

Mass and isospin breaking effects in the Witten-Sakai-Sugimoto model

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2025年7月15日：国科大国际研讨会

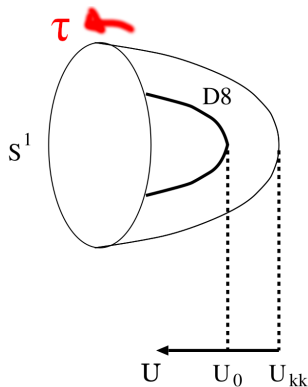
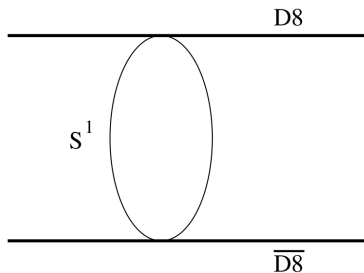
Sakai-Sugimoto – AdS/QCD

The Sakai-Sugimoto (SS) model [[Prog.Theor.Phys.113, 843 \(2005\)](#)] builds on the work of Witten [[Adv.Theor.Math.Phys.2, 253 \(1998\)](#)] where N_f D8- and $\overline{\text{D8}}$ -branes are intersecting N_c D4-branes

	0	1	2	3	(4)	5	6	7	8	9
D4	○	○	○	○	○					
D8- $\overline{\text{D8}}$	○	○	○	○		○	○	○	○	○

- The SS model: type IIA string theory
- SUSY-breaking by anti-periodic boundary conditions on the S^1 for the fermions on D4
- Chiral symmetry is explicit by the two 8-branes, when they stretch
- Chiral symmetry breaking is string geometric as the 8-branes touch and merge – the low-energy supergravity geometry is that of a cigar-shaped space, which is AdS_5 -like

Sakai-Sugimoto – chiral symmetry breaking



[Sakai-Sugimoto, Prog.Theor.Phys.113, 843 (2005)]

- Notice that the confined geometry ends at U_{KK}

Sakai-Sugimoto – AdS/QCD

- The 't Hooft limit is considered $N_c \gg N_f$, so that the 8-branes can be considered in the probe branes embedded in the D4-background (color d.o.f.)

$$ds^2 = \left(\frac{u}{R}\right)^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu + f(u) d\tau^2) + \left(\frac{R}{u}\right)^{3/2} \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right)$$

$$e^\phi = g_s \left(\frac{u}{R}\right)^{3/4}, \quad F_4 = dC_3 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \frac{u_{KK}^3}{u^3}.$$

- The flavor d.o.f. are described by the DBI action and the Chern-Simons term at level N_c – both scale as N_c
- The leading order approximation to the DBI action is the 5-dimensional Yang-Mills term, hence the Sakai-Sugimoto model:

$$S = -\kappa \operatorname{Tr} \int_{\text{AdS}_5} \mathcal{F} \wedge * \mathcal{F} + N_c \int_{\text{AdS}_5} \omega_5,$$

with YM coefficient $\kappa = \frac{\lambda N_c}{216\pi^3}$, and 't Hooft coupling $\lambda = g_{\text{YM}}^2 N_c$ (fixed),

$$g = h(z)k(z)dx^\mu dx_\mu + h^2(z)dz^2, \quad k(z) = h^{-3}(z) = 1 + z^2.$$

- The U(2) gauge fields is split as $\mathcal{A}_\alpha = A_\alpha^a T^a + \hat{A}_\alpha \frac{1}{2}$

Sakai-Sugimoto – Mesons

- The flavor fields are expanded as

$$\mathcal{A}_\mu = \sum_n v_\mu^{2n-1}(x) \psi_{2n-1}(z) + \sum_n a_\mu^{2n}(x) \psi_{2n}(z),$$
$$\mathcal{A}_z = \Pi(x) \phi_0(z) + \sum_n H^n(x) \phi_n(z),$$

with profile functions

$$\text{pions : } \phi_0(z) = \frac{1}{\sqrt{\pi\kappa}} \frac{1}{k(z)}, \quad \text{vectors : } -h^{-1}(z) \partial_z(k(z) \partial_z \psi_n) = \lambda_n \psi_n,$$
$$\phi_n(z) = \frac{1}{\sqrt{\lambda_n}} \partial_z \psi_n(z),$$

