ϕ^4 Loops in AdS_3/CFT_2

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Acknowledgement

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- This work is also thanks to sponsorship by the China Scholarship Council.

AdS/CFT, but without conjecture

- Unlike most other researches in AdS/CFT, we start instead by putting a ϕ^4 theory in fixed Euclidean AdS_3 background without any assumption on its dual.
- Ultilizing the fact that the orientation preserving isometry group in AdS_3 is the same as the conformal group in CFT_2, we can say without conjecture that when taken the boundary limit, the ϕ^4 theory gives rise to a conformal field theory.
- The goal is then to study this dual conformal field theory.
- A similar work has been carried out in AdS_4/CFT_3 in [Bertan, Sachs, Skvortsov, 1810.00907].

Theory Setup

We study a simple ϕ^4 interactive theory in (E)AdS$_{d+1}$

$$S = \int_{\mathrm{AdS}_3} \sqrt{g} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right)$$

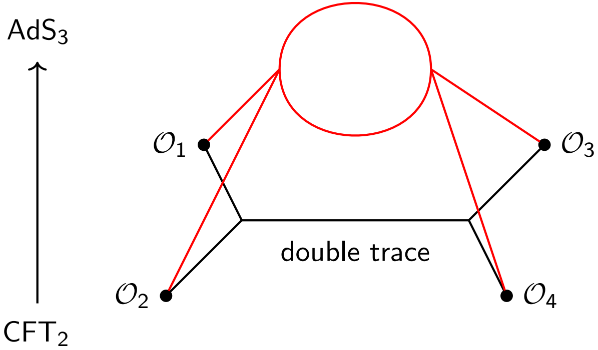
with the mass conformally coupled,

$$m^2 = \xi R = \frac{d^2-1}{4}a^2.$$

When taken the boundary limit, such field corresponds to a dual conformal operator with conformal weight

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - \frac{m^2}{a^2}} = 1 \pm \frac{1}{2} \text{ in AdS}_3.$$

Throughout this talk, we focus on the case where $\Delta = \frac{3}{2}$.



- We intend to study the spectrum of the double trace operator. For a free ϕ^4 theory, the double trace operator takes the form $\mathcal{O}_\Delta \Box^n \partial^l \mathcal{O}_\Delta$.
- With λ being turned on, the double trace operators start to pick up anomalous dimensions, with their total conformal weight being $2\Delta + 2n + l + \gamma^1_{n,l} + \gamma^2_{n,l}$ up to second order in λ. It is $\gamma^2_{n,l}$ that we are insterested in.

Motivation

Why doing this?

- To develop methodology for computing loop diagrams in curved spacetime and extracting information out of a dual conformal field theory.
- Cosmology: Under a Wick rotation, (E)AdS turns into dS. Amplitudes in AdS are then closely related to cosmological correlators under Schwinger-Keldysh formalism.

Why 3 dimensions?

- Unique simplicity and complexity: One loop diagram with $\Delta = \frac{3}{2}$ in AdS_3 is completely without UV or IR divergences. It sets the groundwork for more complicated computations. On the other hand, fractional conformal weights complexify the loop calculations.
- Links to 2D CFT such as Ising model.

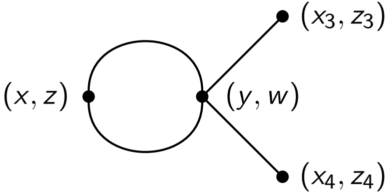
One Loop Computation

For a $\Delta = \frac{3}{2}$ scalar field in the bulk, its bulk to boundary propagator reads

$$\overline{\Lambda}(y, x_i) = (\overline{K}_{yx_i})^{\frac{3}{2}}$$

in which $\overline{K}_{yxi} = \frac{2w}{(y^i - x^i)^2 + w^2}$. After some Schwinger parametrizations and conformal transformations, the fish diagram is computed to be

$$\text{fish} \propto (\overline{K}_{xx_3}\overline{K}_{xx_4})\left(\text{EllipticE}[1 - (\overline{K}_{xx_3}\overline{K}_{xx_4})^{-1}] - (\overline{K}_{xx_3}\overline{K}_{xx_4})^{-\frac{1}{2}}\right)$$

