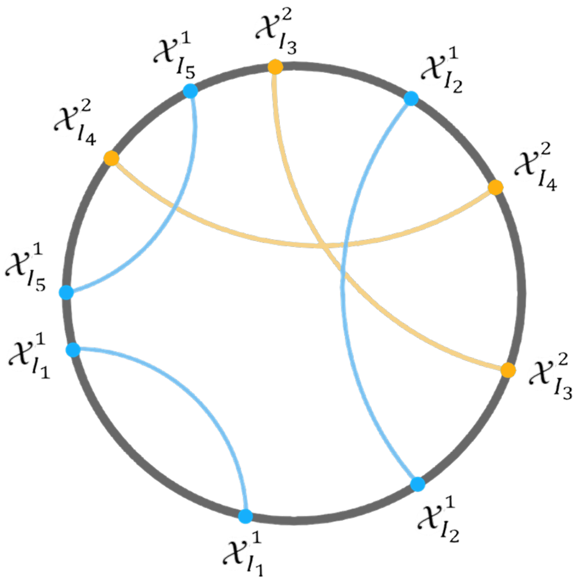
The spectrum can be evaluated either by resolvent or partition function, both quantities are related to the **moments**:

$$M_n \equiv \frac{1}{d^{n/2}} \sum_{a_i} \mathbb{E}[\mathrm{tr}\tilde{H}_{a_1} \cdots \tilde{H}_{a_n}]$$

The n-moment M_n can be expanded in terms of products of n X_I^a and random couplings:

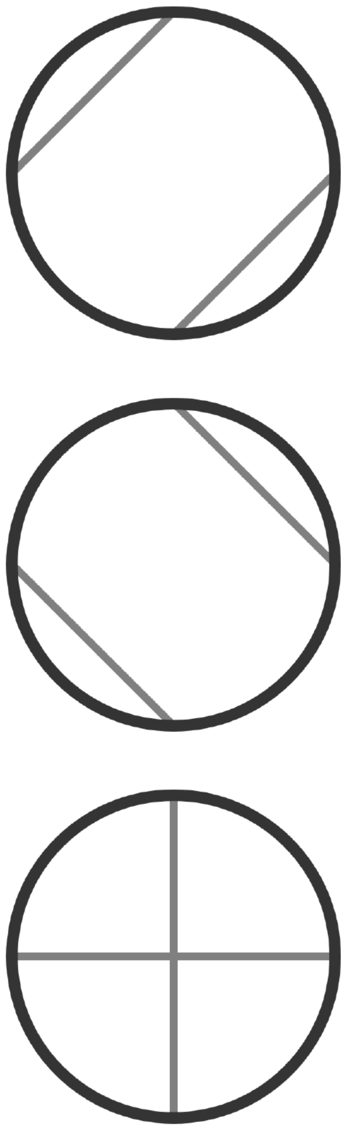
$$\mathbb{E}[\mathrm{tr}\tilde{H}_{a_1} \cdots \tilde{H}_{a_n}] = \sum_{I_i} \mathbb{E}[J_{I_1}^{a_1} \cdots J_{I_n}^{a_n}]\mathrm{tr}X_{I_1}^{a_1} \cdots X_{I_n}^{a_n}$$

The **average over Gaussian ensemble enforces the index to appear in pairs,** accordingly, the X_I^a are contracted pairwise in all possible ways.



A chord diagram within M_{10}

We can represent M_{2n} as summing over all **chord diagrams** on a disk with n chords. One chord diagram represents a scheme of contraction. **Different a are labeled by different colors.**



For example, Consider M_4:

$$M_4 = \frac{1}{d^2}\sum_{a,b}(\mathbb{E}[\mathrm{tr}\tilde{H}_a\tilde{H}_a\tilde{H}_b\tilde{H}_b] + \mathbb{E}[\mathrm{tr}\tilde{H}_a\tilde{H}_b\tilde{H}_b\tilde{H}_a] + \mathbb{E}[\mathrm{tr}\tilde{H}_a\tilde{H}_b\tilde{H}_a\tilde{H}_b])$$
$$= 2\cdot 2^N + \frac{\sigma^4}{d^2}\sum_{a,b,I,I'}\mathrm{tr}X_I^aX_{I'}^bX_I^aX_{I'}^b$$

Which corresponds to the sum of three chord diagrams on the left, with all possible color assignments. Since X_I^a is a string of ψ_i, they obey the following commutation relation

$$X_I^aX_{I'}^b = (-)^{|I\cap I'|}X_{I'}^bX_I^a$$

$|I\cap I'|$ means the total numbers of overlapped Majorana fermions in the two strings X_I^a and $X_{I'}^b$.

