Connecting boundary entropy and effective central charge at interfaces

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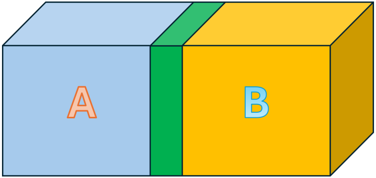
Introduction

Motivation

Interfaces are ubiquitous:

- In condensed matter: boundaries between materials (QHE, topological insulators), Josephson junctions, impurities (in wires),...
- In relativistic quantum field theories: domain walls, Wilson lines (in 1+1), ...
- In string theory: branes ending on branes, orientifolds, ...



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Example of an interface:

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We are interested in the entanglement entropy

Entanglement entropy

- To a state $|\psi\rangle$ we associate a density operator $\hat{\rho} = |\psi\rangle\langle\psi|$
- The Hilbert space s a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, the reduced density operator in A is
$$\hat{\rho}_A = \mathrm{Tr}_{\bar{A}}(\hat{\rho})$$
In general, $\hat{\rho}_A$ takes the form of the density operator of a mixed state in the subspace A
- The Entanglement Entropy of A is defined as the von Neumann entropy of the reduced density operator
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$$|\psi\rangle = |\uparrow_A \downarrow_{\bar{A}}\rangle \;\Rightarrow\; \hat{\rho}_A = |\uparrow_A\rangle\langle\uparrow_A| \;\Rightarrow\; S_A = 0$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A \downarrow_{\bar{A}}\rangle - |\downarrow_A \uparrow_{\bar{A}}\rangle\right) \;\Rightarrow\; \hat{\rho}_A = \frac{1}{2}|\uparrow_A\rangle\langle\uparrow_A| + \frac{1}{2}|\downarrow_A\rangle\langle\downarrow_A|$$

$$\Rightarrow\; S_A = \ln 2$$

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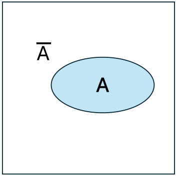
In our case $A =$ compact region of space

$\mathrm{Tr}_{\bar{A}} =$ 'trace' over d.o.f. outside of A *



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- What is a c-theorem?

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- What makes the EE good for proofs of c-theorems?

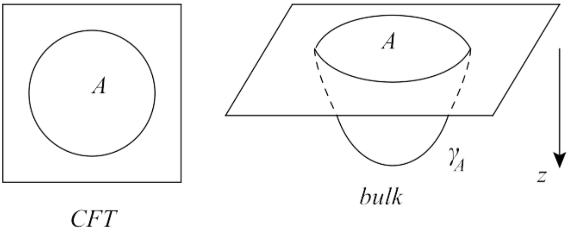
- A property known as Strong Subadditivity (SSA). For two regions of space A and B, there is the following inequality

$$S(A) + S(B) - S(A \cap B) - S(A \cup B) \geq 0$$

Our setup

- We focus on codimension one interfaces in $1+1$ dimensions
- The interfaces preserve some conformal invariance: ICFTs
- Concrete examples will be given in terms of their holographic duals
- We study spacelike intervals and apply the Ryu-Takayanagi prescription

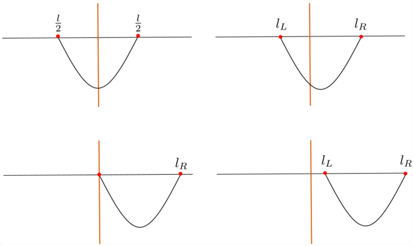
$$S_A = \frac{1}{4G}\,(\text{Minimal area of surfaces homologous to } A)$$



[Imseis '21]

Regularized entanglement entropy

The EE of an interval depends on the distance of the endpoints to the interface



Crossing intervals

$$l = l_R + l_L$$

Non-crossing intervals

$$l = l_R - l_L$$

$$S_A = \frac{c}{3} \log \left(\frac{l}{\epsilon} \right) + \log g^{(1)}$$

CFT: $\log g^{(1)} = 0$

BCFT: $\log g^{(1)} = \log g$

$$S_A = \frac{c}{6} \log \left(\frac{2 l_L}{\epsilon} \right) + \frac{c}{6} \log \left(\frac{2 l_R}{\epsilon} \right) + \log g^{(2)}$$