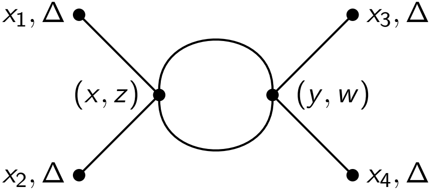


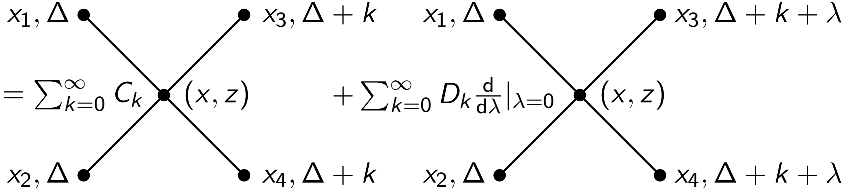
Elliptic Integral Expansion

A key observation is that the elliptic integral has a series expansion such that

$$\begin{aligned}\text{fish} \propto & (\overline{K}_{xx_3}\overline{K}_{xx_4})\left(\text{EllipticE}[1-(\overline{K}_{xx_3}\overline{K}_{xx_4})^{-1}]-(\overline{K}_{xx_3}\overline{K}_{xx_4})^{-\frac{1}{2}}\right)\\ &\propto \sum_{k=0}^{\infty}(C_k(\overline{K}_{xx_3}\overline{K}_{xx_4})^{\frac{3}{2}+k}+D_k(\overline{K}_{xx_3}\overline{K}_{xx_4})^{\frac{3}{2}+k}\log(\overline{K}_{xx_3}\overline{K}_{xx_4}))\\ &\propto \sum_{k=0}^{\infty}(C_k(\overline{K}_{xx_3}\overline{K}_{xx_4})^{k}+D_k\frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=0}((\overline{K}_{xx_3}\overline{K}_{xx_4})^{k+\lambda})).\end{aligned}$$

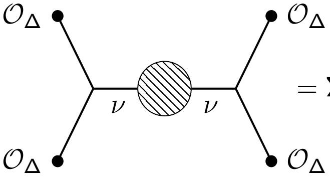
Recall that $\overline{\Lambda}(y, x_i) = (\overline{K}_{yx_i})^{\frac{3}{2}}$, we have



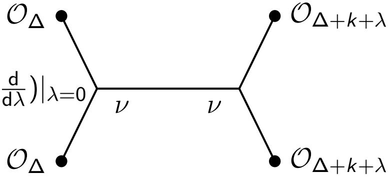


s-channel: Spectral Representation

In the s-channel, the result above allows us to read off the (Källén–Lehmann) spectral function [Sachs, Vanhove, 2303.03491]:



$$= \Sigma_{k=0}^{\infty}(C_k + D_k \tfrac{\mathrm{d}}{\mathrm{d}\lambda})|_{\lambda=0}$$



$$\int_0^\infty \mathrm{d}\nu\, b_{\Delta,\Delta,\nu} b_{\Delta,\Delta,\nu} B(\nu) \Psi^{\Delta,\Delta,\Delta,\Delta}_{\nu,(x_1,x_2,x_3,x_4)}$$

$$=\Sigma_{k=0}^{\infty}(C_k + D_k \frac{\mathrm{d}}{\mathrm{d}\lambda})|_{\lambda=0} \int_0^\infty \mathrm{d}\nu\, b_{\Delta,\Delta,\nu} b_{\Delta+k+\lambda,\Delta+k+\lambda,\nu} \Psi^{\Delta,\Delta,\Delta+k+\lambda,\Delta+k+\lambda}_{\nu,(x_1,x_2,x_3,x_4)}$$

in which $b_{\Delta,\Delta,\nu}$ and $b_{\Delta+k+\lambda,\Delta+k+\lambda,\nu}$ are known quantities.

