Hydrodynamics with multiple charges and holography

Andrei Starinets

The Rudolf Peierls Centre for Theoretical Physics
University of Oxford

Liam Gladden, Victor Ivo, Pavel Kovtun and AOS, 2412.12353 [hep-th], 2507.XXXXX [hep-th]
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Plan

Motivation and overview

The Gubser-Mitra conjecture

Finite-temperature QFT with multiple conserved charges

Relativistic fluid dynamics with multiple charges

$\mathcal{N} = 4$ SYM at finite density of R-charges

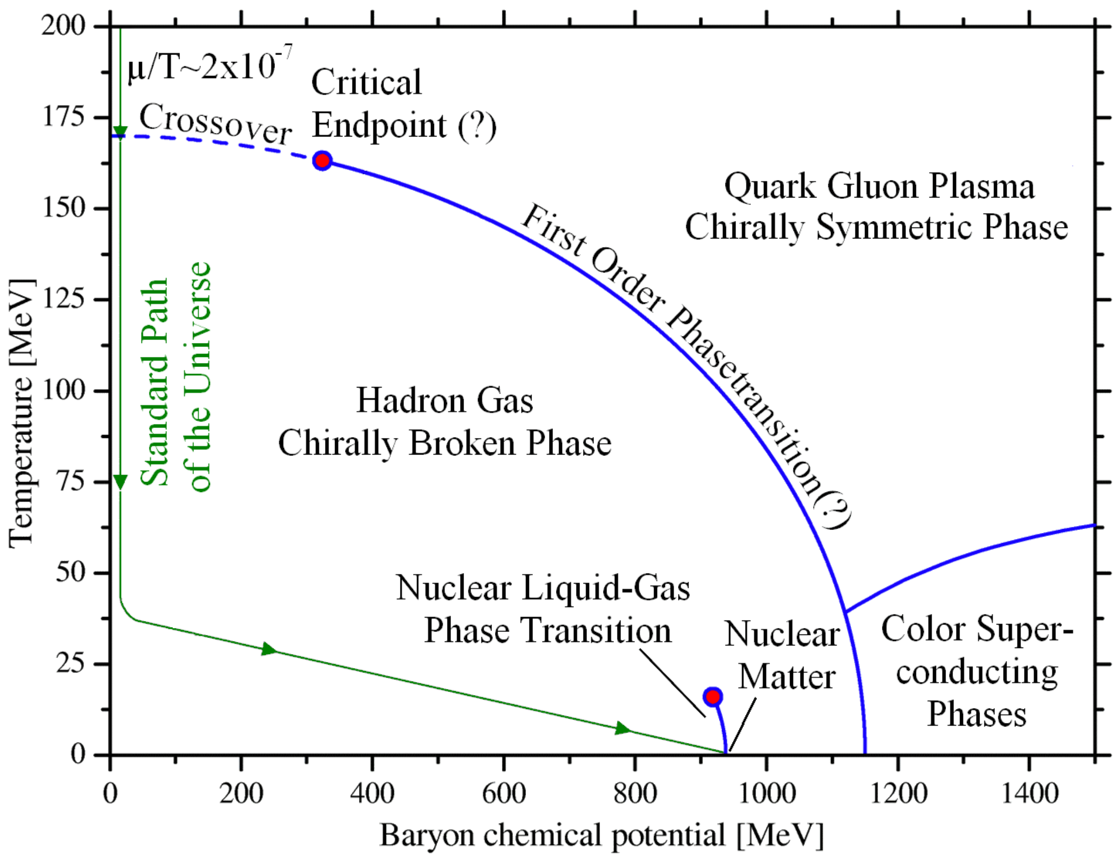
The thermodynamic instability

The dynamic instability

Conclusions

Motivation and overview

Strongly interacting QFTs at finite temperature and density are of interest for heavy ion collision physics (RHIC, NICA, FAIR…) and relativistic astrophysics (cores of neutron stars etc)



Motivation and overview (continued)

Holography was instrumental in providing insights into transport in strongly interacting QFTs at finite temperature (e.g. shear viscosity)

Theories at finite density were considered from 1998-1999
(Cai and Soh, hep-th/9812121, Cvetic and Gubser, hep-th/9902195, hep-th/9903132)

For N=4 SYM theory, the top-down construction involved rotating black 3-branes in 10d whose dimensional reduction on a five-sphere gave the STU solution of 5d gauged SUGRA
(Behrndt, Cvetic, Sabra, hep-th/9810227)

A special case of the STU solution is the Reissner-Nordstrom black hole (brane) of Einstein-Maxwell gravity – used later in many bottom-up scenarios

However, the STU solution is unstable w.r.t. fluctuations of neutral scalars.
Here, we shall describe thermodynamic and hydrodynamic aspects of this instability

The Gubser-Mitra conjecture (Correlated Stability Conjecture)

For black branes, classical instabilities (e.g. the Gregory-Laflamme instability) occur if and only if the system is thermodynamically unstable, typically indicated by a negative specific heat.

Finite-temperature QFT with multiple conserved charges

Consider a QFT with a global symmetry group G

(for N=4 SYM, this is the R-symmetry SU(4)_R)

The grand canonical partition function involves a maximal set of *commuting* conserved charges

$$Z(\beta, \mu_A) = \mathrm{Tr}\ \exp\left[-\beta(H - \mu_A Q^A)\right]$$

$$Z(\beta, g) = \mathrm{Tr}\ \left[U(g)e^{-\beta H}\right] = \mathrm{Tr}\ \left[U(\eta)U(g)U(\eta)^{-1}e^{-\beta H}\right]$$
$$= \mathrm{Tr}\ \left[U(\eta g \eta^{-1})e^{-\beta H}\right] = Z(\beta, \eta g \eta^{-1})$$

Any group element g is equivalent under conjugation to an element h of a maximal Abelian subgroup of G, which, in turn, can be written as an exponential of a sum of generators of a Cartan subalgebra

For SU(4)_R, need to introduce 3 chemical potentials

Yaffe and Yamada, hep-th/0602074; Haber and Weldon, 1982

Probability of a fluctuation in a thermodynamic system in thermal equilibrium is given in the microcanonical ensemble by Einstein's formula

$$w_\Delta = e^{\Delta S}$$

$\Delta S = S' - S$ is the difference between the entropy of a near-equilibrium state emerging as a result of the fluctuation and the entropy of the system in thermal equilibrium.

For small fluctuations characterised by the parameters $\xi_1, \ldots \xi_n$ we have

$$w_\Delta \sim \exp\left\{-\frac{1}{2}\sum_{ik}\lambda_{ik}\xi_i\xi_k\right\}$$

Here, the coefficients are: $\lambda_{ik} = -(\partial^2 S/\partial\xi_i\partial\xi_k)\mid_{\xi_i=0}$

For a stable thermodynamic equilibrium, the quadratic form should be positive definite.

