The compactified D-brane cylinder amplitude and T duality

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Outline

- A brief introduction to T-duality in Type II superstrings
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- The cylinder amplitude and T duality

T-duality

Consider a Type II (closed) superstring moving in 10D with one spatial direction X compactified on a circle with a radius R.

The T-duality implemented along this circle is

$$R \to \frac{\alpha'}{R}, \quad K \leftrightarrow W, \quad g_s \to \frac{\sqrt{\alpha'}}{R}\, g_s. \tag{1.1}$$

Note that

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}}\left(\frac{K}{R} - \frac{WR}{\alpha'}\right), \quad \tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}\left(\frac{K}{R} + \frac{WR}{\alpha'}\right). \tag{1.2}$$

So the above T duality amounts to the following

$$\alpha_0 \to -\alpha_0, \quad \psi_0 \to -\psi_0; \quad \tilde{\alpha}_0 \to \tilde{\alpha}_0, \quad \tilde{\psi}_0 \to \tilde{\psi}_0. \tag{1.3}$$

This can be generalized to the oscillation modes and to have

$$\begin{cases} \alpha_n \to -\alpha_n, \quad x \leftrightarrow \hat{x}, \quad \psi_t \to -\psi_t; \\ X_R(\tau - \sigma) \to -X_R(\tau - \sigma), \quad \psi_R(\tau - \sigma) \to -\psi_R(\tau - \sigma) \end{cases} \tag{1.4}$$

for the right-mover and

T-duality

while leave the left-mover untouched

$$\begin{cases} \tilde{\alpha}_n \to \tilde{\alpha}_n, \quad x \leftrightarrow \hat{x}, \quad \tilde{\psi}_t \to \tilde{\psi}_t; \\ X_L(\tau+\sigma) \to X_L(\tau+\sigma), \quad \psi_L(\tau+\sigma) \to \psi_L(\tau+\sigma). \end{cases} \tag{1.5}$$

In the above the index n is an integer while the index t is an integer in R-sector and a half-integer in NS-sector.

Under T duality, the generators of Virasoro algebra of the matter part for either left or right mover remain invariant. We can see this easily from the following zero modes as examples.

$$\begin{aligned} L_0^{\text{matter}} &= \frac{\alpha'}{4}\left(\frac{K}{R} - \frac{WR}{\alpha'}\right)^2 + \frac{\alpha'}{4}\hat{p}^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \sum_{t>0}^{\infty} t\,\psi_{-t}\cdot\psi_t, \\ \tilde{L}_0^{\text{matter}} &= \frac{\alpha'}{4}\left(\frac{K}{R} + \frac{WR}{\alpha'}\right)^2 + \frac{\alpha'}{4}\hat{p}^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \sum_{t>0}^{\infty} t\,\tilde{\psi}_{-t}\cdot\tilde{\psi}_t, \end{aligned} \tag{1.6}$$

where $\hat{p}$ labels the momentum along the un-compactified directions.

T-duality

Note that in R-sector, we have fermionic zero modes ψ_0 and $\tilde\psi_0$ and they will transform under the T-duality along the compactified direction as

$$\psi_0 \to -\psi_0, \qquad \tilde\psi_0 \to \tilde\psi_0. \tag{1.7}$$

Note also $\psi_0 \sim \Gamma$ with Γ the Dirac matrix along the compact direction (Note $\tilde\psi_0 \sim \Gamma_{11}\Gamma$). This implies that $\Gamma \to -\Gamma$ under T-duality. So this further implies that the chiral operator $\Gamma^{11} \to -\Gamma^{11}$. So under T-duality, the chirality for one of the two chiral spinors in Type II flips which gives

$$\mathrm{IIA} \leftrightarrow \mathrm{IIB}. \tag{1.8}$$

D-brane boundary state and T duality

It is known that a T-duality along a Dp brane worldvolume direction will change it to a D(p -1) brane while a T duality along a direction transverse to it will change it to a D(p + 1) brane.

Then how to implement this T duality on a Dp brane boundary state and to the cylinder D brane amplitude?

We here list the (GSO projected) D-brane boundary state representation carrying no worldvolume flux which is used in computing the closed string tree cylinder amplitude between two D-branes. We have two sectors, namely, NS-NS sector and R-R sector, to consider. They are

$$|B\rangle_{\rm NSNS} = \frac{1}{2}\left[|B,+\rangle_{\rm NSNS} - |B,-\rangle_{\rm NSNS}\right],$$
$$|B\rangle_{\rm RR} = \frac{1}{2}\left[|B,+\rangle_{\rm RR} + |B,-\rangle_{\rm RR}\right]. \quad (2.1)$$

Here the boundary state $|B,\eta\rangle$ with $\eta = \pm$ for a Dp-brane, say from Di Vecchia et al hep-th/9912161, can be expressed as the product of a matter part and a ghost part, i.e.

$$|B,\eta\rangle = |B_{\rm mat},\eta\rangle|B_{\rm g},\eta\rangle, \quad (2.2)$$

D-brane boundary state and T duality

where

$$|B_{\rm mat},\eta\rangle = |B_X\rangle|B_\psi,\eta\rangle, \quad |B_{\rm g},\eta\rangle = |B_{\rm gh}\rangle|B_{\rm sgh},\eta\rangle, \tag{2.3}$$

and

$$|B_X\rangle = \exp(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\cdot S\cdot\tilde\alpha_{-n})|B_X\rangle_0, \tag{2.4}$$

and

$$|B_\psi,\eta\rangle_{\rm NS} = -{\rm i}\,\exp(i\eta\sum_{m=1/2}^{\infty}\psi_{-m}\cdot S\cdot\tilde\psi_{-m})|0\rangle, \tag{2.5}$$

for the NS-NS sector and

$$|B_\psi,\eta\rangle_{\rm R} = -\exp(i\eta\sum_{m=1}^{\infty}\psi_{-m}\cdot S\cdot\tilde\psi_{-m})|B,\eta\rangle_{0{\rm R}}, \tag{2.6}$$

for the R-R sector. The ghost boundary states are the standard ones as given in Billo et al hep-th/9802088.