with vector meson masses $M_n \sim \sqrt{\lambda_n}$

- Fitting the pion decay constant and the rho meson mass, one obtains

$$M_{\text{KK}} = 949 \text{ MeV}, \quad \lambda = 16.63$$

Sakai-Sugimoto – The Skyrme model

- If we truncate to the pions, one gets the Skyrme model

$$S = \tilde{T}(2\pi\alpha')^2 \int d^4x \operatorname{Tr} \left[AL_\mu^2 + B[L_\mu, L_\nu]^2 \right],$$

with the left-invariant chiral current $L_\mu = U^{-1}\partial_\mu U$, and the constants determined by string theory

$$A = \frac{9u_{\text{KK}}}{4\pi}, \quad B = \frac{R^3 b}{2\pi^4}, \quad b \sim 15.25$$

- The Skyrme coupling is determined by the model

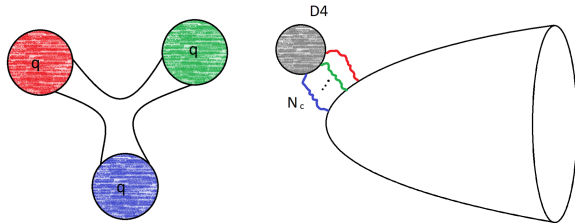
$$e^2 = \frac{27\pi^7}{2b} \frac{1}{\lambda N_c} \sim (7.32 \dots)^2$$

which can be compared to [Adkins-Nappi-Witten, NPB228, 552 (1983)], where they find $e = 5.45$ by fitting to the masses of the nucleon and Delta resonance

Sakai-Sugimoto – The baryon

The baryon is:

- the coupling of N_c strings from the D4-branes to the 8-branes



- the instanton in an (x^1, x^2, x^3, z) slice of the AdS_5 -like geometry
- the Skyrminion in the pion effective theory, which is a soliton of 1 dimension less than the instanton, but same S^3 target space

$$\pi_3(S^3) = \mathbb{Z} \ni k = B$$

- The 3rd homotopy group is due to the mappings being from $\sim \partial\mathbb{R}^4 \simeq S^3$ in the instanton case, and from $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$ in the Skyrminion case
- The precise mathematical relation is given by the instanton holonomy of [Atiyah-Manton, PLB, 438 (1989)]

The instanton in SS

- In the large- λ limit, the curved-space instanton is well approximated by the flat-space BPST instanton solution in the non-Abelian fields
[Hata-Sakai-Sugimoto-Yamato, Prog.Theor.Phys.117, 1157 (2007)]

$$A_M = -i f(\xi) g \partial_M g^{-1}, \quad f(\xi) = \frac{\xi^2}{\xi^2 + \rho^2},$$
$$\xi^2 = (x - X)^2 + (z - Z)^2, \quad g(x) = \frac{(z - Z) - i(\mathbf{x} - \mathbf{X}) \cdot \boldsymbol{\tau}}{\xi}.$$

- The Abelian electric field is new, this field acts as a size stabilization against gravitational collapse

$$\hat{A}_0 = \frac{N_c}{8\pi^2\kappa} \frac{1}{\xi^2} \left(1 - \frac{\rho^4}{(\xi^2 + \rho^2)^2} \right).$$

- Minimization of the pseudo-moduli Z, ρ gives

$$Z = 0, \quad \rho^2 = \frac{N_c}{8\pi^2\kappa} \sqrt{\frac{6}{5}}.$$

- Rotation of the instanton gives rise to spin and isospin quantum numbers

Quark and pion masses in SS

- The issue with the quark mass in SS, is that it involves both the left- and right-handed fermions

$$m_q(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L),$$

which must be nonlocal in the bulk!