(The leading principal minors must be positive.)

For the entropy and charges as independent variables, $\varepsilon = \varepsilon(s, n_i)$

$$w_{\Delta} \sim \exp\left\{ -\frac{1}{2T} \sum_{ij} \frac{\partial^2 \varepsilon}{\partial x_i \partial x_j} \delta x_i \delta x_j \right\}$$

In this formula, $x_i \equiv (s, n_k)$ are the entropy density and the densities of charges

The eigenvalues and eigenvectors of the Hessian identify the unstable hydrodynamic *and dual quasinormal modes of the gravitational background*

$$H^{\epsilon} = \frac{\partial^2 \varepsilon}{\partial x_i \partial x_j}$$

To see this explicitly, we need to develop relativistic hydro with multiple charges

For now, we can just see what happens when we have an equilibrium state at finite temperature and finite density of one charge

Hydrodynamics at finite density of one charge: predictions

(for details, see e.g. P.Kovtun, 1205.5040 [hep-th])

With the spatial momentum along z direction, we have

1) The shear momentum diffusion pole in correlators of currents and energy-momentum tensor such as $\langle J_x J_x \rangle \; \langle J_x T_{zx} \rangle \; \langle T_{tx} T_{zx} \rangle$

$$\omega = -i \frac{\eta}{\varepsilon + P} q^2 + \cdots$$

2) The sound and the charge diffusion poles in correlators of currents and energy-momentum tensor such as $\langle T_{tt} T_{tt} \rangle \; \langle T_{tt} T_{tz} \rangle \; \langle T_{tt} J_z \rangle \; \langle J_t J_z \rangle \; \langle J_z J_z \rangle \; \langle J_t J_t \rangle$

$$\omega = \pm v_s q - i \frac{\Gamma}{2} q^2 + \cdots$$

$$\omega = -i D q^2 + \cdots$$

We may suspect instabilities arising in the sound channel

Sound attenuation constant:

$$\Gamma = \frac{1}{w}\left(\frac{2d-2}{d}\eta + \zeta\right) + \frac{\sigma w}{v_s^2(\det\chi)^2}\left(n\chi_{11} - w\chi_{13}\right)^2$$

Charge diffusion constant:

$$D = \frac{\sigma w^2}{v_s^2 \det\chi}$$

$w = \varepsilon + P :$ enthalphy density

$n :$ charge density

$\eta > 0 :$ shear viscosity

$\zeta > 0 :$ bulk viscosity

$\sigma > 0 :$ conductivity

$\chi_{ab} = \partial n_a / \partial \mu_b :$ susceptibility matrix related to the Hessian

Note: charge diffusion constant is sensitive to the determinant changing sign

Hydrodynamics with multiple conserved charges

Conservation laws: $\partial_\mu T^{\mu\nu} = 0$ $\partial_\mu J_a^\mu = 0$ $a = 1, 2, \ldots N_f$

Conservation laws are supplemented by the constitutive relations:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta S^{\mu\nu} - \zeta \Delta^{\mu\nu}\partial_\lambda u^\lambda + O(\partial^2)$$

$$J_a^\mu = n_a u^\mu - \sigma_{ab} T \Delta^{\mu\nu}\partial_\nu \gamma_b + O(\partial^2)$$

First order transport coefficients: η ζ σ_{ab}

(shear viscosity, bulk viscosity, conductivity matrix)

Parameters and tensor structures

$$\gamma_a \equiv \mu_a / T$$

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$$

$$S^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta}\partial_\lambda u^\lambda)$$

Kubo formulas for transport coefficients

$$\eta = \frac{1}{2T} \lim_{\omega \to 0} G_{T^{xy}T^{xy}}(\omega, k = 0)$$

$$\zeta = \frac{1}{2Td^2} \lim_{\omega \to 0} G_{T^{ii}T^{kk}}(\omega, k = 0)$$

$$\sigma_{ab} = \frac{1}{2Td} \lim_{\omega \to 0} G_{J_a^i J_b^i}(\omega, k = 0)$$

For small fluctuations around equilibrium, one has the system of equations

$$\left(\frac{\partial\epsilon}{\partial T}\right)_\gamma \partial_t\delta T + T\left(\frac{\partial\epsilon}{\partial\mu_a}\right)_T \partial_t\delta\gamma_a + (\epsilon{+}p)\partial_x\delta u^x = 0$$

$$(\epsilon{+}p)\,\partial_t\delta u^x + \left(\frac{\partial p}{\partial T}\right)_\gamma \partial_x\delta T + T\left(\frac{\partial p}{\partial\mu_a}\right)_T \partial_x\delta\gamma_a - (\frac{2d-2}{d}\eta + \zeta)\partial_x^2\delta u^x = 0$$

$$\left(\frac{\partial n_a}{\partial T}\right)_\gamma \partial_t\delta T + n_a\,\partial_x\delta u^x + T\left(\left(\frac{\partial n_a}{\partial\mu_b}\right)_T \partial_t - \sigma_{ab}\,\partial_x^2\right)\delta\gamma_b = 0$$

This system can be written in a matrix form

$$\left[X_{AB}\partial_t + Y_{AB}\partial_x - Z_{AB}\partial_x^2\right] V_B = 0$$

$$X = \begin{pmatrix} C & 0 & v_a \\ 0 & \epsilon+p & 0_a \\ v_a & 0_a & \chi_{ab} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \epsilon+p & 0_a \\ \epsilon+p & 0 & n_a \\ 0_a & n_a & 0_{ab} \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0_a \\ 0 & \eta_s & 0_a \\ 0_a & 0_a & \sigma_{ab} \end{pmatrix}.$$

The parameters are

$$C \equiv T\left(\frac{\partial \epsilon}{\partial T}\right)_\gamma, \quad v_a \equiv \left(\frac{\partial \epsilon}{\partial \mu_a}\right)_T = T\left(\frac{\partial n_a}{\partial T}\right)_\gamma, \quad \chi_{ab} \equiv \left(\frac{\partial n_a}{\partial \mu_b}\right)_T$$

The hydro modes dispersion relations are determined from the condition

$$\det\left[-\omega X + kY - ik^2 Z\right] = 0$$

There are N_f diffusion modes

$$\omega_{(a)}(k) = -iD_{(a)}k^2 + \dots$$

and two sound modes

$$\omega_{\pm}(k) = \pm c_s|k| - \frac{i}{2}\Gamma k^2 + \dots$$

whose parameters are determined by the matrices X,Y,Z, e.g.