The above matrix S for a Dp brane is, with $\alpha,\beta = 0,1,\cdots p$ and $i,j = p+1,\cdots 9-p$,

$$S_p = (\eta_{\alpha\beta}, -\delta_{ij}). \tag{2.7}$$

D-brane boundary state and T duality

The zero-mode boundary state is

$$|B_X\rangle_0 = \frac{c_p}{2}\,\delta^{(9-p)}(q^i - y^i)|k^\mu = 0\rangle, \tag{2.8}$$

for the bosonic sector with the overall normalization $c_p = \sqrt{\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p}$, and

$$|B_\psi, \eta\rangle_{0\mathrm{R}} = \left(C\Gamma^0\Gamma^1\cdots\Gamma^p\frac{1+\mathrm{i}\eta\Gamma_{11}}{1+\mathrm{i}\eta}\right)_{AB}|A\rangle|\tilde{B}\rangle, \tag{2.9}$$

for the R-R sector.

In the above, the Greek indices $\alpha, \beta, \cdots$ label the world-volume directions $0, 1, \cdots, p$ along which the Dp brane extends, while the Latin indices $i, j, \cdots$ label the directions transverse to the brane, i.e., $p+1, \cdots, 9$.

We also have denoted by y^i the positions of the D-brane along the transverse directions, by C the charge conjugation matrix. $|A\rangle|\tilde{B}\rangle$ stands for the spinor vacuum of the R-R sector.

Note that the η in the above denotes either sign $\pm$ or the worldvolume Minkowski flat metric and should be clear from the content.

D-brane boundary state and T duality

We come now to examine how the boundary state transforms under a T-duality.

For this, let us first examine how each of the exponential factors due to either the bosonic or the fermionic oscillators in the respective boundary states of matter part transforms under the T -duality.

It is clear that the T-duality amounts to the change of the matrix S_p given in (2.7) as

$$S_p = (\eta_{\alpha\beta}, -\delta_{ij}) \to S_{p-1} = (\eta_{\alpha'\beta'}, -\delta_{i'j'}), \tag{2.10}$$

with now $\alpha', \beta' = 0, 1, \cdots p - 1$ and $i', j' = p, \cdots 10 - p$ if we choose for simplicity, for example, the compactified $X = X^p$ or

$$S_p = (\eta_{\alpha\beta}, -\delta_{ij}) \to S_{p+1} = (\eta_{\alpha'\beta'}, -\delta_{i'j'}), \tag{2.11}$$

with now $\alpha', \beta' = 0, 1, \cdots p + 1$ and $i', j' = p + 2, \cdots 8 - p$ if we choose for simplicity, for example, the compactified $X = X^{p+1}$.

D-brane boundary state and T duality

If choose $X = X^p$, under the T duality, $\Gamma^p \to -\Gamma^p \Rightarrow \Gamma_{11} \to -\Gamma_{11}$. We have then from (2.9), noticing under T duality $C \to C$,

$$
\begin{aligned}
|B^p_\psi, \eta\rangle_{0\mathrm{R}} &= \left(C\Gamma^0\Gamma^1\cdots\Gamma^p \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AB} |A\rangle|\tilde{B}\rangle \\
&\to -\left(C\Gamma^0\Gamma^1\cdots\Gamma^p \frac{1 - i\eta\Gamma_{11}}{1 + i\eta}\right)_{AB} |A\rangle|\tilde{B}\rangle \\
&= -\left(C\Gamma^0\Gamma^1\cdots\Gamma^{p-1} \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AD} |A\rangle \left(\Gamma^p\right)_{DB} |\tilde{B}\rangle. \quad (2.12)
\end{aligned}
$$

Note also that we can choose $(\Gamma_{11})_{AB}|B\rangle = |A\rangle$ and $(\Gamma_{11})_{AB}|\tilde{B}\rangle = \pm|\tilde{A}\rangle$ with the '$+$' sign corresponding to IIB while the '$-$' sign to IIA. Denoting $|\tilde{\mathcal{B}}\rangle = -\left(\Gamma^p\right)_{BD}|\tilde{D}\rangle$, we have $\left(\Gamma_{11}\right)_{AB}|\tilde{\mathcal{B}}\rangle = \mp|\tilde{\mathcal{A}}\rangle$. In other words, after the T duality, we transform $|\tilde{B}\rangle$ to $|\tilde{\mathcal{B}}\rangle$ with an opposite chirality, i.e. $\mathrm{IIA} \leftrightarrow \mathrm{IIB}$, as expected. We have then

$$
\begin{aligned}
|B^p_\psi, \eta\rangle_{0\mathrm{R}} &\to -\left(C\Gamma^0\Gamma^1\cdots\Gamma^{p-1} \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AD} |A\rangle \left(\Gamma^p\right)_{DB} |\tilde{B}\rangle \\
&= \left(C\Gamma^0\Gamma^1\cdots\Gamma^{p-1} \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AB} |A\rangle|\tilde{\mathcal{B}}\rangle = |B^{p-1}_\psi, \eta\rangle_{0\mathrm{R}}.
\end{aligned}
\quad (2.13)
$$