When $a=b$ this is simply 1. But when $a\neq b$, it becomes a complicated combinatorial problem.

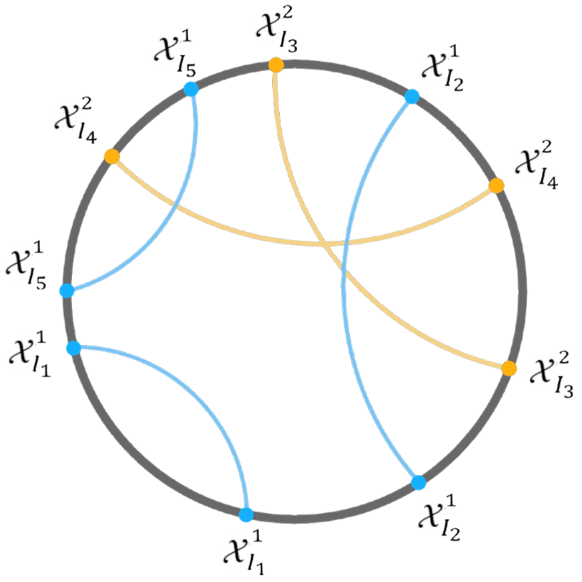
A useful limit is **double scaled limit**, where

$$N \to \infty, \quad \lambda \equiv 4p^2/N \quad \text{fixed}$$

In this limit, the summation can be dealt with analytically. As the result, for intersection of different colors, we need only to multiply a **penalty factor**

$$q = e^{-\lambda}$$

In general, we have the following **chord rules**:
For each crossing in a colored chord diagram, **if the crossing is by two different colors, we give penalty factor q, or otherwise give penalty factor 1**.
The value of each colored chord diagram is the product of all penalty factors.



This chord diagram gives q^3

There exists **two special limits of q** that the problem can be tackled.

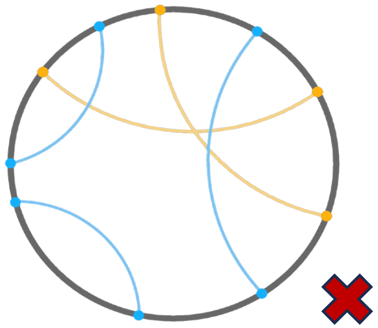
q=0: Free Probability Theory

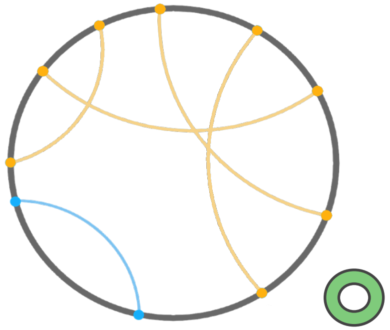
When q=0, connected parts within a chord diagram must have the same color.

This non-crossing feature between different colored chords reflects the fact that **all Ha can be regarded as freely independent random variables**.

In this situation, the theory of **free probability** implies that **the cumulants simply add up**.

Therefore, from the Gaussian spectrum of cSYK, we can obtain an integral representation for the resolvent of dcSYK:





$$\rho_c(x) = \sqrt{\frac{d}{2\pi}} e^{-dx^2/2}$$

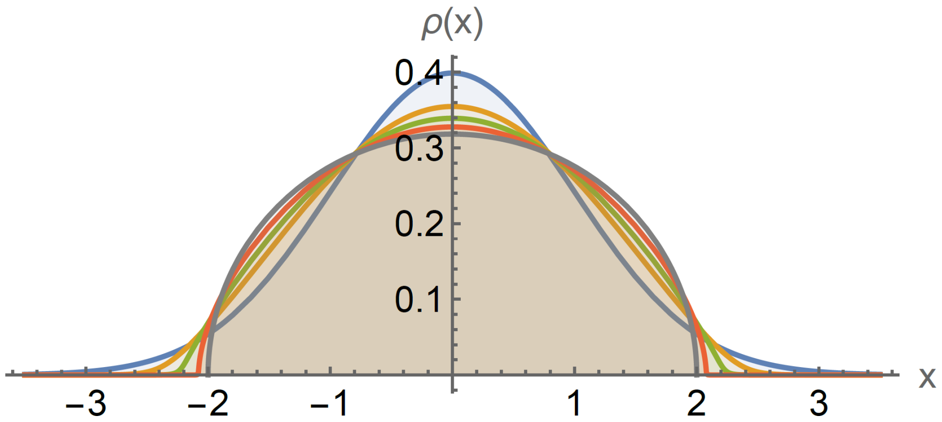
$$R(z) = \sqrt{\frac{d}{2\pi}} \int_{\mathbb{R}} \frac{dx}{z/d + (1 - 1/d)/R(z) - x} e^{-dx^2/2}$$

The previous integral equation can be solved implicitly in terms of Dawson function D:

$$r = \sqrt{2d} D(\sqrt{d/2}w) \mp i\sqrt{\pi d/2} e^{-dw^2/2} \quad w = z/d + (1 - 1/d)/r$$

The spectrum is related to the discontinuity of resolvent across the real axis:

$$\rho(x) = \frac{R(x + i\epsilon) - R(x - i\epsilon)}{-2\pi i}$$



In the plot, blue, yellow, green, red and gray are for d = 1, 3, 6, 15, ∞ respectively.

We see clearly that **the spectrum becomes tighter and has a smaller tail as we increase d.**

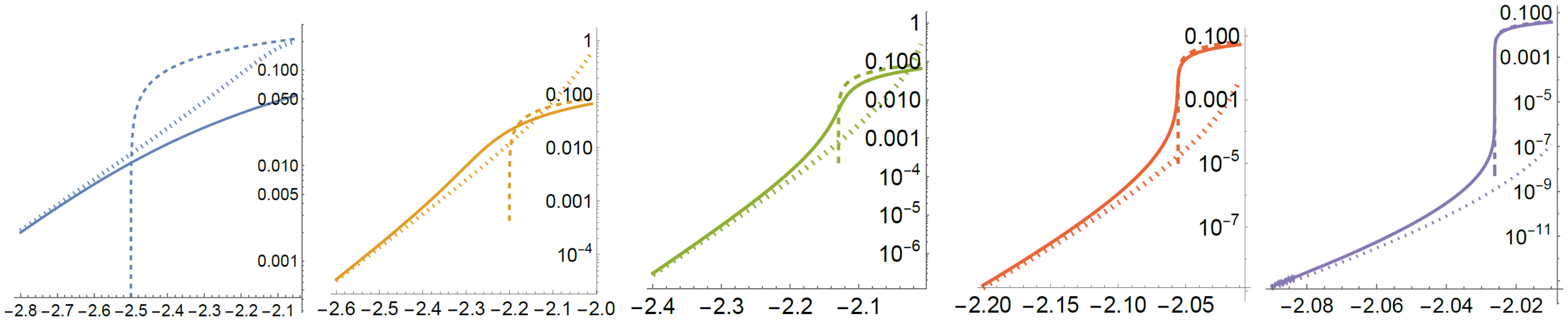
From the implicit relation, we can also use some approximations to obtain asymptotic results.

Zoom to the edge and consider large d asymptotics:

$$\rho(z) = \frac{\sqrt{z - E_0}}{\pi}(1 + O(1/d))$$

Away from the edge, carefully consider nonperturbative exponentially small contributions:

$$\rho(z) \sim \frac{d^{3/2} e^{-d-1}}{\sqrt{2\pi}} e^{-2d(-y) - dy^2/2}$$



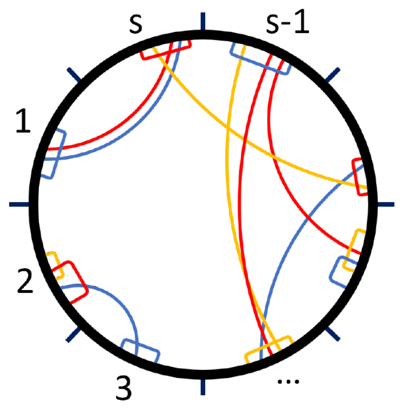
The figures with blue, yellow, green, red and purple curves are for d = 2, 5, 10, 20, 40 respectively.