$$c_s^2 = \frac{1}{(\epsilon+p)^{2(N_f-1)}}\frac{\det M_X}{\det X}$$

$$\Gamma = \frac{\frac{2d-2}{d}\eta + \zeta}{\epsilon+p} + \frac{V_\pm^a \sigma_{ab} V_\pm^b}{(\epsilon+p)\det(X)^2\, c_s^2}$$

The analysis shows that positive definite Hessian (thermodynamic stability) implies hydrodynamic stability

$\mathcal{N} = 4$ supersymmetric SU(N) YM theory

Gliozzi,Scherk,Olive'77
Brink,Schwarz,Scherk'77

- Field content:

$$A_\mu \qquad \Phi_I \qquad \Psi_\alpha^A \qquad \text{all in the adjoint of } SU(N)$$

$$I = 1 \ldots 6 \qquad A = 1 \ldots 4$$

- Action:

$$S = \frac{1}{g_{YM}^2} \int d^4x \; \mathrm{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_I)^2 - \frac{1}{2} [\Phi_I, \Phi_J]^2 + i \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma^I [\Phi_I, \Psi] \right\}$$

- Large N: effective coupling = 't Hooft coupling $\lambda = g_{YM}^2 N$

(super)conformal field theory = coupling doesn't run

$\mathcal{N} = 4$ supersymmetric SU(N) YM theory

Consider the theory at finite temperature and finite density of three R-charges
(or three chemical potentials)

a) In 3+1 Minkowski space

b) On a three-sphere (plus time)

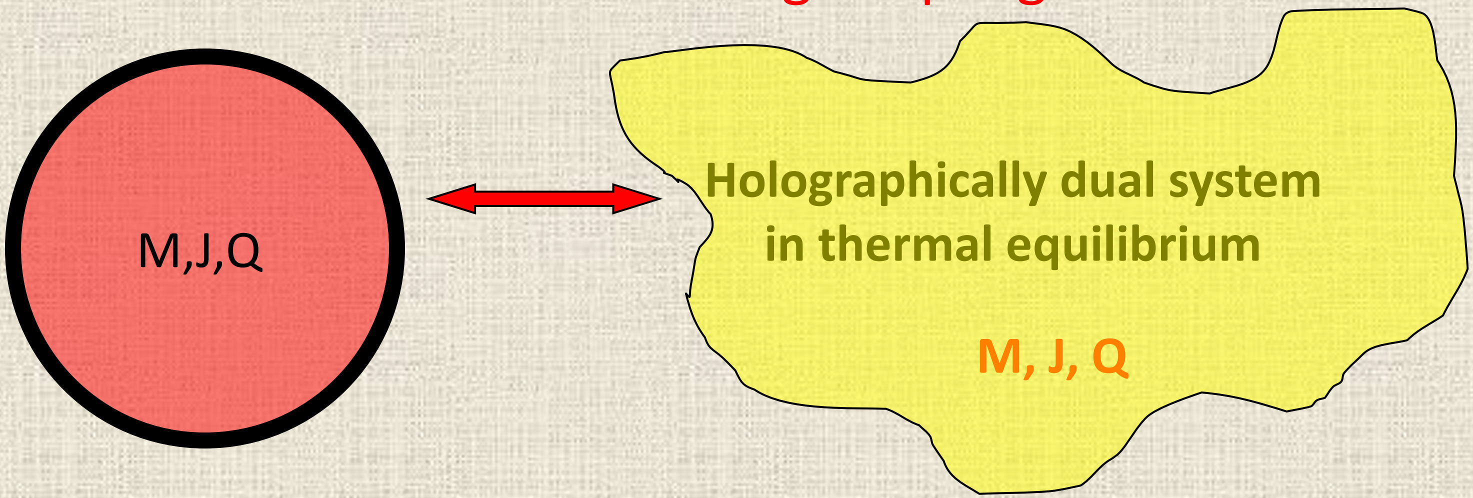
Do we know the equilibrium state at all values of T/μ_i ,
including the low-temperature limit?

Holography may help to answer this question in the limit of infinite N and large ‘t Hooft coupling. But we can also consider the perturbative regime, and compare

Here, we’ll focus on the scenario a). Scenario b) involves Hawking-Page transition
(see e.g. Yamada and Yaffe, hep-th/0602074)

10-dim gravity

4-dim gauge theory – large N, strong coupling



T_{Hawking} $S_{\text{Bekenstein-Hawking}}$ $\Longleftrightarrow$ T S

Gravitational fluctuations
(and fluctuations of other fields) $\Longleftrightarrow$ Deviations from equilibrium

$g^{(0)}_{\mu\nu} + h_{\mu\nu}$ $\Longleftrightarrow$????

$A^{(0)}_{\mu} + a_{\mu}$

$j_i = -D\partial_i j^0 + \cdots$

$''\Box''\, h_{\mu\nu} = 0 \quad ''\Box''\, a_\mu = 0$ $\Longleftrightarrow$ $\partial_t j^0 + \partial_i j^i = 0$

and B.C.

$\partial_t j^0 = D\nabla^2 j^0$

Quasinormal spectrum $\Longleftrightarrow$ $\omega = -iDq^2 + \cdots$

Birmingham, Sachs, Solodukhin, 2001; Son and AOS, 2002

In quantum field theory, the dispersion relations such as

$$\omega = \pm v_s\, q - \frac{i}{2(\epsilon + P)} \left(\frac{4}{3}\eta + \zeta\right) q^2$$

appear as poles of the retarded correlation functions, e.g.

$$\langle T_{00}(k)\, T_{00}(-k)\rangle \sim \frac{q^2\, T^4}{\omega^2 - q^2/3 + i\omega q^2/3\pi T}$$