D-brane boundary state and T duality

On the other hand, if choose $X = X^{p+1}$, still $\Gamma_{11} \to -\Gamma_{11}$ due to $\Gamma^{p+1} \to -\Gamma^{p+1}$ under the T duality. We then have, from (2.9),

$$
\begin{aligned}
|B^p_\psi, \eta\rangle_{0\mathrm{R}} &= \left(C\Gamma^0\Gamma^1\cdots\Gamma^p \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AB} |A\rangle|\tilde{B}\rangle \\
&\to \left(C\Gamma^0\Gamma^1\cdots\Gamma^p \frac{1 - i\eta\Gamma_{11}}{1 + i\eta}\right)_{AB} |A\rangle|\tilde{B}\rangle \\
&= \left(C\Gamma^0\Gamma^1\cdots\Gamma^p \frac{1 - i\eta\Gamma_{11}}{1 + i\eta}\left(\Gamma^{p+1}\right)^2\right)_{AB} |A\rangle|\tilde{B}\rangle \\
&= \left(C\Gamma^0\Gamma^1\cdots\Gamma^p\Gamma^{p+1} \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AD} |A\rangle \left(\Gamma^{p+1}\right)_{DB} |\tilde{B}\rangle \\
&= \left(C\Gamma^0\Gamma^1\cdots\Gamma^p\Gamma^{p+1} \frac{1 + i\eta\Gamma_{11}}{1 + i\eta}\right)_{AB} |A\rangle|\tilde{\mathcal{B}}\rangle \\
&= |B^{p+1}_\psi, \eta\rangle_{0\mathrm{R}},
\end{aligned}
\tag{2.14}
$$

where $|\tilde{\mathcal{B}}\rangle \equiv \left(\Gamma^{p+1}\right)_{BD} |\tilde{D}\rangle$.

So under the T duality, we still have the expected $|B^p_\psi, \eta\rangle_{0\mathrm{R}} \to |B^{p+1}_\psi, \eta\rangle_{0\mathrm{R}}$ and $\mathrm{IIA} \leftrightarrow \mathrm{IIB}$.

D-brane boundary state and T duality

So the only thing left is to check how the bosonic zero mode boundary state transforms under the T duality either along a longitudinal or transverse direction.

Note that the bosonic zero-mode boundary state changes from the non-compactified one (2.8) to the following compactified one

$$
\begin{aligned}
|\Omega_p\rangle &= \mathcal{N}_p \prod_{i=1}^{k} \left[\sum_{\omega_{\alpha_i}} e^{-i\, y^{\alpha_i} \omega_{\alpha_i} r_{\alpha_i}/\alpha'} |n_{\alpha_i} = 0, \omega_{\alpha_i}\rangle \right] |k^0 = 0, \hat{k}^{\|} = 0\rangle \\
&\quad \times \prod_{m=1}^{l} \left[\sum_{n_{j_m}} e^{-i\, y^{j_m} n_{j_m}/R_{j_m}} |n_{j_m}, \omega_{j_m} = 0\rangle \right] \hat{\delta}^{(\perp)} \left(\hat{q}^{\perp} - \hat{y}^{\perp} \right) |\hat{k}^{\perp} = 0\rangle,
\end{aligned} \tag{2.15}
$$

where we consider a Dp brane with k longitudinal compactified directions of radii r_{α_i} $(i = 1, \cdots k)$ and l transverse compactified directions of radii R_{j_m} $(m = 1, \cdots, l)$, generalized to different radii from that given in DiVecchia & Liccardo hep-th/9912275 and we use '^' to denote those non-compactified directions, either longitudinal or transverse.

D-brane boundary state and T duality

In the above, the normalization factor

$$\mathcal{N}_p = \frac{c_p}{2} \prod_{i=1}^{k} \left(\frac{2\pi r_{\alpha_i}}{\Phi_{\alpha_i}} \right)^{1/2} \prod_{m=1}^{l} \left(\frac{1}{2\pi R_{j_m} \Phi_{j_m}} \right)^{1/2}. \quad (2.16)$$

Following DiVecchia & Liccardo hep-th/9912275 , we introduce, for convenience, the 'position' and 'momentum' operators, respectively, for the momentum and winding degrees of freedom, as

$$[q^\mu_\omega, p^\nu_\omega] = i\delta^{\mu\nu}, \quad [q^\mu_n, p^\nu_n] = i\delta^{\mu\nu}, \quad (2.17)$$

where μ, ν are along either the longitudinal or the transverse compactified spatial directions. We then have

$$p^\alpha_n |\Omega_p\rangle = 0, \qquad p^j_\omega |\Omega_p\rangle = 0, \quad (2.18)$$

where α represents one of α_i and j represents one of j_m. By denoting the eigenstate $|n_\nu, \omega_\nu\rangle$ of the respective 'momentum' operators, we have

$$p^\mu_n |n_\mu, \omega_\mu\rangle = \frac{n_\mu}{a_\mu} |n_\mu, \omega_\mu\rangle, \quad p^\mu_\omega |n_\mu, \omega_\mu\rangle = \frac{\omega_\mu a_\mu}{\alpha'} |n_\mu, \omega_\mu\rangle, \quad (2.19)$$

where a_μ is the radius of the compactified direction which can be either one of r_{α_i} or one of R_{j_m} mentioned earlier.

D-brane boundary state and T duality

So

$$|n_\mu, \omega_\mu\rangle \equiv e^{i q_n^\mu n_\mu / a_\mu} \, e^{i q_\omega^\mu \omega_\mu a_\mu / \alpha'} |0, 0\rangle, \tag{2.20}$$

with $|0,0\rangle$ denoting the zero-momentum and zero-winding state.