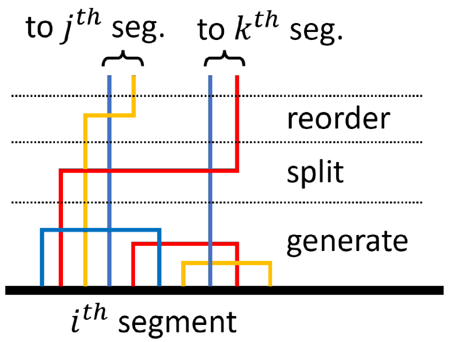
q→1: A Liouville-Like Theory

In this limit we evaluate the spectrum from the partition function: $Z(\beta) = \mathbb{E}[\mathrm{tr} e^{-\beta \tilde{H}}]$

A **coarse graining method** (Berkooz et. al, Phys. Rev. D **110**, 106015 (2024)) for multicolor chord diagrams cuts the circle into s segements. And separately consider the chord distributions near and away from the boundary.

As the result, a chord diagram can be characterized by n_{ij}^a, the number of color-a chords stretching between segment i and j:





$$Z(\beta) = \sum_{\{n_{ij}^a\}} q^{\sum_{a\neq b}\sum_{i>k>j>l} n_{ij}^a n_{kl}^b} \frac{e^{\beta^2/(2s)}}{\prod_{i>j}\prod_a (n_{ij}^a)!} \frac{\beta^{2n}}{(ds^2)^n}$$

After performing a Hubbard-Stratonovich transformation and going to the continuous limit s→infty, the partition function becomes a path integral of a Liouville-like theory:

$$\mathcal{L} = \frac{1}{4} J^a(\tau,\tau') K^{-1}_{ab} \partial_\tau \partial_{\tau'} J^b(\tau,\tau') + \frac{1}{d} \sum_a e^{J^a(\tau,\tau')} \qquad K_{ab} = 1 - \delta_{ab}$$

In the q→1 or lambda→0 limit, this path integral becomes semiclassical and is controlled by a homogeneous SYK saddle.

$$J^a = J \qquad e^J = \frac{\cos^2 \omega\beta_r/2}{\cos^2 \omega(\tau_{12} - \beta_r/2)} \qquad \omega = \sqrt{1 - 1/d}\cos\omega\beta_r/2$$

The partition function is given by the **on-shell** part and **1-loop determinant** around the saddle. The on-shell partition function is

$$Z \sim e^{\frac{1}{\lambda}S_0} \sim \exp\left[\frac{2\beta}{\sqrt{\lambda(1-1/d)}} + \frac{\pi^2}{[\lambda(1-1/d)]^{\frac{3}{2}}\beta}\right]$$

For the 1-loop determinant, we have a quadratic action for the fluctuations:

$$S_2 = \int_0^{\beta_r} d\tau \int_0^{\tau} d\tau' \left[\frac{1}{4(d-1)}\delta J(\tau,\tau')\partial_\tau\partial_{\tau'}\delta J(\tau,\tau') + \frac{1}{2d}e^{J(\tau-\tau')}\delta J(\tau,\tau')\delta J(\tau,\tau')\right.$$
$$\left. + \sum_{i=1}^{d-1}\left(-\frac{1}{4}\delta J^i(\tau,\tau')\partial_\tau\partial_{\tau'}\delta J^i(\tau,\tau') + \frac{1}{2d}e^{J(\tau-\tau')}\delta J^i(\tau,\tau')\delta J^i(\tau,\tau')\right)\right]$$

This 1-loop action contains essential difference to SYK: there are **d-1 non-Schwarzian** modes in addition to the usual **Schwarzian mode**. They obey the same type of eigenequation:

$$\left[\partial_1\partial_2 + \frac{h(h-1)\omega^2}{\cos^2\omega(\tau_{12}-\beta_r/2)}\right] f(\tau_1,\tau_2) = \eta f(\tau_1,\tau_2)$$