- in the hydro approximation - $\omega/T \ll 1\,, \quad q/T \ll 1$

Singularities of a (retarded) Green's function in the complex frequency plane

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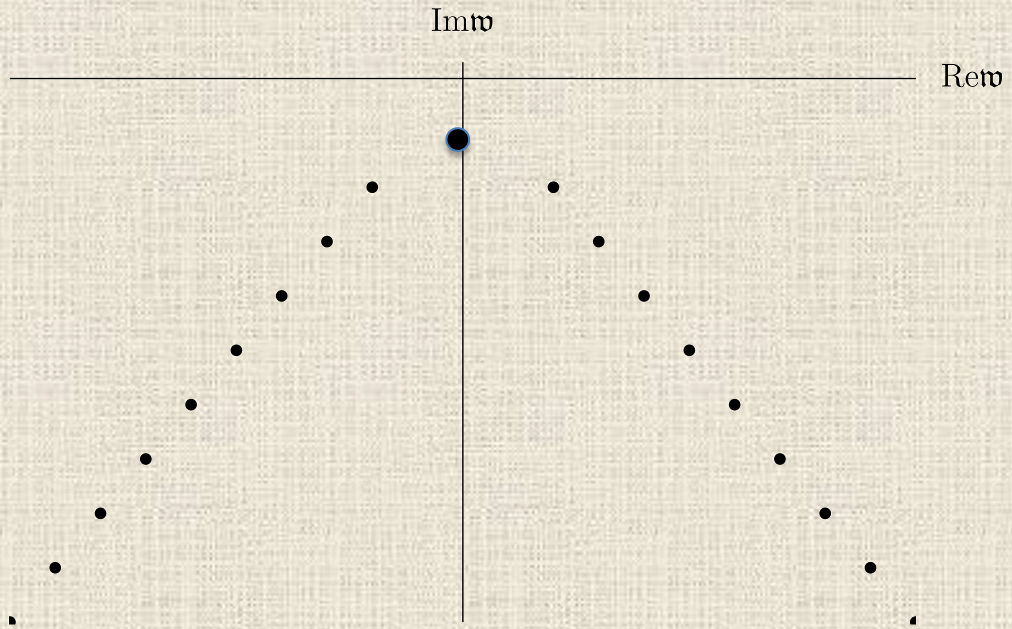
Quasinormal modes of dual black holes

Shear channel

$$\mathfrak{w} = -iD\mathfrak{q}^2 + \cdots$$



Strong (infinite) coupling

Real spatial momentum q

Gravity dual to N=4 SYM

$$\mathcal{L} = \sqrt{-g}\Big(R + \frac{4}{L^2}H^{-\frac{1}{3}}\sum_{i=1}^{3} H_i - \frac{1}{4}H^{-\frac{2}{3}}\sum_i H_i^2 F^i_{\mu\nu}F^{\mu\nu i} - \frac{1}{3}\sum_{k=1}^{3}\frac{\partial_\mu H_k \partial^\mu H_k}{H_k^2}$$
$$+\frac{1}{3}\sum_{i<j}^{3}\frac{\partial_\mu H_i \partial^\mu H_j}{H_i H_j}\Big) + \frac{1}{24}\epsilon^{\mu\nu\rho\sigma\lambda}C_{ijk}F^i_{\mu\nu}F^j_{\rho\sigma}A^k_\lambda$$

The Lagrangian of 5d SUGRA contains the metric, three Abelian fields and three scalars

The STU background depends on the non-extremality parameter r_+ and three charges Q_i

$$ds_5^2 = -H^{-2/3}\frac{(\pi T_0 L)^2}{u}f(u)\,dt^2 + H^{1/3}\frac{(\pi T_0 L)^2}{u}\left(dx^2+dy^2+dz^2\right) + H^{1/3}\frac{L^2}{4fu^2}du^2$$

$$A^i_\mu(u) = \delta^t_\mu\left(\frac{1}{1+\kappa_i} - \frac{u}{H_i(u)}\right)\pi T_0\sqrt{2\kappa_i}\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}$$

$$H_i = 1+\kappa_i u \qquad\qquad H(u) = H_1H_2H_3 = (1+\kappa_1 u)(1+\kappa_2 u)(1+\kappa_3 u)$$

$$\kappa_i \equiv Q_i/r_+^2 \qquad T_0 = r_+/\pi L^2$$

Gravity dual to N=4 SYM (continued)

Thermodynamics follows from the standard black brane thermodynamics

$$s = \frac{1}{2}\pi^2 N_c^2 T_0^3 \sqrt{1+\kappa_1}\sqrt{1+\kappa_2}\sqrt{1+\kappa_3}$$

$$n_i = \frac{1}{8}\pi N_c^2 T_0^3 \sqrt{2\kappa_i}\sqrt{1+\kappa_1}\sqrt{1+\kappa_2}\sqrt{1+\kappa_3}$$

$$T_H = \frac{2+\kappa_1+\kappa_2+\kappa_3-\kappa_1\kappa_2\kappa_3}{2\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}} T_0$$

$$\mu_i = \pi T_0 \frac{\sqrt{2\kappa_i}}{1+\kappa_i}\sqrt{1+\kappa_1}\sqrt{1+\kappa_2}\sqrt{1+\kappa_3}$$

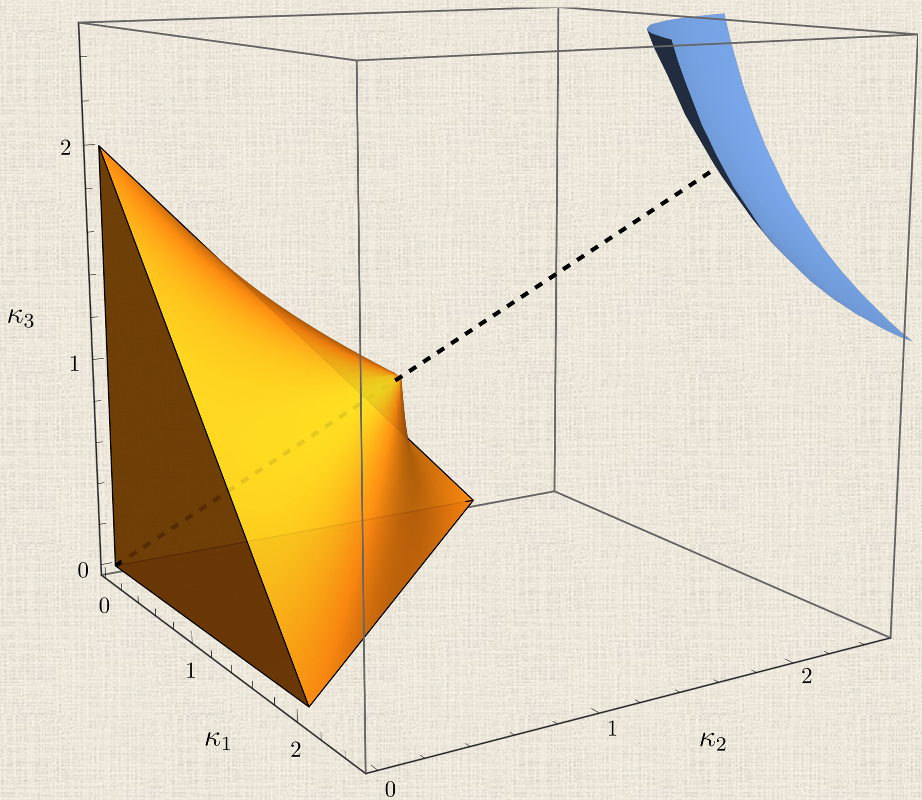
The equation of state

$$\epsilon(s, n_1, n_2, n_3) = \frac{3}{2(2\pi N_c)^{2/3}}\, s^{4/3} \prod_{i=1}^{3} \left(1 + \frac{8\pi^2 n_i^2}{s^2}\right)^{1/3}$$

allows to find the stability region via the Hessian

The thermodynamic stability region

Computing the leading principal minors of the Hessian (rather than the eigenvalues), we find the stability conditions in the space of 3 chemical potentials