The two conformal partial waves, $\Psi^{\Delta,\Delta,\Delta,\Delta}_{\nu,(x_1,x_2,x_3,x_4)}$ and $\Psi^{\Delta,\Delta,\Delta+k+\lambda,\Delta+k+\lambda}_{\nu,(x_1,x_2,x_3,x_4)}$ differ only by a coefficient. Hence we read off

$$B(\nu) = \Sigma_{k=0}^{\infty}(C_k + D_k \frac{\mathrm{d}}{\mathrm{d}\lambda})|_{\lambda=0} \left(\frac{b_{\Delta+k+\lambda,\Delta+k+\lambda,\nu}}{b_{\Delta,\Delta,\nu}} \right).$$

On the other hand, the spectral function has been decided in [Carmi, Di Pietro, Komatsu, 1810.04185] using conformal bootstrap to be

$$\mathcal{B}(\nu) = \frac{i\left(\psi[\Delta - \frac{1+i\nu}{2}] - \psi[\Delta - \frac{1-i\nu}{2}]\right)}{8\pi\nu}.$$

We have checked that the two spectral functions have the same Laurent series expansion in the -1, 0 and 1st order, i.e. they give rise to the same anomalous dimensions,

$$\gamma^s_{n,0} \propto \frac{1}{n^3}.$$

t- and u-channel

In the case of t-channel, for example, the equation for spectral function reads

$$\int_0^\infty \mathrm{d}\nu\, b_{\Delta,\Delta,\nu} b_{\Delta,\Delta,\nu} B(\nu) \Psi^{\Delta,\Delta,\Delta,\Delta}_{\nu,(x_1,x_2,x_3,x_4)}$$

$$=\Sigma_{k=0}^{\infty}(C_k + D_k \frac{\mathrm{d}}{\mathrm{d}\lambda})|_{\lambda=0} \int_0^\infty \mathrm{d}\nu\, b_{\Delta,\Delta+k+\lambda,\nu} b_{\Delta,\Delta+k+\lambda,\nu} \Psi^{\Delta,\Delta+k+\lambda,\Delta,\Delta+k+\lambda}_{\nu,(x_1,x_2,x_3,x_4)}.$$

However, the relation between $\Psi^{\Delta,\Delta,\Delta,\Delta}_{\nu,(x_1,x_2,x_3,x_4)}$ and $\Psi^{\Delta,\Delta+k+\lambda,\Delta,\Delta+k+\lambda}_{\nu,(x_1,x_2,x_3,x_4)}$ is now well known. Therefore we are unable to extract the spectral functions for t- and u-channel.

We have to resort to conformal block expansion.

Conformal Block Expansion

Conformal blocks form a basis of correlation functions. We compare correlation functions with conformal blocks to extract the anomalous dimensions.

Luckily, tree level correlation function (the cross diagram) of the ϕ^4 theory is well studied in [Dolan, Osborn, 0006098].

The $\log(u)$ term of t-channel one loop correlation function, expressed as a sum over cross diagrams, reads

$$\sum_{i,j,k=0}^{\infty} \frac{4\Gamma\left[\frac{3}{2}+i\right]\Gamma\left[\frac{3}{2}+i+j\right]\Gamma\left[\frac{3}{2}+i+k\right]\Gamma\left[\frac{3}{2}+i+j+k\right]\left(-2+(1+2k)\left(-H[k]+H\left[\frac{1}{2}+i+k\right]+H\left[\frac{1}{2}+i+j+k\right]-H[2+2i+j+k]\right)\right)}{(1+2k)^2\pi^2\Gamma[1+i]^2\Gamma[1+j]\Gamma[1+k]\Gamma[3+2i+j+k]} \log(u) u^i Y^j$$

in which

$$u = \frac{r_{12}^2 r_{34}^2}{r_{14}^2 r_{23}^2} \quad Y = 1 - \frac{r_{13}^2 r_{24}^2}{r_{14}^2 r_{23}^2}$$

are the conformal cross ratios.