- [Aharony-Kutasov, PRD78, 026005 (2008)] solved this problem with a Wilson line

$$S_{AK} = c \int d^4x \text{Tr} P \left[M \left(e^{i\varphi} + e^{-i\varphi} - 2\mathbb{1} \right) \right], \quad \varphi = - \int dz A_z,$$

- It can be interpreted also as an effect from world-sheet instantons
- Inclusion of the AK action (quark masses) deforms the size of the instanton and the mass of the instanton
- It also induces a pion mass (k_0 of ϕ_0 becomes nonvanishing) and in turn yields the Gell-Man-Oakes-Renner relation

$$4m_q c = f_\pi^2 m_\pi^2.$$

i.e. the pion mass squared is proportional to the quark mass

Chiral symmetry breaking induced by the Yukawas

- We will introduce the chiral symmetry breaking along the lines of the Standard Model: i.e. by the mass term (Yukawa interactions):

$$S_{\text{AK}} = c \int d^4x \operatorname{Tr} \left[M \left(e^{i\varphi} + e^{-i\varphi} - 2\mathbb{1} \right) \right], \quad e^{i\varphi} = \mathcal{P} \exp \left(\frac{i}{2} \int dz A_z \right),$$

$$M = m_q \mathbb{1}_2 + \epsilon m_q \tau^3, \quad m_q = \frac{1}{2}(m_u + m_d), \quad \epsilon = \frac{m_u - m_d}{m_u + m_d}.$$

- Phenomenologically $m_q \approx 3 \text{ MeV}$ and $\epsilon \approx -0.36$.

Quantum baryons

- Turning on isospin moduli:

$$\hat{A}_z = -\frac{N_c}{16\pi^2\kappa} \frac{\rho^2}{(\xi^2 + \rho^2)^2} \chi \cdot \mathbf{x}, \quad \hat{A}_i = -\frac{N_c}{16\pi^2\kappa} \frac{\rho^2}{(\xi^2 + \rho^2)^2} \epsilon^{iab} \chi^a x^b.$$

- Protons: $|p \uparrow\rangle \propto (a_1 + ia_2)$ and $|p \downarrow\rangle \propto (a_4 - ia_3)$
- Neutrons: $|n \uparrow\rangle \propto (a_4 + ia_3)$ and $|n \downarrow\rangle \propto (a_1 - ia_2)$
- Angular velocity: $\chi^a = -i \text{Tr}(A^\dagger \dot{A} \tau^a)$ with $A = \mathbb{1}_2 a_4 + ia_i \tau^i$.
- Matching linearized EOM with BPST core [[Hashimoto-Sakai-Sugimoto, Prog.Theor.Phys.120, 1093 \(2008\)](#)]

Isospin breaking: modified tails of the soliton

- Linearized EOMs:

$$\begin{aligned}\kappa k(z) \left(\partial_i^2 \hat{A}_z - \partial_i \partial_z \hat{A}_i \right) &= m_q c \left[\int_{-\infty}^{+\infty} dz \hat{A}_z + \epsilon \int_{-\infty}^{+\infty} dz A_z^{a=3} \right], \\ \kappa k(z) \left(\partial_i^2 A_z^{a=3} - \partial_i \partial_z A_i^{a=3} \right) &= m_q c \left[\int_{-\infty}^{+\infty} dz A_z^{a=3} + \epsilon \int_{-\infty}^{+\infty} dz \hat{A}_z \right].\end{aligned}$$

- Diagonalize into the mass basis (π and η):

$$A_\eta = \frac{1}{\sqrt{2}} (A_z^{a=3} + \hat{A}_z), \quad A_\pi = \frac{1}{\sqrt{2}} (A_z^{a=3} - \hat{A}_z).$$

- Solutions:

$$A_\eta = b \chi^j \partial_{X^j} H_+ - \tilde{b} \text{Tr} \left(A \tau^j A^\dagger \tau^3 \right) \partial_{X^j} H_+, \quad A_\pi = d \chi^j \partial_{X^j} H_- - \tilde{d} \text{Tr} \left(A \tau^j A^\dagger \tau^3 \right) \partial_{X^j} H_-.$$

$$b = -d, \quad \tilde{b} = \tilde{d}, \quad b = \frac{\rho^2 N_c}{\sqrt{2} 8\kappa}, \quad \tilde{b} = \sqrt{2} \pi^2 \rho^2.$$

- Modified Green's functions:

$$H_\pm \equiv -\kappa \sum_{n=0} \phi_n(z) \phi_n(Z) \frac{1}{4\pi} \frac{e^{-\sqrt{\lambda_n} r}}{r}, \quad \lambda_0 = m_\pm^2, \quad m_\pm^2 = m^2 (1 \pm \epsilon).$$