Zero temperature limit is shown as a blue surface

$$2 - \kappa_1 - \kappa_2 - \kappa_3 + \kappa_1 \kappa_2 \kappa_3 = 0$$

$$\kappa_1 + \kappa_2 + \kappa_3 < 3$$

We can consider simple examples such as the case of a single chemical potential

$$(\kappa_1, \kappa_2, \kappa_3) = (\kappa, 0, 0)$$

We can compute all parameters of the first-order hydro analytically:

$$v_s = 1/\sqrt{3}$$ speed of sound

$$\eta = \frac{\pi N_c^2 (1+\kappa)^2}{(2+\kappa)^3} T_H^3 = \frac{s}{4\pi}$$ shear viscosity

$$\zeta = 0$$ bulk viscosity

$$\mathcal{D} = \frac{\eta}{\varepsilon + P} = \frac{2+\kappa}{8\pi T_H (1+\kappa)}$$ shear momentum diffusion constant

$$D_R = \frac{4-\kappa^2}{8\pi T_H (1+\kappa)}$$ The R-charge diffusion constant

The full set of hydro modes

In the sound channel - appearing as poles in the correlators

$$G^R_{J_t J_t}, G^R_{J_t J_z}, G^R_{J_z J_z}, G^R_{T_{tt} T_{tt}}, G^R_{T_{tz} T_{tz}}, G^R_{T_{zz} T_{zz}}, G^R_{J_z T_{tt}}$$

$$\omega_{\text{sound}} = \pm \frac{1}{\sqrt{3}} q - i \frac{2+\kappa}{12\pi T_H (1+\kappa)} q^2 + \cdots$$

$$\omega_{\text{charge diffusion}} = -i \frac{4-\kappa^2}{8\pi T_H (1+\kappa)} q^2 + \cdots$$

In the shear channel - appearing as poles in the correlators

$$G^R_{J_a J_a}, G^R_{T_{ta} T_{ta}}, G^R_{T_{ta} T_{za}}, G^R_{T_{za} T_{za}}, G^R_{J_z T_{xx}}$$

$$\omega_{\text{momentum diffusion}} = -i \frac{2+\kappa}{8\pi T_H (1+\kappa)} q^2 + \cdots .$$

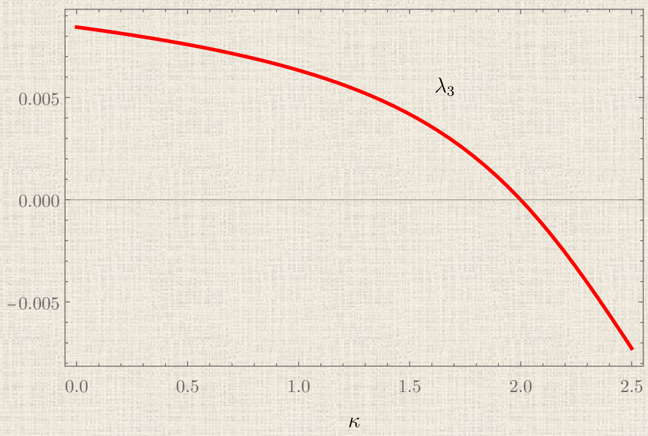
Thermodynamic and dynamic instability

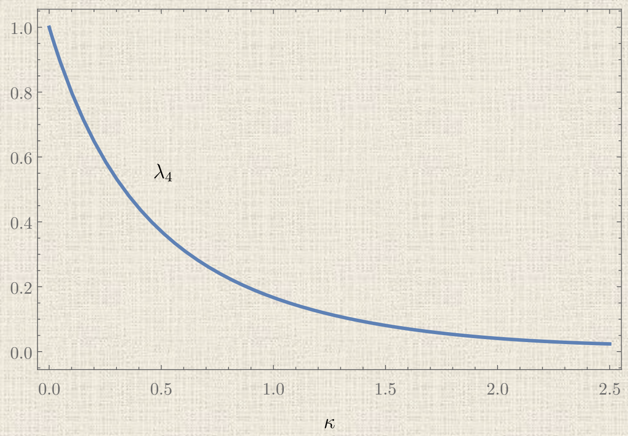
The Hessian in variables s, n_1, n_2, n_3

$$\bar{h}_{ab} = \begin{pmatrix} \frac{2-(-5+\kappa)\kappa}{24\pi^2(1+\kappa)^2} & \frac{(-1+\kappa)\sqrt{\kappa}}{3\sqrt{2}\pi(1+\kappa)^2} & 0 & 0 \\ \frac{(-1+\kappa)\sqrt{\kappa}}{3\sqrt{2}\pi(1+\kappa)^2} & -\frac{-3+\kappa}{3(1+\kappa)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \bar{h}_{ab} = \frac{2-\kappa}{24\pi^2(1+\kappa)^2}$$

Eigenvalues: $\lambda_1 = \lambda_2 = 1$





There is a thermodynamic instability at $\kappa = 2$

Thermodynamic and dynamic instability

Fluctuation equations of the STU background follow from the equations of motion of 5d SUGRA

$$R_{\mu\nu} = T^{(m)}_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T^{(m)}$$

$$\partial_\nu \left(\sqrt{-g} H^{-\frac{2}{3}} H_i^2 F^{\mu\nu i} \right) = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\lambda} C_{ijk} F^j_{\rho\sigma} F^k_{\nu\lambda}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} \left(-\frac{2}{3} \frac{\partial^\mu H_i}{H_i} + \frac{1}{3} \sum_{j \neq i} \frac{\partial^\mu H_j}{H_j} \right) \right] = -\frac{4}{3L^2} H^{-\frac{1}{3}} \sum_{j \neq i} H_j$$

$$+ \frac{8}{3L^2} H^{-\frac{1}{3}} H_i - \frac{1}{3} H^{-\frac{2}{3}} H_i^2 F^i_{\mu\nu} F^{\mu\nu i} + \frac{1}{6} H^{-\frac{2}{3}} \sum_{j \neq i} H_j^2 F^j_{\mu\nu} F^{\mu\nu j}$$

Generically, fluctuations of the metric, the three U(1) fields and the three scalars are coupled.

However, eigenvectors of the Hessian help to isolate the set of fluctuations the unstable mode couples to. In the single charge case, the relevant U(1) field couples to the metric, other two U(1) fields decouple.