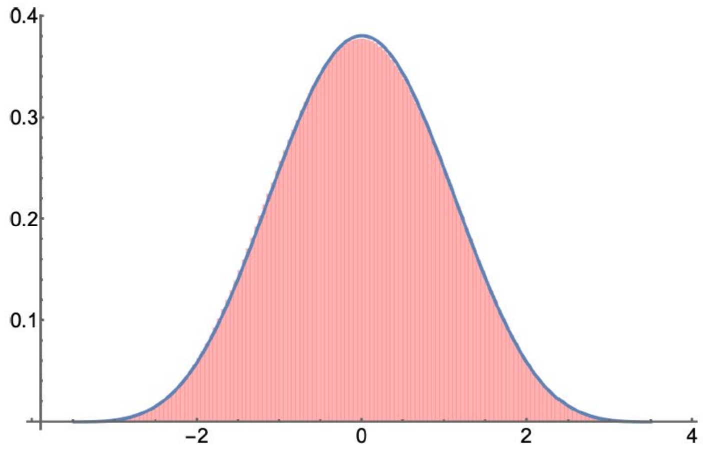
With **h=2 for Schwarzian mode** and **h=** $\frac{1}{2}\left(1 \pm \sqrt{\frac{d-9}{d-1}}\right)$ **for non Schwarzian modes**. The 1-loop determinant is the product of all eigenvalues, which can be evaluated via Sommerfeld-Watson resummation. The full result is:

$$\rho(\zeta) \sim \frac{e^\zeta}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{dt e^{\zeta t}}{(1+t)^{3/2}(-t)^{(d-1)/2}} = \begin{cases} \frac{e^\zeta U(3/2-d/2,1-d/2,-\zeta)}{\Gamma((d-1)/2)} & \zeta < 0 \\ \frac{U(-1/2,1-d/2,\zeta)}{\Gamma(3/2)} & \zeta > 0 \end{cases}$$

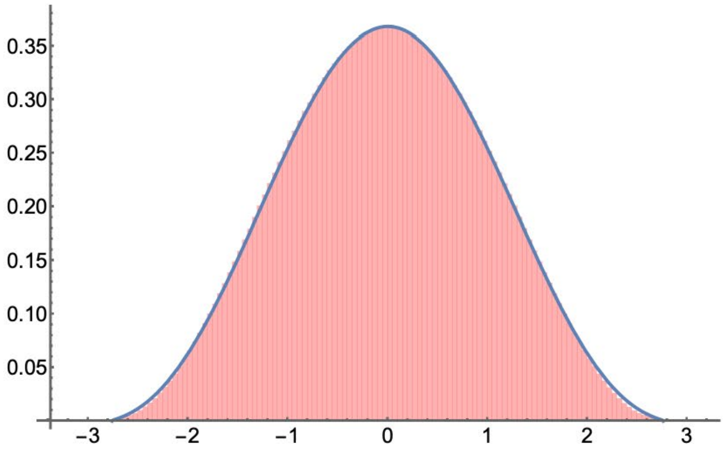
Numerical Results-Spectrum

We compare these analytical results to **exact diagonalization**:

The spectrum agrees pretty good.