BCFT$_L$ $\times$ BCFT$_R$:

$\log g^{(2)} = 0$

Special characteristics of EE in ICFTs

Both in free field theories and in holographic models:

1) g_{eff}-theorem for crossing intervals (follows from SSA) [Afxonidis, Karch, Murdia '24]

$$\lim_{\frac{l_L}{l_R}\to 0} \frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}} \leq \frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}} \leq 0$$

$\log g^{(2)}$ is a monotonically decreasing function of l_L/l_R

2) Effective central charge when an endpoint is at the interface

[Sakai, Satoh '08; Brehm, Brunner '15; Karch, Luo, Sun '21; Karch, Wang '22]

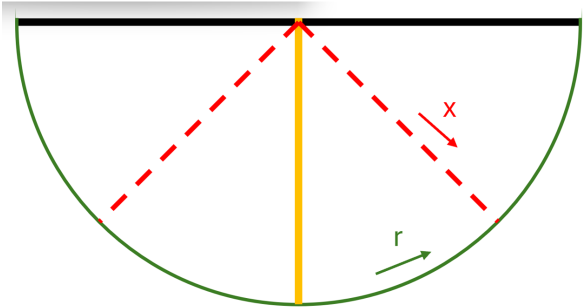
$$S_A \sim \frac{c + c_{\text{eff}}}{6} \log\left(\frac{2l_R}{\epsilon}\right)$$

Bound $c_{\text{eff}} \leq \min(c_L, c_R)$ [Karch, Kusuki, Ooguri, Sun Wang '23, '24]

Holographic ICFTs and EE

Holographic dual geometries of ICFTs

$$ds^2 = R^2\left(dr^2 + e^{2A(r)}\left(\frac{-dt^2+dx^2}{x^2}\right)\right)$$



Holographic dual geometries of ICFTs

$$ds^2 = R^2 \left(dr^2 + e^{2A(r)} \left(\frac{-dt^2 + dx^2}{x^2} \right) \right)$$

- Pure AdS_3 of radius R for $e^A = \cosh r \to CFT_2$
- Simpler bottom-up model: Randal-Sundrum braneworld
- Non-supersymmetric backreacted model: Janus solution
- Supersymmetric top-down model: super Janus solution
- Minimal value of the warp factor at $r = r_*$: $A_* = A(r_*)$
  Determines the effective central charge $c_{\text{eff}} = ce^{A_*}$

Holographic dual geometries of ICFTs

- RS braneworld $(r_* \neq 0)$ [Karch, Randall '00, '01]

$$e^A = \cosh(|r| - r_*), \quad e^{A_*} = 1$$

- Janus $(r_* = 0)$ [Bak, Gutperle, Hirano '07]

$$e^A = \left[\frac{1}{2}\left(1 + \sqrt{1-2\gamma^2}\cosh(2r)\right)\right]^{1/2}, \quad e^{A_*} = \left(\frac{1}{2}\left(1 + \sqrt{1-2\gamma^2}\right)\right)^{1/2}$$

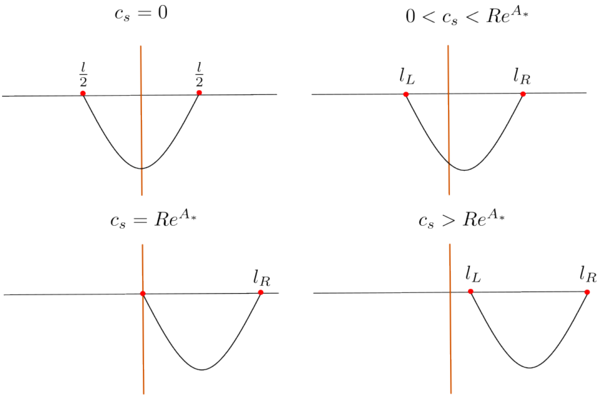
- super Janus $(r_* = 0)$ [Chiodaroli, Gutperle, Krym '09; Chiodaroli, Gutperle, Hung '10; Baig, Karch, Wang '24]

$$e^A = \frac{\cosh r}{\cosh\psi \cosh\theta}, \quad e^{A_*} = \frac{1}{\cosh\psi\cosh\theta}$$