Isospin breaking: modified inertia

- Inserting the solutions into the Yang-Mills:

$$T = \frac{\pi^2 \rho^4 \kappa}{2} m \left[\left(\frac{8}{3mR} - \left(2 + \sqrt{1+\epsilon} + \sqrt{1-\epsilon} \right) \right) \chi^2 - \left(2 - \sqrt{1+\epsilon} - \sqrt{1-\epsilon} \right) \chi_\zeta^2 \right].$$

- Inertia tensor:

$$\mathcal{I} = I_0 \mathbb{1}_3 + \delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \delta = \frac{\pi^2 \rho^4 \kappa}{12} m \epsilon^2 + \mathcal{O}(m^2, \epsilon^4).$$

- I_0 requires numerical computation!
- δ is computed analytically to leading order in ϵ !

Isospin breaking: new quantization

- The AK term contains $\mathcal{O}(\epsilon)$ modification of the angular momentum ($\mathcal{J}_2 \approx 1.054$):

$$\begin{aligned} L_{\text{AK}} &= c \operatorname{Tr} \int d^3x \left[M e^{i\varphi} A U A^\dagger + e^{-i\varphi} A U^\dagger A^\dagger M - 2\mathbb{1}_2 \right] \\ &= \frac{m_q c \epsilon N_c}{12\kappa} \rho^3 \mathcal{J}_2 \chi^i R_{3i}, \end{aligned}$$

- The usual angular momentum gets modified:

$$J_i = \mathcal{I}_{ij} \chi_j \quad \longrightarrow \quad J_i = \mathcal{I}_{ij} \chi_j - K_i,$$

- The AK shift

$$K_i = -\frac{\epsilon m^2 N_c}{12\pi} \rho^3 R_{3i} \mathcal{J}_2.$$

- Hamiltonian

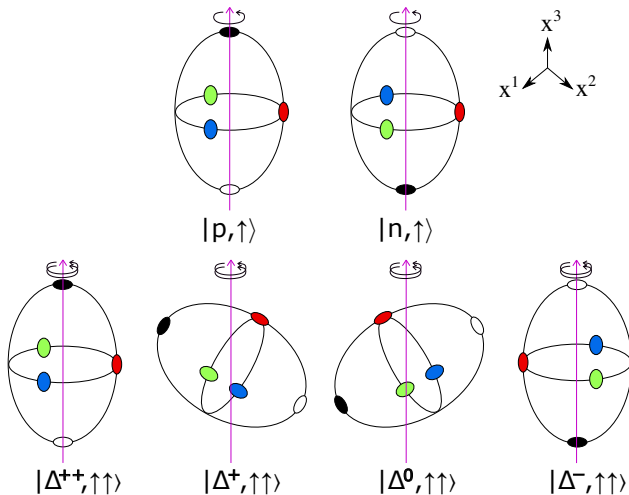
$$H_{\text{full}} = \frac{1}{2I_A} \left(J_\xi^2 + J_\eta^2 \right) + \frac{1}{2I_C} \left(J_\zeta^2 + K_\zeta^2 + 2J_\zeta K_\zeta \right) + M_0.$$

- Quantized energy

$$E_{\text{full}} = \underbrace{\frac{1}{2I_A} j(j+1)}_{\text{ANW}} + \underbrace{\frac{1}{2} \left(\frac{1}{I_C} - \frac{1}{I_A} \right) i_3^2}_{\text{BBGR}} - \underbrace{\frac{1}{I_0} K_\zeta i_3}_{\text{Bigazzi-Niro}} + \frac{1}{2I_0} K_\zeta^2 + M_0,$$

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Isospin breaking: new quantization



Isospin breaking: proton-neutron splitting

- Proton-neutron mass splitting [Bigazzi-Niro, PRD98 (2018), 046004]

$$M_p - M_n = \frac{\epsilon m^2 N_c}{48\pi^3 \kappa} \rho \mathcal{J}_2 \simeq -4.74 \text{ MeV}.$$

- PDG value

$$(M_p - M_n)^{\text{PDG}} = -1.293 \text{ MeV}$$

- Baryon observables always **overestimated** by meson-fitted parameters in large- N_c theories.
- It's better to compare ratios:

$$2 \frac{M_p - M_n}{M_p + M_n} \simeq -2.5 \times 10^{-3},$$

- PDG:

$$\left(2 \frac{M_p - M_n}{M_p + M_n} \right)^{\text{PDG}} \simeq -1.4 \times 10^{-3}.$$