Conjecture

- For any $i, j \in \mathbb{N}$,

$$X = \sum_{k=0}^{\infty} \frac{4\Gamma\left[\frac{3}{2}+i\right]\Gamma\left[\frac{3}{2}+i+j\right]\Gamma\left[\frac{3}{2}+i+k\right]\Gamma\left[\frac{3}{2}+i+j+k\right]\left(-2+(1+2k)\left(-H[k]+H\left[\frac{1}{2}+i+k\right]+H\left[\frac{1}{2}+i+j+k\right]-H[2+2i+j+k]\right)\right)}{(1+2k)^2\pi^2\Gamma[1+i]^2\Gamma[1+j]\Gamma[1+k]\Gamma[3+2i+j+k]}$$

  is rational.

- For any $i, j \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \frac{4\Gamma\left[\frac{3}{2}+i\right]\Gamma\left[\frac{3}{2}+i+j\right]\Gamma\left[\frac{3}{2}+i+k\right]\Gamma\left[\frac{3}{2}+i+j+k\right]\left(-2+(1+2k)\left(H[k]_H\left[\frac{1}{2}+i+k\right]+H\left[\frac{1}{2}+i+j+k\right]-H[2+2i+j+k]\right)\right)}{(1+2k)^2\pi^2\Gamma[1+i]^2\Gamma[1+j]\Gamma[1+k]\Gamma[3+2i+j+k]}$$

  can be uniquely written as $q_1 + \frac{q_2}{\pi}$, $q_1, q_2 \in \mathbb{Q}$, such that $X = q_1$.

- $\pi \to \infty$

Step1: Obatining the Correlation Function

- We have Mathematica perform the summation on a cluster computer for all $2i + j \leq 100$.
- Special thanks to the IT service of LMU Department of Physics.

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Step2: Obatining the Conformal Block

- We cite [Li, 1912.01168] and again compute the conformal block for all $2i + j \leq 100$.
- By comparing it with the correlation function, anomalous dimensions can be extracted recursively using code developed in [Heckelbacher, Sachs, 2009.06511].
- Anomalous dimensions for all $2n + l \leq 100$ have been successfully extracted for both t- and u-channel.

Fine Tuning?

An interesting observation is that anmoulamous dimensions in both the t- and u-channel diverges with n at very high order. At $n = 50$ for example:

$$\gamma^{t}_{50,0} \sim \gamma^{u}_{50,0} \sim -10^{192}$$

Their sum, however, converges to 0 as n increases.

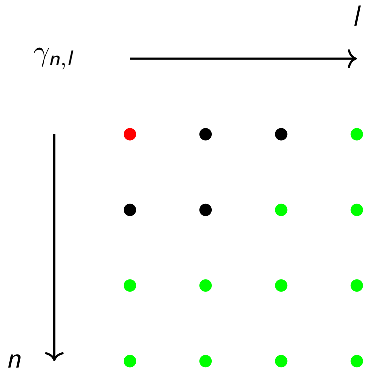
$$\gamma^{t}_{50,0} + \gamma^{u}_{50,0} \sim -0.002$$

This might have something to do with the 6-j symbol, but exact physical interpretation is not clear as of this time.