Minimal area surface

The profile $x(r)$ is determined by

$$\frac{x'}{x} = \pm \frac{|c_s| e^{-A}}{\sqrt{R^2\, e^{2A} - c_s^2}}$$



Minimal area surface

Finite interface entropy contribution

$$\log g^{(2)} = \frac{c}{3}\left[\int_{r_{\mathsf{min}}}^{\infty} dr \left(\frac{e^{A}}{R\sqrt{e^{2A}R^2 - c_s^2}} - 1\right) - r_{\mathsf{min}}\right]$$

Ratio of the distance of the endpoints to the interface

$$\log\left(\frac{l_L}{l_R}\right) = -2\int_{r_{\mathsf{min}}}^{\infty} \frac{c_s e^{-A}}{\sqrt{e^{2A}R^2 - c_s^2}} dr$$

Crossing intervals have $r_{\mathsf{min}} = 0$ and non-crossing $|r_{\mathsf{min}}| \geq r_*$.

Monotonicity properties

Change of interface entropy with ratio of distance of endpoints

$$\frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}} = \frac{d\log g^{(2)}}{dc_s}\left(\frac{d\log\frac{l_L}{l_R}}{dc_s}\right)^{-1}$$

The interface entropy is a monotonically decreasing function of the ratio

$$\frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}} = -\frac{c}{6}\frac{c_s}{R} \le 0$$

The change is faster for non-crossing intervals

$$\left|\frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}}\right|_{\text{crossing}} \le \left|\frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}}\right|_{\text{non-crossing}}$$

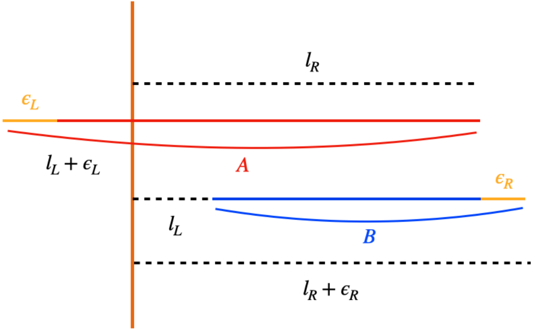
Interval with endpoints at the boundary

$$\lim_{\frac{l_L}{l_R}\to 0}\frac{d\log g^{(2)}}{d\log\frac{l_L}{l_R}} = -\frac{ce^{A_*}}{6} = -\frac{c_{\text{eff}}}{6}$$

Interface entropy and effective central charge

g-theorem for non-crossing intervals

$$S_c\left(l_L+\epsilon_L,l_R\right)+S_{nc}\left(l_L,l_R+\epsilon_R\right)-S_c\left(l_L+\epsilon_L,l_R+\epsilon_R\right)-S_{nc}\left(l_L,l_R\right)\geq 0$$



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$$-\epsilon_R\left(\frac{\partial S_c(l_L,l_R)}{\partial l_R}-\frac{\partial S_{nc}(l_L,l_R)}{\partial l_R}\right)+\mathcal{O}\left(\epsilon^2\right)\geq 0$$

$$\frac{d\log g_{nc}^{(2)}}{d\log\frac{l_L}{l_R}}\leq\frac{d\log g_c^{(2)}}{d\log\frac{l_L}{l_R}}\leq 0$$

In agreement with monotonicity properties

One can reverse the argument and use the monotoncity properties to show that the SSA is satisfied

Intervals with an endpoint at the interface

- $\frac{l_L}{l_R} = 0$ corresponds to $c_s = Re^{A_*}$
- The integrands in $\log g^{(2)}$ and $\log \frac{l_L}{l_R}$ become divergent

$$e^{A(r)} \approx e^{A_*} + b_2(r - r_*)^2 + \dots$$

Then

$$\frac{1}{\sqrt{e^{2A}R^2 - c_s^2}} \approx \frac{1}{R\sqrt{(e^{A_*} + b_2(r - r_*)^2)^2 - e^{2A_*}}} \approx \frac{1}{R\sqrt{b_2}|r - r_*|}$$