In 1005.0819 [hep-th], Buchel numerically found an unstable mode of the type

$$\omega = -i\, \mathcal{A}\, \frac{(2-\kappa)}{2\pi T}\, q^2 + O(q^4)\,, \qquad \mathcal{A} \approx 0.33333$$

This is exactly the hydro diffusion mode postdicted by our analysis

Another interesting case to consider is $(\kappa_1, \kappa_2, \kappa_3) = (\kappa, \kappa, \kappa)$

The STU solution in this case is given by

$$ds_5^2 = -\frac{(\pi T_0 L)^2}{u\mathcal{H}^2} f dt^2 + \frac{(\pi T_0 L)^2 \mathcal{H}}{u}(dx^2 + dy^2 + dz^2) + \frac{\mathcal{H} L^2}{4fu^2} du^2$$

$$A_t(u) \equiv A_t^1(u) = A_t^2(u) = A_t^3(u) = \delta_\mu^t \frac{1-u}{\mathcal{H}(u)} \pi T_0 \sqrt{2\kappa(1+\kappa)}$$

$$\mathcal{H}(u) \equiv H_1(u) = H_2(u) = H_3(u) = 1 + \kappa u$$

The background metric can be brought to the AdS-Reissner-Nordstrom form, with background scalars trivial:

$$ds_5^2 = -\frac{(\pi T_0 L)^2 (1+\kappa)}{\tilde{u}} \tilde{f} dt^2 + \frac{(\pi T_0 L)^2 (1+\kappa)}{\tilde{u}}(dx^2 + dy^2 + dz^2) + \frac{L^2}{4\tilde{f}\tilde{u}^2} d\tilde{u}^2$$

$$A_t = \frac{\sqrt{2Q(1+\kappa)}\tilde{u}}{L^2} = \mu \tilde{u}$$

Numerous bottom-up constructions use this background as a basic finite density gravity dual

Thermodynamic and dynamic instability in the case of three chemical potentials

Thermodynamic instability is seen by computing the Hessian:

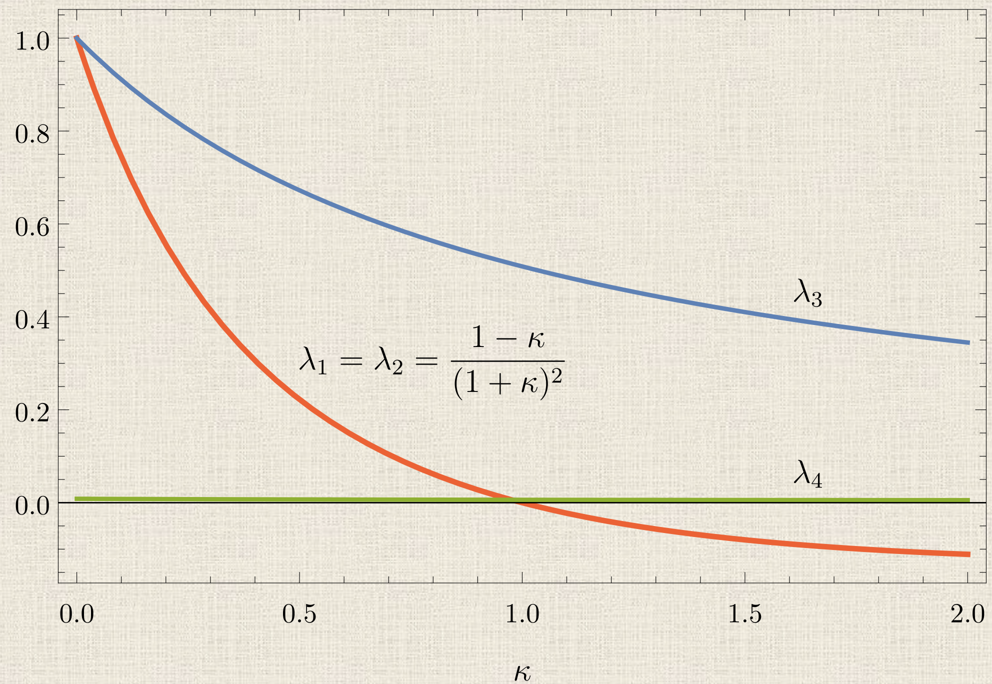
$$\bar{h}_{ab} = \begin{pmatrix} \frac{2+5\kappa}{24\pi^2(1+\kappa)} & -\frac{\sqrt{\kappa}}{\sqrt{2}(3\pi+3\pi\kappa)} & -\frac{\sqrt{\kappa}}{\sqrt{2}(3\pi+3\pi\kappa)} & -\frac{\sqrt{\kappa}}{\sqrt{2}(3\pi+3\pi\kappa)} \\ -\frac{\sqrt{\kappa}}{\sqrt{2}(3\pi+3\pi\kappa)} & -\frac{-3+\kappa}{3(1+\kappa)^2} & \frac{2\kappa}{3(1+\kappa)^2} & \frac{2\kappa}{3(1+\kappa)^2} \\ -\frac{\sqrt{\kappa}}{\sqrt{2}(3\pi+3\pi\kappa)} & \frac{2\kappa}{3(1+\kappa)^2} & -\frac{-3+\kappa}{3(1+\kappa)^2} & \frac{2\kappa}{3(1+\kappa)^2} \\ -\frac{\sqrt{\kappa}}{\sqrt{2}(3\pi+3\pi\kappa)} & \frac{2\kappa}{3(1+\kappa)^2} & \frac{2\kappa}{3(1+\kappa)^2} & -\frac{-3+\kappa}{3(1+\kappa)^2} \end{pmatrix}$$

The eigenvalues of the Hessian are:

$$\lambda_{1,2} = \frac{1-\kappa}{(1+\kappa)^2}$$

$$\lambda_{3,4} = \frac{2+24\pi^2+5\kappa \mp \sqrt{576\pi^4+48\pi^2(3\kappa-2)+(5\kappa+2)^2}}{48\pi^2(1+\kappa)}$$

Two identical eigenvalues change sign, the determinant remain non-negative



The eigenvectors of the Hessian suggest the unstable mode decouples from metric fluctuations

$$V_1^{(3)} = (0, -1, 0, 1)$$

$$V_2^{(3)} = (0, -1, 1, 0)$$

$$V_3^{(3)} = (r_-(\kappa), 1, 1, 1)$$

$$V_4^{(3)} = (r_+(\kappa), 1, 1, 1)$$

Dynamic instability in the case of three chemical potentials

We now consider linear fluctuations of the background

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$
$$A^i_\mu \to A^i_\mu + \delta A^i_\mu$$
$$H_i \to H_i + \delta H_i$$