The normalization of this state is given

$$\langle n'_\mu, \omega'_\mu | n_\mu, \omega_\mu\rangle = \Phi_\mu \, \delta_{n'_\mu, n_\mu} \, \delta_{\omega'_\mu, \omega_\mu}, \tag{2.21}$$

where Φ_μ is the so called 'self-dual' volume which has the following properties

$$\Phi_\mu = 2\pi a_\mu \text{ if } a_\mu \to \infty; \quad \Phi_\mu = \frac{2\pi\alpha'}{a_\mu} \text{ if } a_\mu \to 0. \tag{2.22}$$

Note also that in the decompactification limit we have when $r_\alpha \to \infty$

$$\sum_{\omega_\alpha} e^{i\left(q_\omega^\alpha - y^\alpha\right)\omega_\alpha r_\alpha / \alpha'} |0, 0\rangle \to |0, 0\rangle \tag{2.23}$$

where α is one of α_i,

D-brane boundary state and T duality

and when $R_j \to \infty$

$$\sum_{n_j} e^{i\left(q^j_n - y^j\right) n_j/R_j} |0, 0\rangle \to R_j \int dk^j\, e^{i\left(q^j - y^j\right) k^j} |0, 0\rangle =$$

$$2\pi R_j \int \frac{dk^j}{2\pi} e^{i\left(q^j - y^j\right) k^j} |0, 0\rangle = 2\pi R_j \delta\left(q^j - y^j\right) |0, 0\rangle, \qquad (2.24)$$

where j is one of j_m. In the above, when taking $r_\alpha \to \infty$, only the $\omega_\alpha = 0$ term survives in the sum in (2.23) while in (2.24) we have replaced the sum by an integral of k given by $k^j = n_j/R_j$ when $R_j \to \infty$.

With the above, we now come to perform a T duality along either a longitudinal or a transverse direction to the zero-mode state $|\Omega_p\rangle$ to see if it is consistent with our expectation.

Let us begin with a T duality along a longitudinal direction first. Without loss of generality, let us perform this T-duality along the α_k-direction. We then need to send $r_{\alpha_k} \to \alpha'/r_{\alpha_k}$ and $n_{\alpha_k} \leftrightarrow \omega_{\alpha_k}$.

D-brane boundary state and T duality

We have then

$$
\begin{aligned}
|\Omega_p\rangle \quad \rightarrow \quad & \tilde{\mathcal{N}}_p \prod_{i=1}^{k-1}\left[\sum_{\omega_{\alpha_i}} e^{-iy^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}/\alpha'} |n_{\alpha_i}=0,\omega_{\alpha_i}\rangle\right] |k^0=0,\hat{k}^{\parallel}=0\rangle \\
& \times \left(\sum_{n_{\alpha_k}} e^{-iy^{\alpha_k} n_{\alpha_k}/r_{\alpha_k}} |n_{\alpha_k},\omega_{\alpha_k}=0\rangle\right) \\
& \times \prod_{m=1}^{l}\left[\sum_{n_{j_m}} e^{-i\,y^{j_m} n_{j_m}/R_{j_m}} |n_{j_m},\omega_{j_m}=0\rangle\right] \hat{\delta}^{(\perp)}\left(\hat{q}^{\perp}-\hat{y}^{\perp}\right)|\tilde{k}^{\perp}=0\rangle \\
= \quad & \tilde{\mathcal{N}}_p \prod_{i=1}^{k-1}\left[\sum_{\omega_{\alpha_i}} e^{-i\,y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}/\alpha'} |n_{\alpha_i}=0,\omega_{\alpha_i}\rangle\right] |k^0=0,\hat{k}^{\parallel}=0\rangle \\
& \times \prod_{m=1}^{l+1}\left[\sum_{n_{j_m}} e^{-i\,y^{j_m} n_{j_m}/R_{j_m}} |n_{j_m},\omega_{j_m}=0\rangle\right] \hat{\delta}^{(\perp)}\left(\hat{q}^{\perp}-\hat{y}^{\perp}\right)|\tilde{k}^{\perp}=0\rangle,
\end{aligned}
\tag{2.25}
$$

where in the last equality we have taken $n_{j_{l+1}} = n_{\alpha_k}$, $j_{l+1} = \alpha_k$, $R_{j_{l+1}} = r_{\alpha_k}$.

D-brane boundary state and T duality

Note also under this T duality, we have

$$\begin{aligned}
\mathcal{N}_p \to \tilde{\mathcal{N}}_p &= \frac{c_p}{2}\left(\frac{2\pi\alpha'}{r_{\alpha_k}\Phi_{\alpha_k}}\right)^{1/2}\prod_{i=1}^{k-1}\left(\frac{2\pi r_{\alpha_i}}{\Phi_{\alpha_i}}\right)^{1/2}\prod_{m=1}^{l}\left(\frac{1}{2\pi R_{j_m}\Phi_{j_m}}\right)^{1/2} \\
&= \frac{c_p}{2}\left(2\pi\sqrt{\alpha'}\right)\left(\frac{1}{2\pi r_{\alpha_k}\Phi_{\alpha_k}}\right)^{1/2}\prod_{i=1}^{k-1}\left(\frac{2\pi r_{\alpha_i}}{\Phi_{\alpha_i}}\right)^{1/2}\prod_{m=1}^{l}\left(\frac{1}{2\pi R_{j_m}\Phi_{j_m}}\right)^{1/2} \\
&= \frac{c_{p-1}}{2}\prod_{i=1}^{k-1}\left(\frac{2\pi r_{\alpha_i}}{\Phi_{\alpha_i}}\right)^{1/2}\prod_{m=1}^{l+1}\left(\frac{1}{2\pi R_{j_m}\Phi_{j_m}}\right)^{1/2} = \mathcal{N}_{p-1}. \qquad (2.26)
\end{aligned}$$

With the above, under this T duality, we have the expected transformation

$$|\Omega\rangle_p \to |\Omega\rangle_{p-1}. \qquad (2.27)$$

By a similar token, one can also show, when a T duality is performed along a transverse compactified direction,

$$|\Omega\rangle_p \to |\Omega\rangle_{p+1}, \qquad (2.28)$$

which is also expected.