(a) $N = 15\ d = 2$



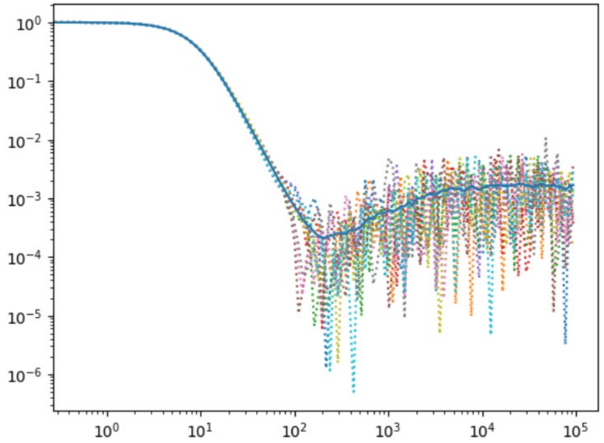
(b) $N = 15\ d = 5$

Numerical Results-SFF

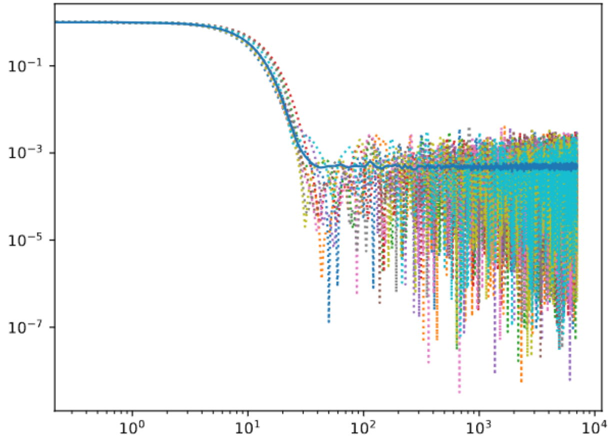
The **spectral form factor (SFF)** is an independent diagnosis for quantum chaos:

$$\mathrm{SFF}(\beta, t) = \mathbb{E}[Z(\beta + it)Z(\beta - it)]$$

For chaotic systems (e.g. SYK model, left), **the shape of SFF exhibits a "dip-ramp-plateau" structure** which hints random matrix level statistics, while integrable systems (e.g. commuting SYK model, right) does not.

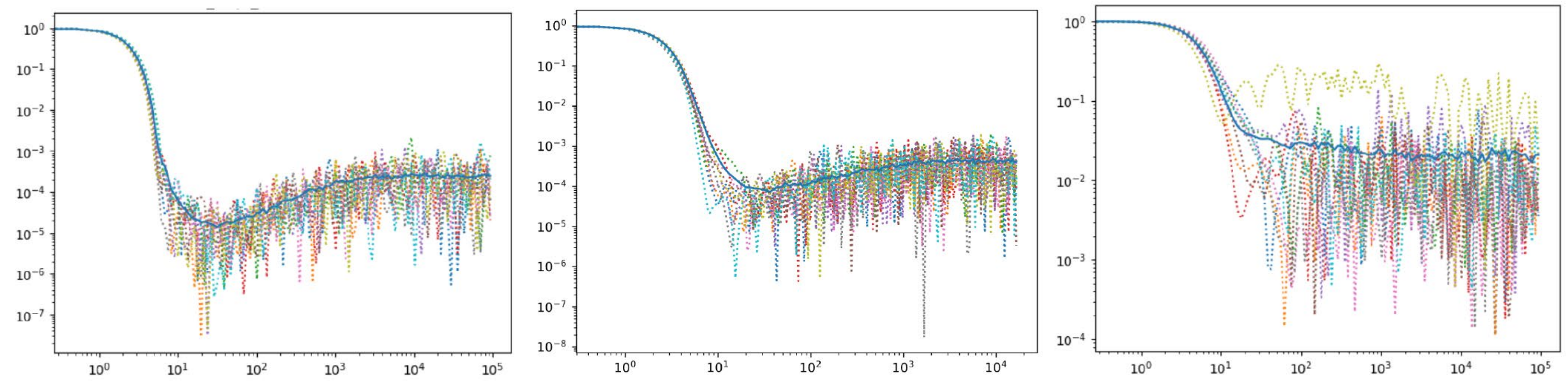


SFF of SYK model



SFF of commuting SYK model

We numerically studied the SFF of dcSYK under different temperatures: (colorful dashed lines for different realizations, blue solid line for the ensemble average)



(b) $d = 2, \beta = 0,$ (c) $d = 2, \beta = 2,$ (d) $d = 2, \beta = 10,$

These numerical results clearly show **chaotic-integrable transition under decreasing temperatures.**

Summary

We constructed a new family of quantum chaotic models that exhibits chaotic-integrable transitions under sufficiently low temperature.

We study the spectrum of this model in the double scaled limit with $\lambda \equiv 4p^2/N$ fixed and $N \rightarrow \infty$. The spectrum is non-compact for generic d and becomes compact only when $d \rightarrow \infty$. These results are confirmed numerically.

To simulate an authentic signal of traversable wormholes in a holographic system, besides increasing N, we need to **break integrability by including adequate numbers of non-commuting terms** and **make sure the system is in a state exhibiting chaotic behavior**.

Thanks for Listening!