Step3: Guessing a recursive relation

- We applied a Mathematica package 'Guess' written by Dr.Manuel Kauers, as part of RISCErgoSum package developed by RISC, Johannes Kepler University of Linz.
- A recurrence relation was found by extrapolating anomalous dimensions with $0 \leq n \leq 20$, $0 \leq l \leq 40$.
- The recurrence relation then sucessfully reproduced all other anomalous dimensions with $2n + l \leq 100$. The computation time for a high order anomalous dimension was reduced from aroound 10000 hours to 1 second.
- A closed form has also been found for $n = 0$: $\gamma_{0,l}^{t+s} = -\frac{4}{(l+1)(2l+1)(2l+3)}$

Recurrence Relation: Tetris



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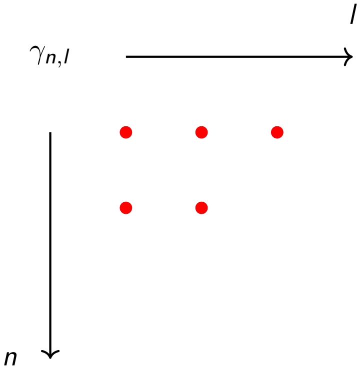

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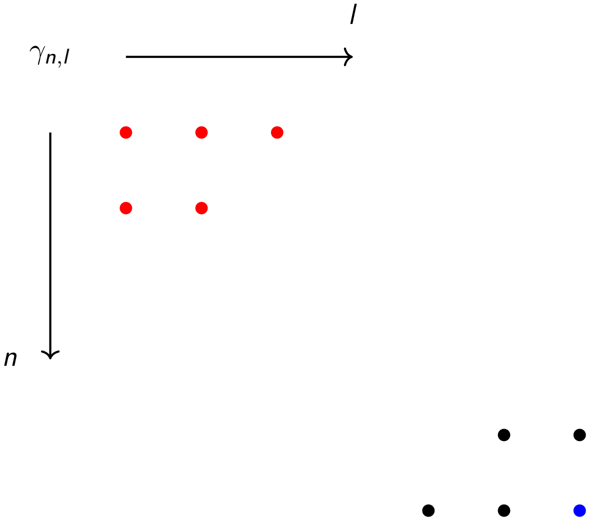

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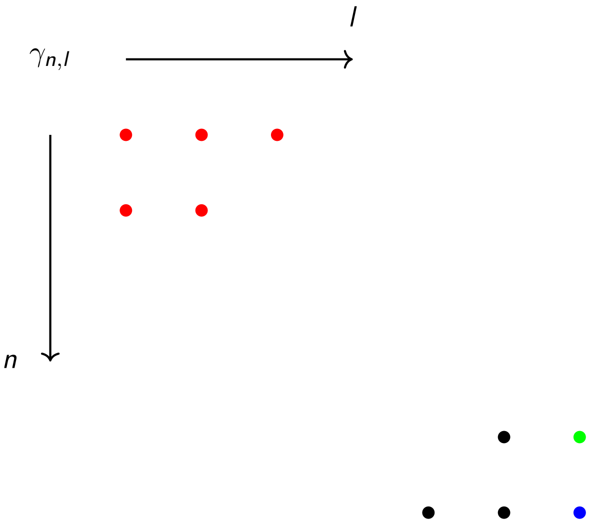

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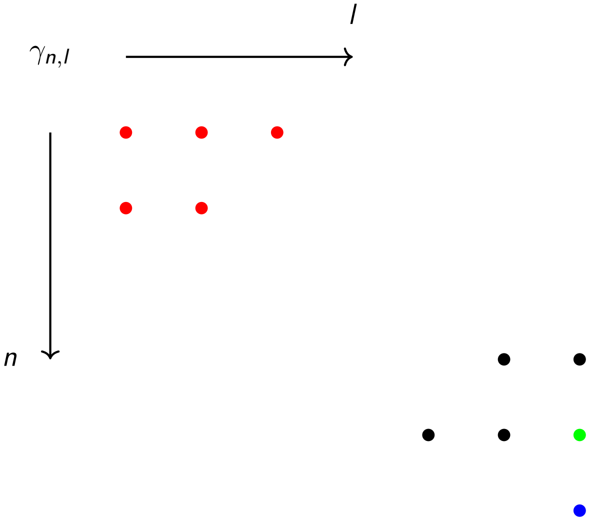