The compactified D-brane cylinder amplitude

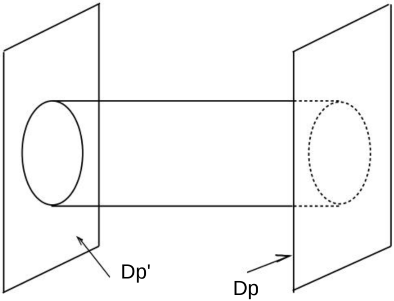
The vacuum amplitude between a Dp' brane and a Dp brane, placed parallel at a separation y, can be calculated via

$$\Gamma_{\mathrm{Dp'|Dp}} = \langle B_{p'}|D|B_p\rangle, \tag{3.1}$$

where D is the closed string propagator defined as

$$D = \frac{\alpha'}{4\pi}\int_{|z|\le 1}\frac{d^2 z}{|z|^2} z^{L_0}\bar{z}^{\tilde{L}_0}. \tag{3.2}$$

Picture-wise, it is



The compactified D-brane cylinder amplitude

In the above, L_0 and $\tilde{L}_0$ are the respective left and right mover total zero-mode Virasoro generators of matter fields, ghosts and superghosts. For example, $L_0 = L_0^X + L_0^\psi + L_0^{\rm gh} + L_0^{\rm sgh}$ where $L_0^X, L_0^\psi, L_0^{\rm gh}$ and $L_0^{\rm sgh}$ represent contributions from matter fields X^μ, matter fields ψ^μ, ghosts b and c, and superghosts β and γ, respectively, and their explicit expressions can be found in any standard discussion of superstring theories, for example in Di Vecchia et al hep-th/9912161, therefore will not be presented here.

The above total vacuum amplitude has contributions from both NS-NS and R-R sectors, respectively, and can be written as $\Gamma_{{\rm Dp}'/{\rm Dp}} = \Gamma^{\rm NSNS}_{{\rm Dp}'/{\rm Dp}} + \Gamma^{\rm RR}_{{\rm Dp}'/{\rm Dp}}$. In calculating either $\Gamma^{\rm NSNS}_{{\rm Dp}'/{\rm Dp}}$ or $\Gamma^{\rm RR}_{{\rm Dp}'/{\rm Dp}}$, we need to keep in mind that the boundary state used should be the GSO projected one as given earlier.

Computing each of them is boiled down to the following one in each sector

$$\Gamma_{{\rm Dp}'|{\rm Dp}}(\eta', \eta) = \langle B^{p'}, \eta' | D | B^p, \eta\rangle, \tag{3.3}$$

with the respective boundary state given by (2.2) for which we also include the compactified case into consideration.

The compactified D-brane cylinder amplitude

Note that $\Gamma_{\mathrm{Dp}'|\mathrm{Dp}}(\eta', \eta) = \Gamma_{\mathrm{Dp}'|\mathrm{Dp}}(\eta'\eta)$ and this amplitude can be factorized as

$$\Gamma_{\mathrm{Dp}'|\mathrm{Dp}}(\eta'\eta) = \frac{\alpha'}{4\pi}\int_{|z|\le 1}\frac{d^2 z}{|z|^2} A^X\, A^{\mathrm{bc}}\, A^{\psi}(\eta'\eta)\, A^{\beta\gamma}(\eta'\eta). \qquad (3.4)$$

In the above, we have

$$A^X = \langle B_X^{p'}|z^{L_0^X}\bar z^{\tilde L_0^X}|B_X^p\rangle, \quad A^\psi(\eta'\eta) = \langle B_\psi^{p'}, \eta'||z|^{2L_0^\psi}|B_\psi^p, \eta\rangle,$$
$$A^{\mathrm{bc}} = \langle B_{\mathrm{gh}}||z|^{2L_0^{\mathrm{gh}}}|B_{\mathrm{gh}}\rangle, \quad A^{\beta\gamma}(\eta'\eta) = \langle B_{\mathrm{sgh}}, \eta'||z|^{2L_0^{\mathrm{sgh}}}|B_{\mathrm{sgh}}, \eta\rangle (3.5)$$

The total amplitude has a contribution from the R-R sector only when $p = p'$ for which this amplitude vanishes due to the cancellation between the contribution from the NS-NS sector and that from the R-R sector because of the 1/2 BPS nature of this system. This certainly still holds true when a T duality is performed.

The compactified D-brane cylinder amplitude

We can understand this easily as follows:

- The bosonic zero-mode contribution to the boundary state remains the same to both sectors.
- The oscillator contributions to the amplitude from all sectors remain the same as before and after the T duality due to that the only quantity relevant to the T duality is the matrix SS'^{T} Jia et al NPB953 (2020) 114947, with the respective S and S' given by (2.7) for the Dp brane and Dp′ brane, which remains invariant under the T duality along any direction.
- As demonstrated in the previous section, the fermionic zero mode boundary state in R-R sector will have a sign change in front of Γ_{11}, for example, see the second line in (2.12) or (2.14), under the T duality. We can absorb this '-' sign by defining $\tilde{\eta} = -\eta$ and $\tilde{\eta}' = -\eta'$. Since this zero-mode contribution to the amplitude depends only on the product $\tilde{\eta}\tilde{\eta}' = \eta\eta'$, not individual $\tilde{\eta}$ and $\tilde{\eta}'$, therefore this contribution will also remain invariant under the T duality.