Note that although $A^1_\mu = A^2_\mu = A^3_\mu$, generically, $\delta A^1_\mu \neq \delta A^2_\mu \neq \delta A^3_\mu$, and similarly for scalars

$$h_{tt} = -g_{tt}(u) e^{-i\omega t + iqz} H_{tt}(u)$$
$$h_{zz} = g_{zz}(u) e^{-i\omega t + iqz} H_{zz}(u)$$
$$h_{xx} = \frac{1}{2} g_{zz}(u) e^{-i\omega t + iqz} H_{xx}(u)$$
$$h_{yy} = \frac{1}{2} g_{zz}(u) e^{-i\omega t + iqz} H_{yy}(u)$$
$$h_{tz} = g_{zz} e^{-i\omega t + iqz} H_{tz}(u)$$

$$\delta A^i_z = \pi T_0 \sqrt{2} \left(\prod_{i=1}^{3} (1+\kappa_i)^{1/2} \right) e^{-i\omega t + iqz} a^i_z(u) \,,$$
$$\delta A^i_t = \pi T_0 \sqrt{2} \left(\prod_{i=1}^{3} (1+\kappa_i)^{1/2} \right) e^{-i\omega t + iqz} a^i_t(u).$$

$$E^i_z = \mathfrak{w}_0 \, a^i_z + \mathfrak{q}_0 \, a^i_t$$

$$\delta H_i = e^{-i\omega t + iqz} \cdot s_i(u)$$

Generically, coupled fluctuations of the metric, 3 gauge fields and 3 scalars

Dynamic instability
in the case of three chemical potentials (continued)

Using the eigenvectors of the Hessian, we can re-arrange the fluctuations as

$$E_z^i = E_z^{\mathrm{CM}} + \mathfrak{E}_z^i \qquad\qquad s_i = s_{\mathrm{CM}} + \mathfrak{s}_i$$

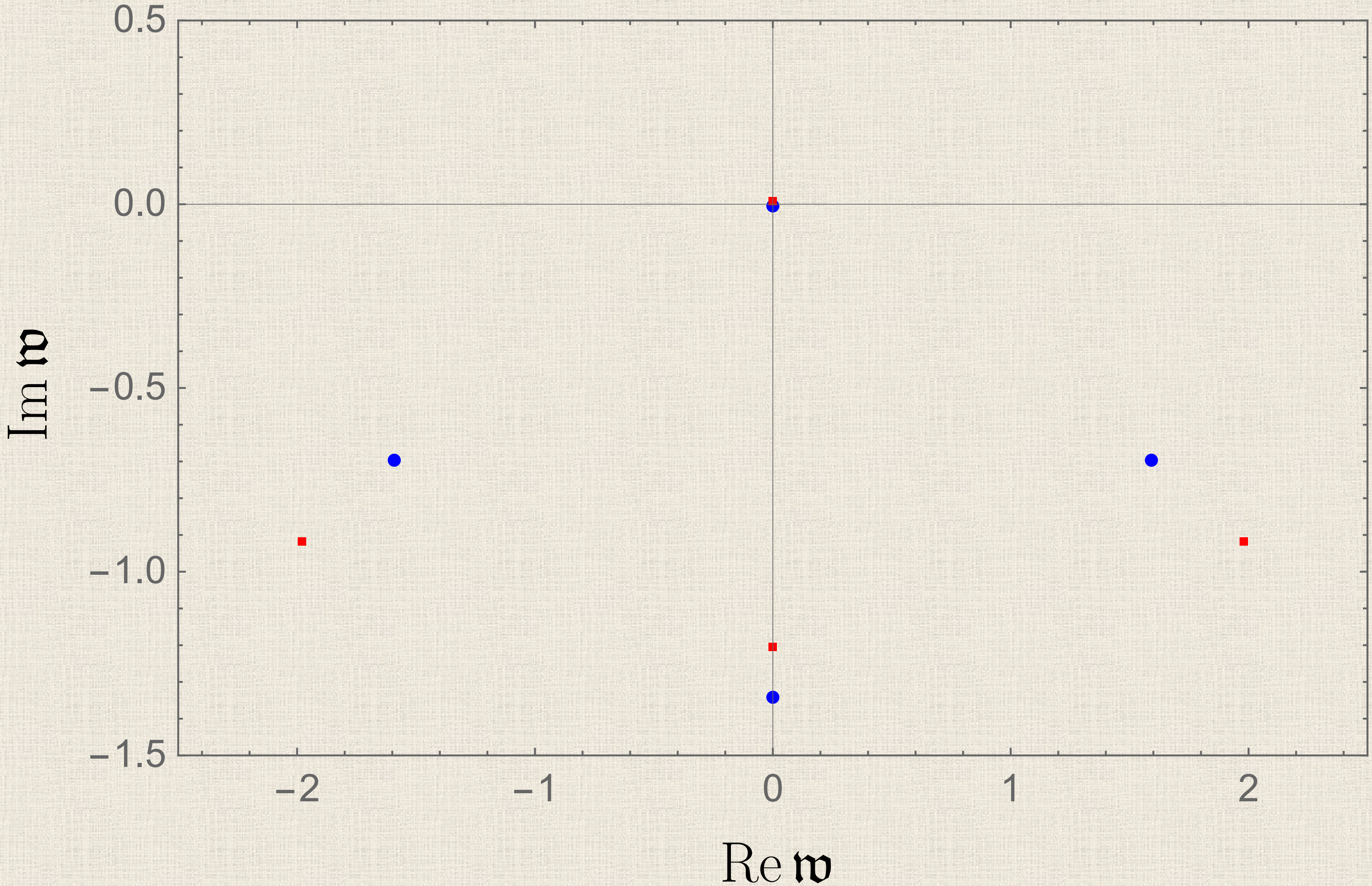
where the “center of mass” variables are defined as

$$E_z^{\mathrm{CM}} = \left(E_z^1 + E_z^2 + E_z^3\right)/3 \qquad\qquad s_{\mathrm{CM}} = \left(s_1 + s_2 + s_3\right)/3$$