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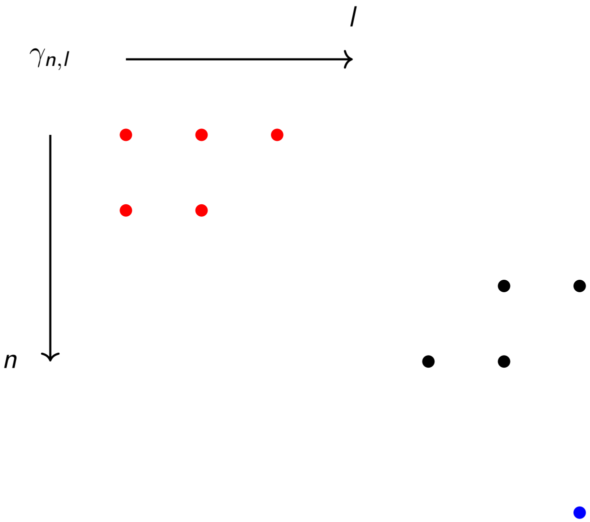

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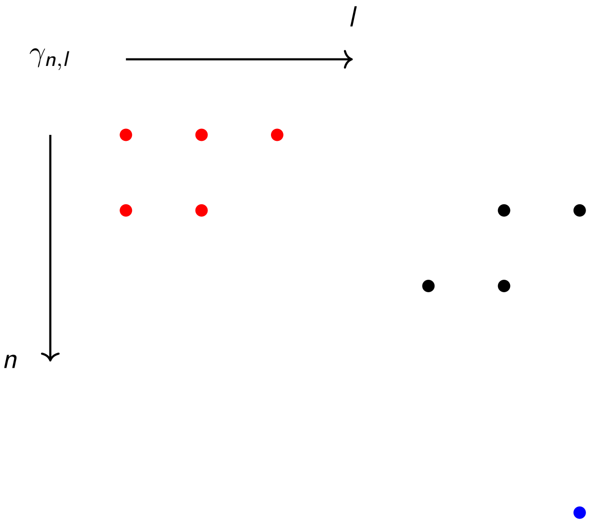

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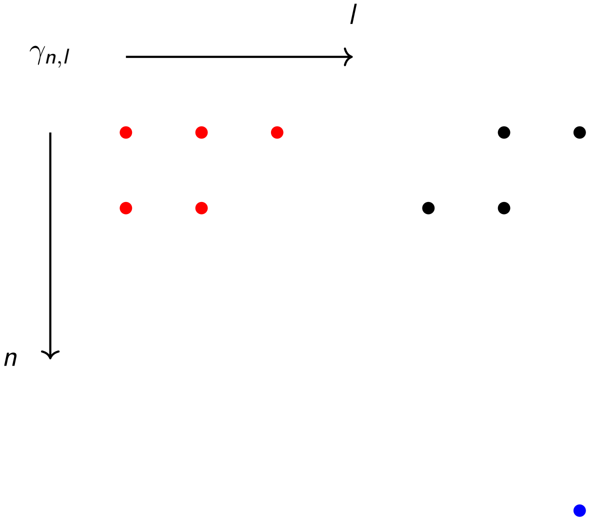