The compactified D-brane cylinder amplitude

When $p \neq p'$, the cylinder amplitude comes from the NS-NS sector only. As discussed above, only the bosonic zero-mode contribution to the amplitude will change under the T duality along any direction. So this is our focus in computing the closed string tree cylinder amplitude for $p - p' = 2n$ with $n = 0, 1, 2, 3$ though the amplitude vanishes when $n = 0, 2$.

The corresponding non-compactified cylinder amplitude for any of these cases has been given before, say, in Jia et al NPB953 (2020) 114947 when we turn off the worldvolume fluxes:

$$\Gamma_{\mathrm{D(p-2n)|Dp}}(y) = \frac{2^{2-n}\, V_{1+p-2n}}{(8\pi^2\alpha')^{\frac{1+p-2n}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-p}{2}}}\, e^{-\frac{y^2}{2\pi\alpha' t}}\, Z_{\mathrm{(p-2n)|p}}(t), \tag{3.6}$$

where $Z_{\mathrm{(p-2n)|Dp}}(t) = 0$ for $n = 0, 2$,

$$Z_{\mathrm{(p-2)|p}}(t) = \frac{\left(1 + |z|^{4n}\right)^4}{\left(1 - |z|^{2n}\right)^6 \left(1 + |z|^{2n}\right)^2}, \tag{3.7}$$

for $n = 1$ and

$$Z_{\mathrm{(p-6)|p}}(t) = -\prod_{n=1}^{\infty} \frac{(1 + |z|^{4n})^4}{(1 - |z|^{2n})^2 (1 + |z|^{2n})^6}, \tag{3.8}$$

for $n = 3$.

The compactified D-brane cylinder amplitude

Note that the above $Z_{(\mathrm{p-2n})|\mathrm{Dp}}(t)$, due to the oscillator contribution, remains the same whether we have compactifications or not or further under T-duality.

Given what has been described, in order to compute the compactified cylinder amplitude, all we need is to compute the corresponding bosonic zero-mode contribution to this amplitude.

In other words, we need to compute

$$A_0^X = {}_0\langle B_X^{p'}| z^{\frac{1}{2}\alpha_{0R}^2 + \frac{1}{2}\hat{\alpha}_0^2}\, \bar{z}^{\frac{1}{2}\alpha_{0L}^2 + \frac{1}{2}\hat{\alpha}_0^2} |B_X^p\rangle_0, \tag{3.9}$$

where α_{0R}, α_{0L} denote the right-mover and left-mover bosonic zero modes, respectively, along the compactified directions while $\hat{\alpha}_0 = \hat{\tilde{\alpha}}_0$ denote the respective bosonic zero modes along the non-compactified directions. We will use this A_0^X to replace the corresponding one in the non-compactified case in the amplitude (3.4).

Note that in the compactified case $|B_X^p\rangle_0 = |\Omega_p\rangle$, $|B_X^{p'}\rangle_0 = |\Omega_{p'}\rangle$, and also the following

$$\alpha_{0R} = \frac{l_s}{2} p_R = \frac{l_s}{2}\left(p_n - p_\omega\right), \quad \alpha_{0L} = \frac{l_s}{2} p_L = \frac{l_s}{2}\left(p_n + p_\omega\right), \tag{3.10}$$

along with $\hat{\alpha}_0 = \hat{\tilde{\alpha}}_0 = \frac{l_s}{2}\hat{p}$ ($l_s = \sqrt{2\alpha'}$).

The compactified D-brane cylinder amplitude

After long computations (see Lu'25 for detail), we end up with

$$A_0^X = \frac{c_{p-2n} c_p}{4(2\pi^2\alpha')^{\frac{9-p-l}{2}}} V_{1+p-2n-k} \prod_{i=1}^{k} \left(2\pi r_{\alpha_i} \sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\, \omega_{\alpha_i}^2 r_{\alpha_i}^2}{2\alpha'} - i \frac{y^{\alpha_i} \omega_{\alpha_i} r_{\alpha_i}}{\alpha'}} \right) \times \prod_{m=1}^{l} \left(\frac{1}{2\pi R_{j_m}} \sum_{n_{j_m}} e^{-\frac{\pi t \alpha' n_{j_m}^2}{2R_{j_m}^2} - i \frac{y^{j_m} n_{j_m}}{R_{j_m}}} \right) \frac{e^{-\frac{(\hat{y}^\perp)^2}{2\pi\alpha' t}}}{t^{\frac{9-p-l}{2}}}. \tag{3.11}$$

When we take the decompactification limit $r_{\alpha_i} \to \infty$ and $R_{j_m} \to \infty$,

$$A_0^X \to \frac{c_{p-2n} c_p V_{1+p-2n}}{4(2\pi^2\alpha')^{\frac{9-p}{2}}} \frac{e^{-\frac{y^2}{2\pi\alpha' t}}}{t^{\frac{9-p}{2}}}, \tag{3.12}$$

giving precisely the corresponding decompactification limit. In the above, we have set $V_{1+p-2n} = V_{1+p+2n-k} \prod_{i=1}^{k} 2\pi r_{\alpha_i}$ with $r_{\alpha_i} \to \infty$ and $y^2 = (\hat{y}^\perp)^2 + \sum_{m=1}^{l} (y^{j_m})^2$, the brane separation along the transverse directions to both branes when the decompactification limit is taken.