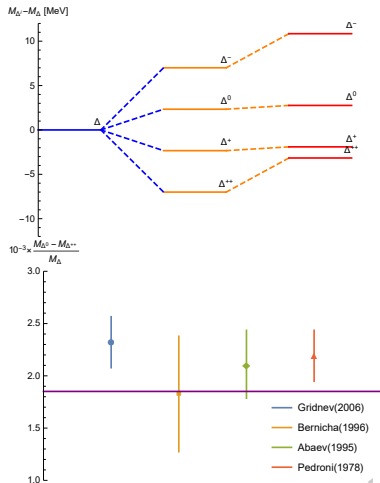
Isospin breaking: application to the Δ s

- Splitting of Δ s

$$\frac{1}{2}(M_{\Delta^{++}} - M_{\Delta^+} - M_{\Delta^0} + M_{\Delta^-}) = \frac{3\delta}{(I + \delta)(I - 2\delta)} \simeq \frac{27\pi}{8} \frac{m\epsilon^2}{N_c\lambda} \approx 3.41 \text{ MeV}.$$

- Numerical values of the splitting

$$\begin{aligned} M_{\Delta^{++}} - M_{\Delta^+} &\simeq \frac{3\delta}{I_0^2} + \frac{\epsilon m^2 N_c}{48\pi^3 \kappa} \rho \mathcal{J}_2 \\ &\approx -1.25 \text{ MeV}, \\ M_{\Delta^+} - M_{\Delta^0} &\simeq \frac{\epsilon m^2 N_c}{48\pi^3 \kappa} \rho \mathcal{J}_2 \\ &\approx -4.67 \text{ MeV}, \\ M_{\Delta^0} - M_{\Delta^-} &\simeq -\frac{3\delta}{I_0^2} + \frac{\epsilon m^2 N_c}{48\pi^3 \kappa} \rho \mathcal{J}_2 \\ &\approx -8.08 \text{ MeV}. \end{aligned}$$



It's hard to calculate I_0 in the WSS (3-dimensional PDE)



we'll do it in the Skyrme model

Detour – the Skyrme model: small pion mass expansion

- The linear tail:

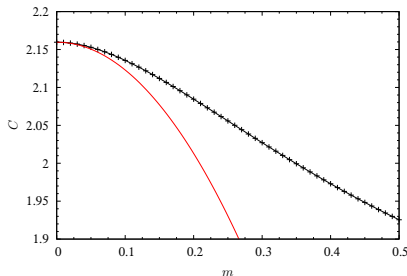
$$f'' + \frac{2f'}{r} - \frac{2f}{r^2} - m^2 f = 0, \quad f(r) = C(m) \frac{mr + 1}{r^2} e^{-mr}, \quad r \gg 1.$$

- Related to $g_{NN\pi}$:

$$g_{NN\pi} = \frac{8\pi M_{\text{Sk}}}{f_\pi e^2} C(m).$$

- $C(m)$'s m -dependence:

$$C(m) = C(0) - 2m^2 C(0)^2 \left(\beta - \frac{1}{4} \right) + \mathcal{O}(m^3), \quad C(0) = R(0)^2.$$



Detour – the Skyrme model: splitting

- Mass matrix for π and η :

$$M^2 = m^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \epsilon \\ 0 & 0 & \epsilon & 1 \end{pmatrix} \sim m^2 \text{diag}(1, 1, 1 + \epsilon, 1 - \epsilon).$$

- Tails:

$$\begin{aligned} \pi^1 &= C \frac{mr + 1}{r^2} e^{-mr} \hat{x}^1, & \pi^2 &= C \frac{mr + 1}{r^2} e^{-mr} \hat{x}^2, \\ \pi^3 &= \frac{D}{2} \frac{m\sqrt{1+\epsilon}r + 1}{r^2} e^{-m\sqrt{1+\epsilon}r} \hat{x}^3 + \frac{E}{2} \frac{m\sqrt{1-\epsilon}r + 1}{r^2} e^{-m\sqrt{1-\epsilon}r} \hat{x}^3, \\ \eta &= \frac{D}{2} \frac{m\sqrt{1+\epsilon}r + 1}{r^2} e^{-m\sqrt{1+\epsilon}r} \hat{x}^3 - \frac{E}{2} \frac{m\sqrt{1-\epsilon}r + 1}{r^2} e^{-m\sqrt{1-\epsilon}r} \hat{x}^3, \end{aligned}$$