Which makes

$$\log g^{(2)} \to \infty, \qquad \log\left(\frac{l_L}{l_R}\right) \to -\infty$$

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- Expanding the integrals for $c_s \to Re^{A_*}$, $r \approx r_*$, we find

$$\log g^{(2)} \sim -\frac{ce^{A_*}}{6} \log\left(\frac{l_L}{l_R}\right) + \text{finite}$$

The origin of c_{eff}

- The position of the endpoint of an interval is determined up to distances of the order of the cutoff $l_L = \lambda\epsilon$

$$S_A \sim \frac{c}{6}\log\left(\frac{2l_R}{\epsilon}\right) + \frac{c}{6}\log\left(\frac{2\lambda\epsilon}{\epsilon}\right) + \log g^{(2)}\Big|_{l_L=\lambda\epsilon}$$

- Using the limit of $\log g^{(2)}$ for $\frac{l_L}{l_R} \to 0$ ($c_s \to Re^{A_*}$)

$$S_A \sim \frac{c + c_{\text{eff}}}{6}\log\left(\frac{2l_R}{\epsilon}\right) + \text{finite}, \qquad c_{\text{eff}} = ce^{A_*}$$

We reproduce previous results

The effective central charge originates in the finite interface entropy!

A c_{eff}-theorem

- Introduce the pseudo-beta function

$$B_g = \frac{d\log g^{(2)}}{d\log \frac{l_L}{l_R}}, \quad B_g \leq 0 \; (g_{eff} - \text{theorem})$$

- We can prove

$$\frac{dB_g}{d\log \frac{l_L}{l_R}} \leq 0$$

- Then, the following quantity behaves as a c-function of the ratio

$$C_{\text{eff}} = -6B_g, \quad \frac{dC_{\text{eff}}}{d\log \frac{l_L}{l_R}} \geq 0,$$

$$\text{UV:} \quad \lim_{\frac{l_L}{l_R}\to 0} C_{\text{eff}} = c_{\text{eff}}, \qquad \text{IR:} \quad \lim_{\frac{l_L}{l_R}\to 1} C_{\text{eff}} = 0$$

Note however that this is not an actual RG flow, the ICFT theory remains fixed

Other results

1) Boundary entropy number in crossing and non-crossing intervals:

$$\lim_{\frac{l_L}{l_R}\to 1} \log g_c^{(2)} = \log g, \qquad \lim_{\frac{l_L}{l_R}\to 1} \log g_{nc}^{(2)} = -\log g$$

2) We can define a finite scheme-independent finite interface entropy:

$$\log g_i \equiv \lim_{\frac{l_L}{l_R}\to 0} \left(S_A^{\text{crossing}}(l_L, l_R) - S_A^{\text{non-crossing}}(l_L, l_R) \right)$$

In holographic models: $\log g_i = \dfrac{c}{3} \displaystyle\int_0^{r_*} dr\, \sqrt{1 - e^{-2(A-A_*)}}$

(super) Janus: $\log g_i = 0$, RS braneworld: $\log g_i = \log \cosh r_*$

3) In an asymmetric ICFT $c_L \neq c_R$ dual to an RS braneworld
Crossing intervals: $c_{\text{eff}} = \min(c_L, c_R)$
Non-crossing intervals: $c_{\text{eff}} = c_L(c_R)$ to the left (right) of the interface

Outlook

- Field theory calculation of interface entropy for non-crossing intervals
- RG flows at the interface or in the bulk (e.g. Kondo-like impurities)

  [Erdmenger, Flory, Hoyos, Newrzella, O'Bannon, Wu '15; Erdmenger, Melby-Thompson, Northe '20]
- c and g-theorems for proper RG flows
  - Boosted intervals [Casini, Huerta '04; Takayanagi '11],
    Covariant EE prescription [Hubeny, Rangamani, Takayanagi '07]
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谢谢