The compactified D-brane cylinder amplitude

With the above, we have the compactified D-brane cylinder amplitude as

$$\Gamma_{\mathrm{D(p-2n)|Dp}}(y) = \frac{2^{2-n-l}\, V_{1+p-2n-k}}{(8\pi^2\alpha')^{\frac{1+p-2n-l}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-p-l}{2}}}\, e^{-\frac{(\hat{y}^\perp)^2}{2\pi\alpha' t}}\, Z_{\mathrm{(p-2n)|p}}(t)$$
$$\times \prod_{i=1}^{k}\left[2\pi r_{\alpha_i}\sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\,\omega_{\alpha_i}^2 r_{\alpha_i}^2}{2\alpha'} - i\frac{y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}}{\alpha'}}\right] \prod_{m=1}^{l}\left[\frac{1}{2\pi R_{j_m}}\sum_{n_{j_m}} e^{-\frac{\pi t\alpha' n_{j_m}^2}{2R_{j_m}^2} - i\frac{y^{j_m} n_{j_m}}{R_{j_m}}}\right], \tag{3.13}$$

which reduces to the expected decompactified one (3.6) when taking the decompactified limit $r_{\alpha_i} \to \infty$ and $R_{j_m} \to \infty$, noticing

$$\prod_{i=1}^{k}\left[2\pi r_{\alpha_i}\sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\,\omega_{\alpha_i}^2 r_{\alpha_i}^2}{2\alpha'} - i\frac{y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}}{\alpha'}}\right] \to \prod_{i=1}^{k} 2\pi r_{\alpha_i},$$
$$\prod_{m=1}^{l}\left[\frac{1}{2\pi R_{j_m}}\sum_{n_{j_m}} e^{-\frac{\pi t\alpha' n_{j_m}^2}{2R_{j_m}^2} - i\frac{y^{j_m} n_{j_m}}{R_{j_m}}}\right] \to \prod_{m=1}^{l}\frac{e^{-\frac{(y^{j_m})^2}{2\pi\alpha' t}}}{\sqrt{2\pi^2\alpha' t}} = \frac{e^{-\sum_{m=1}^{l}\frac{(y^{j_m})^2}{2\pi\alpha' t}}}{(2\pi^2\alpha' t)^{\frac{l}{2}}}. \tag{3.14}$$

The compactified D-brane cylinder amplitude

In summary,

- we have computed the compactified cylinder amplitude (3.13) between a D(p - 2n) and a Dp with $2n \leq p \leq 8$.
- This amplitude has $k \leq p - 2n$ compactified longitudinal directions and $l \leq 9 - p$ compactified transverse directions common to both the branes.
- These two D branes are placed parallel along the non-compactified transverse directions at a separation $\hat{y}^{\perp}$ and along each of the compactified transverse directions at $|y^{j_m}|$.
- y^{α_i} are the Wilson lines turned on along the respective compactified worldvolume directions of the Dp brane while keeping the D(p - 2n) brane absent of these.

T dual along a brane longitudinal direction

Given the cylinder amplitude (3.13), performing a T duality along either a compactified longitudinal direction or a compactified transverse direction becomes easy.

Without loss of generality, let us first perform this T duality specifically along the longitudinal α_k direction. We expect to have

$$\Gamma_{\rm D(p-2n)|Dp} \to \Gamma_{\rm D(p-2n-1|D(p-1)}. \tag{4.1}$$

Let us check if this is indeed true. For this, we send $r_{\alpha_k} \to R_{j_{l+1}} = \alpha'/r_{\alpha_k}$, $\omega_{\alpha_k} \to n_{j_{l+1}} = \omega_{\alpha_k}$, and $y^{\alpha_k} \to y^{j_{l+1}} = y^{\alpha_k}$ to the cylinder amplitude (3.13). We have then

$$2\pi r_{\alpha_k} \sum_{\omega_{\alpha_k}} e^{-\frac{\pi t\, \omega^2_{\alpha_k} r^2_{\alpha_k}}{2\alpha'} - i\frac{y^{\alpha_k}\omega_{\alpha_k} r_{\alpha_k}}{\alpha'}} \to \frac{8\pi^2\alpha'}{2}\frac{1}{2\pi R_{j_{l+1}}}\sum_{n_{j_{l+1}}} e^{-\frac{\pi t \alpha' n^2_{j_{l+1}}}{2R^2_{j_{l+1}}} - i\frac{y^{j_{l+1}} n_{j_{l+1}}}{R_{j_{l+1}}}}. \tag{4.2}$$

T dual along the brane longitudinal direction

We have then from (3.13)

$$
\begin{aligned}
\Gamma_{\mathrm{D(p-2n)|Dp}} &\to \frac{2^{2-n-l}\, V_{1+p-2n-k}}{(8\pi^2\alpha')^{\frac{1+p-2n-l}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-p-l}{2}}}\, e^{-\frac{(\hat{y}^\perp)^2}{2\pi\alpha' t}}\, Z_{\mathrm{(p-2n-1)|(p-1)}}(t) \\
&\times \prod_{i=1}^{k-1}\left[2\pi r_{\alpha_i} \sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\, \omega_{\alpha_i}^2 r_{\alpha_i}^2}{2\alpha'} - i\frac{y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}}{\alpha'}}\right] \prod_{m=1}^{l}\left[\frac{1}{2\pi R_{j_m}} \sum_{n_{j_m}} e^{-\frac{\pi t\alpha' n_{j_m}^2}{2R_{j_m}^2} - i\frac{y^{j_m} n_{j_m}}{R_{j_m}}}\right] \\
&\times \frac{8\pi^2\alpha'}{2} \frac{1}{2\pi R_{j_{l+1}}} \sum_{n_{j_{l+1}}} e^{-\frac{\pi t\alpha' n_{j_{l+1}}^2}{2R_{j_{l+1}}^2} - i\frac{y^{j_{l+1}} n_{j_{l+1}}}{R_{j_{l+1}}}} \\
&= \frac{2^{2-n-(l+1)}\, V_{1+(p-1)-2n-(k-1)}}{(8\pi^2\alpha')^{\frac{1+(p-1)-2n-(l+1)}{2}}} \int_0^\infty \frac{dt}{t^{\frac{9-(p-1)-(l+1)}{2}}}\, e^{-\frac{(\hat{y}^\perp)^2}{2\pi\alpha' t}}\, Z_{\mathrm{(p-2n-1)|(p-1)}}(t) \\
&\times \prod_{i=1}^{k-1}\left[2\pi r_{\alpha_i} \sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\, \omega_{\alpha_i}^2 r_{\alpha_i}^2}{2\alpha'} - i\frac{y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}}{\alpha'}}\right] \prod_{m=1}^{l+1}\left[\frac{1}{2\pi R_{j_m}} \sum_{n_{j_m}} e^{-\frac{\pi t\alpha' n_{j_m}^2}{2R_{j_m}^2} - i\frac{y^{j_m} n_{j_m}}{R_{j_m}}}\right] \\
&= \Gamma_{\mathrm{D(p-2n-1)|D(p-1)}}, \qquad (4.3)
\end{aligned}
$$