- Moments of inertia:

$$\begin{aligned} I &= \frac{4\pi}{9} \frac{D^2 + E^2 + 4C^2}{R_{UV}} - \frac{2\pi}{3} \left(\sqrt{1+\epsilon} D^2 + \sqrt{1-\epsilon} E^2 + 4C^2 \right) m + \mathcal{O}(m^2, m^2\epsilon), \\ \delta &= \frac{2\pi}{9} \frac{D^2 + E^2 - 2C^2}{R_{UV}} - \frac{\pi}{3} \left(\sqrt{1+\epsilon} D^2 + \sqrt{1-\epsilon} E^2 - 2C^2 \right) m + \mathcal{O}(m^2, m^2\epsilon). \end{aligned}$$

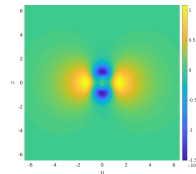
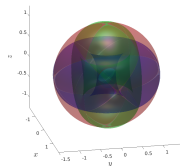
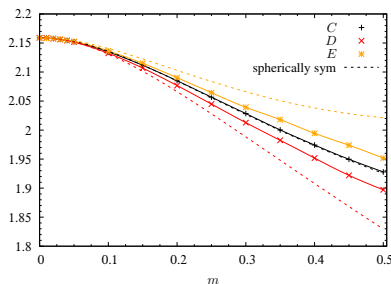
Detour – the Skyrme model: splitting

- Leading order (with $C = D = E$):

$$I(m) = I(0) - 4\pi C(0)^2 \left(1 - \frac{\epsilon^2}{24}\right) m + \mathcal{O}(m^2, m^2\epsilon, m\epsilon^4),$$

$$\delta = \frac{\pi}{12} C(0)^2 \epsilon^2 m + \mathcal{O}(m^2, m^2\epsilon, m\epsilon^4).$$

- Numerical computation in the Skyrme model:

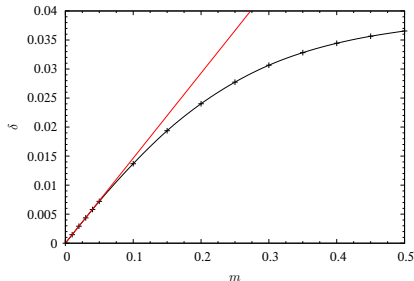
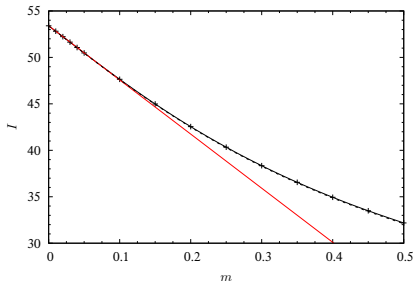


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$$\delta = \frac{\pi}{12} C(0)^2 \epsilon^2 m + \mathcal{O}(m^2, m^2\epsilon, m\epsilon^4).$$

- Expansion vs. full numerics



Conclusions

- The mechanism of mass splitting the Yukawa way is complicated and requires 4 conditions:
 - ① Mass-splitting in the quark/pion mass term
 - ② Including the η (extending SU(2) to U(2))
 - ③ Chern-Simons (or in 4D Wess-Zumino) term
 - ④ Vectors
- Easy to calculate:
 - In the Skyrme model: the zeroth-order moment of inertia (I_0)
 - In holography (WSS): the leading-in- ϵ splitting of the moment of inertia
- Myriads of possibilities for improvement:
 - Full-PDE computation of the baryon with core and tail $\Rightarrow I_0$
 - Higher-order $\epsilon, N_c^{-1}, \lambda^{-1}, \dots$ corrections
 - 3 flavors (U(3))
 - Non-antipodal branes?
 - Gravitational backreaction
 - ...

谢谢大家！！

有问题吗？