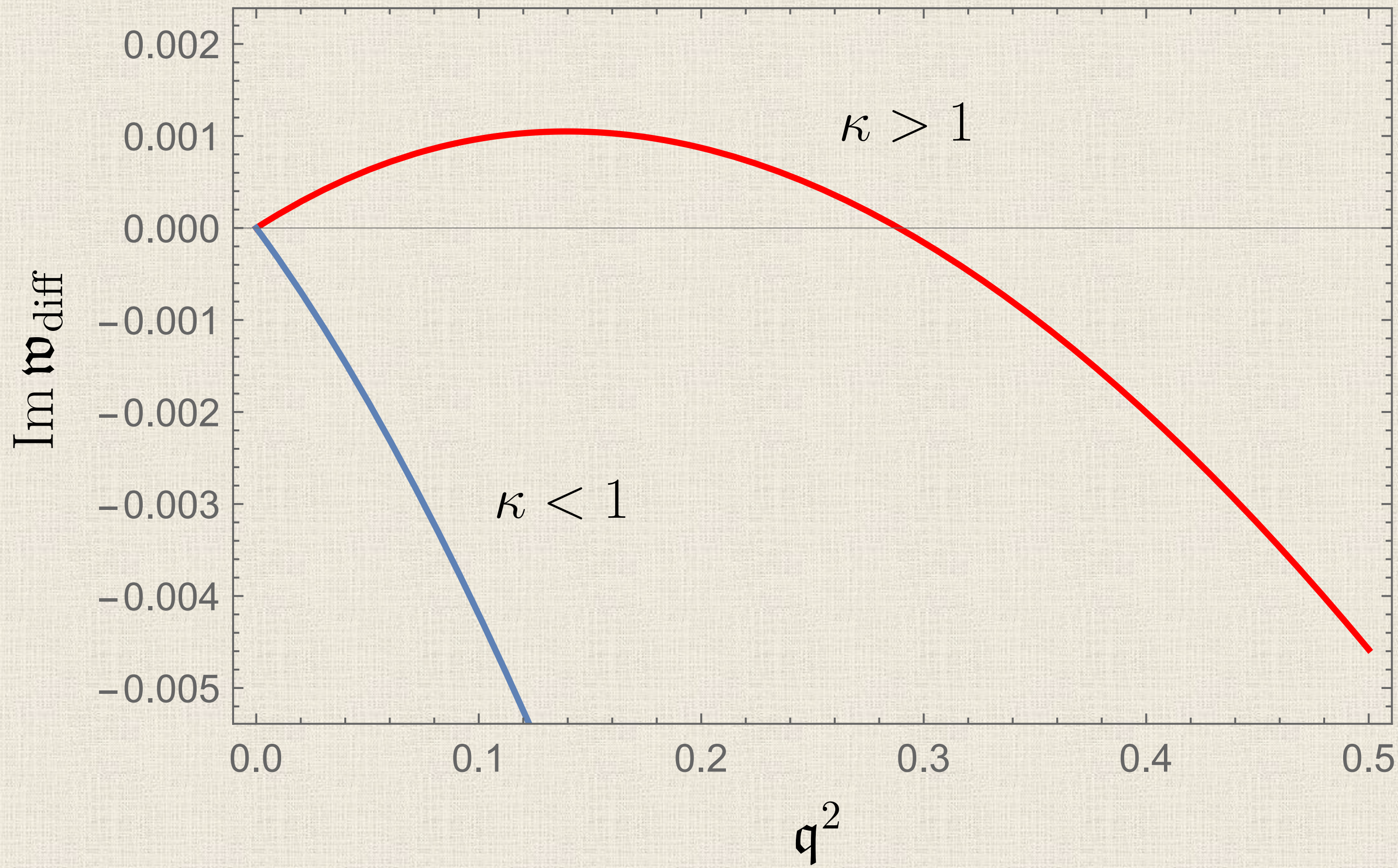
The equations for the new variables decouple:

$$\mathfrak{E}_z^{i''} + \mathfrak{D}^{-1}\left[\mathfrak{w}_0^2(1+\kappa u)^3\left(\frac{f'}{f} - \frac{\kappa}{1+\kappa u}\right) - \frac{2\kappa\cdot\mathfrak{q}_0^2 f}{1+\kappa u}\right]\cdot\mathfrak{E}_z^{i'} + \frac{\mathfrak{D}}{uf^2}\cdot\mathfrak{E}_z^i$$
$$+\frac{2\sqrt{\kappa}\,\mathfrak{q}_0}{(1+\kappa u)^3}\cdot\mathfrak{s}_i' + \frac{2\kappa^{1/2}\mathfrak{q}_0}{\mathfrak{D}}\cdot\left[\mathfrak{w}_0^2\left(\frac{f'}{f} - \frac{4\kappa}{1+\kappa u}\right) + \frac{\kappa\mathfrak{q}_0^2\cdot f}{(1+\kappa u)^4}\right]\mathfrak{s}_i = 0$$

$$\mathfrak{s}_i'' + \left(\frac{f'}{f} - \frac{1+3\kappa u}{u(1+\kappa u)}\right)\cdot\mathfrak{s}_i' + \left[\frac{\mathfrak{D}}{uf^2} + \frac{1+\kappa u}{u^2 f} + \frac{2\kappa(1+\kappa)^3 u}{(1+\kappa u)^2 f}\right.$$
$$\left.-\frac{\kappa}{1+\kappa u}\left(\frac{f'}{f} - \frac{1+3\kappa u}{u(1+\kappa u)}\right) - \frac{4\kappa(1+\kappa)^3(1+\kappa u)u\cdot\mathfrak{w}_0^2}{\mathfrak{D}\cdot f}\right]\cdot\mathfrak{s}_i$$
$$-\frac{2\kappa^{1/2}(1+\kappa)^3(1+\kappa u)u\cdot\mathfrak{q}_0}{\mathfrak{D}}\cdot\mathfrak{E}_z^{i'} = 0$$



The spectrum of linear fluctuations in the complex frequency plane for k<1 (blue) and k>1 (red)



$$\mathfrak{w}_{\mathrm{diff}} = -iD(\kappa)\mathfrak{q}^2 + \cdots$$

$$D(\kappa) = C(1-\kappa)\,, \quad C \approx 0.25 \quad \text{for } \kappa \sim 1$$

In fact, we can compute the diffusion coefficients analytically:

Single chemical potential state:

$$(\kappa, 0, 0)$$

$$D_R = \frac{1}{8\pi T}\frac{(2-\kappa)(2+\kappa)}{1+\kappa}$$

Two equal chemical potentials state:

$$(\kappa, \kappa, 0)$$

$$D_R = \frac{1}{2\pi T}\frac{1-\kappa}{1+\kappa}$$

Three equal chemical potentials state:

$$(\kappa, \kappa, \kappa)$$

$$D_R = \frac{(2-\kappa)(1-\kappa)}{4\pi T(1+\kappa)}$$

In all cases, the change of sign in the diffusion coefficient comes from the matrix of susceptibility (i.e. thermodynamic instability implies hydro instability)

Conclusions

We have constructed relativistic fluid dynamics with multiple charges, including relevant dispersion relations for quasiparticle excitations

We have explicitly demonstrated that due to the instability we identified, the low-temperature equilibrium state of strongly coupled N=4 SYM theory is not described by a dual AdS-Reissner-Nordström (RN-AdS) black hole, contrary to a widely held belief (see also Minwalla et al, 2024).

This finding challenges the common paradigm of using the RN-AdS solution as a benchmark in holographic models of low-temperature condensed matter systems.

We show how predictions from fluid dynamics with multiple charges help establish an explicit connection between thermodynamic and dynamic instability in a quantum field theory with a gravity dual.

THANK YOU!