where in the first line we have sent $Z_{\mathrm{(p-2n)|p}}(t) \to Z_{\mathrm{(p-2n-1)|(p-1)}}(t) = Z_{\mathrm{(p-2n)|p}}(t)$ due to the inert of various oscillator contributions to the amplitude under T duality and $\hat{y}^\perp$ remains the same. So this goes indeed as expected.

T dual along a direction transverse to both branes

Again without loss of generality, we choose to perform the T duality along j_l direction. We then expect to have

$$\Gamma_{\mathrm{D(p-2n)|Dp}} \to \Gamma_{\mathrm{D(p+1-2n)|D(p+1)}}. \tag{4.4}$$

For this, we send $R_{j_l} \to r_{\alpha_{k+1}} = \alpha'/R_{j_l}$, $n_{j_l} \to \omega_{\alpha_{k+1}} = n_{j_l}$ and $y^{j_l} \to y^{\alpha_{k+1}} = y^{j_l}$.

We then have

$$\frac{1}{2\pi R_{j_l}} \sum_{n_{j_l}} e^{-\frac{\pi t \alpha' n_{j_l}^2}{2R_{j_l}^2} - i\frac{y^{j_l} n_{j_l}}{R_{j_l}}} \to$$

$$\frac{2}{8\pi^2\alpha'} 2\pi r_{\alpha_{k+1}} \sum_{\omega_{\alpha_{k+1}}} e^{-\frac{\pi t \omega^2_{\alpha_{k+1}} r^2_{\alpha_{k+1}}}{2\alpha'} - i\frac{y^{\alpha_{k+1}} \omega_{\alpha_{k+1}} r_{\alpha_{k+1}}}{\alpha'}}. \tag{4.5}$$

T dual along a direction transverse to both branes

We have then from (3.13)

$$
\begin{aligned}
\Gamma_{\mathrm{D(p-2n)|Dp}} &\to \frac{2^{2-n-l}\,V_{1+p-2n-k}}{(8\pi^2\alpha')^{\frac{1+p-2n-l}{2}}}\int_0^\infty \frac{dt}{t^{\frac{9-p-l}{2}}}\,e^{-\frac{(\hat{y}^\perp)^2}{2\pi\alpha' t}}\,Z_{\mathrm{(p+1-2n)|(p+1)}}(t)\\
&\times \prod_{i=1}^{k}\left[2\pi r_{\alpha_i}\sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\,\omega^2_{\alpha_i} r^2_{\alpha_i}}{2\alpha'} - i\frac{y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}}{\alpha'}}\right]\prod_{m=1}^{l-1}\left[\frac{1}{2\pi R_{j_m}}\sum_{n_{j_m}} e^{-\frac{\pi t\alpha' n^2_{j_m}}{2R^2_{j_m}} - i\frac{y^{j_m} n_{j_m}}{R_{j_m}}}\right]\\
&\times \frac{2}{8\pi^2\alpha'}\,2\pi r_{\alpha_{k+1}}\sum_{\omega_{\alpha_{k+1}}} e^{-\frac{\pi t\omega^2_{\alpha_{k+1}} r^2_{\alpha_{k+1}}}{2\alpha'} - i\frac{y^{\alpha_{k+1}}\omega_{\alpha_{k+1}} r_{\alpha_{k+1}}}{\alpha'}}\\
&= \frac{2^{2-n-(l-1)}\,V_{1+(p+1)-2n-(k+1)}}{(8\pi^2\alpha')^{\frac{1+(p+1)-2n-(l-1)}{2}}}\int_0^\infty \frac{dt}{t^{\frac{9-(p+1)-(l-1)}{2}}}\,e^{-\frac{(\hat{y}^\perp)^2}{2\pi\alpha' t}}\,Z_{\mathrm{(p+1-2n)|(p+1)}}(t)\\
&\times \prod_{i=1}^{k+1}\left[2\pi r_{\alpha_i}\sum_{\omega_{\alpha_i}} e^{-\frac{\pi t\,\omega^2_{\alpha_i} r^2_{\alpha_i}}{2\alpha'} - i\frac{y^{\alpha_i}\omega_{\alpha_i} r_{\alpha_i}}{\alpha'}}\right]\prod_{m=1}^{l-1}\left[\frac{1}{2\pi R_{j_m}}\sum_{n_{j_m}} e^{-\frac{\pi t\alpha' n^2_{j_m}}{2R^2_{j_m}} - i\frac{y^{j_m} n_{j_m}}{R_{j_m}}}\right]\\
&= \Gamma_{\mathrm{D(p+1-2n)|D(p+1)}}, \qquad (4.6)
\end{aligned}
$$

where in the first line we have also set $Z_{\mathrm{(p+1-2n)|(p+1)}}(t) = Z_{\mathrm{(p-2n)|Dp}}(t)$ for the same reason as explained earlier. Again we obtain the expected one.

THANK YOU!