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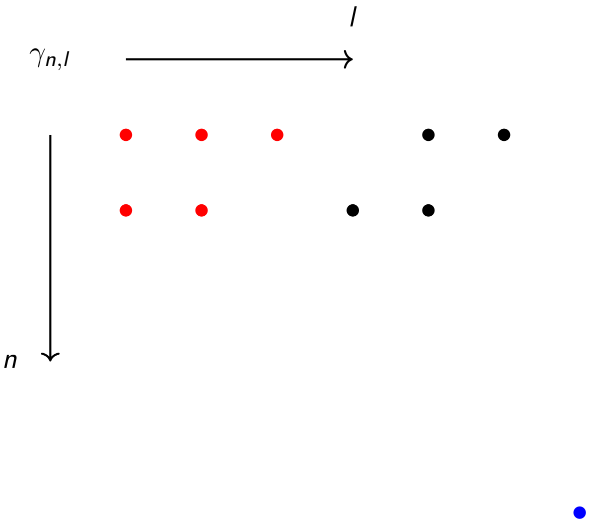
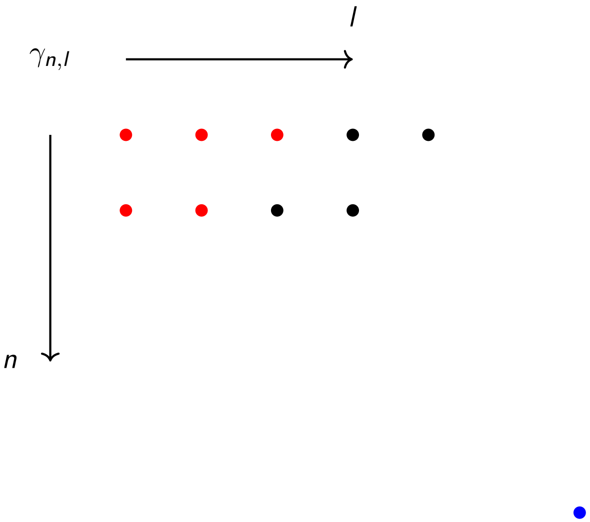
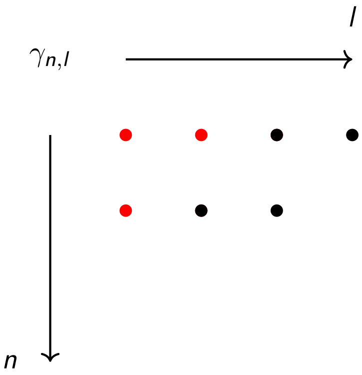



l
$\gamma_{n,l}$
n

Recurrence Relation: Tetris



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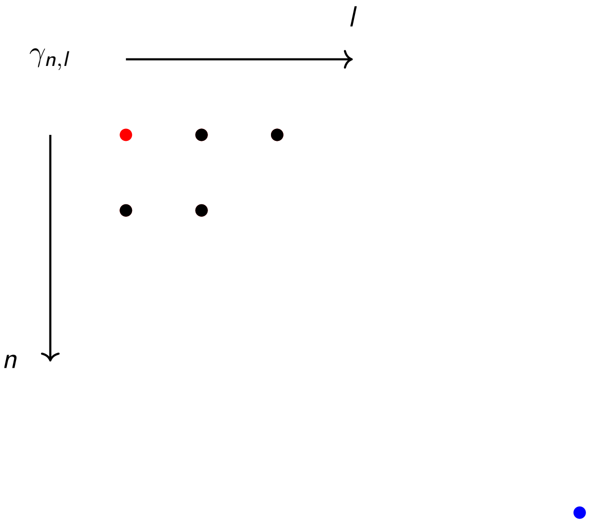

Future Outlook

- For $n = 0$: does $\gamma_{0,l}^{t+s} = \frac{4}{(l+1)(2l+1)(2l+3)}$ fit constraints for large spin expansion from conformal bootstrap?
- Connection to critical Ising model for $\Delta = \frac{1}{2}$?
- Combining both conformal weights: Schwinger-Keldysh formalism for cosmological correlators [2312.13803] [2503.10598]?

Conclusion

- We study the dual conformal field theory to a ϕ^4 theory in AdS_3.
- The one loop diagram in the bulk is found to be a summation over tree level diagrams.
- In the s-channel information about the dual conformal field theory is extracted by reading off the spectral function of the one loop diagram.
- In the t- and u-channel conformal block expansion is performed, and a recurrence relation of the anomalous dimensions is found.
- Questions? Comments? Wish to collaborate or hire a postdoc in 2026? Feel free to contact me at w.xiao@lmu.de